Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 2, 367–372

Persistent URL: http://dml.cz/dmlcz/128409

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

A NOTE ON JOINT CAPACITIES IN BANACH ALGEBRAS

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(Received December 30, 1991)

The concept of capacity of a Banach algebra element was introduced by Halmos [1] and extended by Stirling [9] (for alternative approach see also [5], [6]) to mutually commuting n-tuples $(x_1, \ldots, x_n)$ of elements of a Banach algebra A. The main result of [9] states that $\operatorname{cap}\sigma(x_1, \ldots, x_n) \leqslant \operatorname{cap}(x_1, \ldots, x_n) \leqslant 2^n \operatorname{cap}\sigma(x_1, \ldots, x_n)$.

The aim of this paper is to show that $\operatorname{cap}(x_1, \ldots, x_n) = \operatorname{cap}\sigma(x_1, \ldots, x_n)$ for every commuting n-tuple $(x_1, \ldots, x_n)$ of elements of a Banach algebra, so that there is analogy with the Halmos' result for $n = 1$.

Further we show that the joint essential spectrum and the joint spectrum of an mutually commuting n-tuple of operators on a Banach space have the same capacities, which is again analogy to the case $n = 1$, see [8].

All algebras in this paper will be complex and with the unit element. Let $x_1, \ldots, x_n$ be mutually commuting elements of a Banach algebra A. By $\sigma(x_1, \ldots, x_n)$ we denote the Harte spectrum [2], i.e. the set of all n-tuples $(\lambda_1, \ldots, \lambda_n)$ of complex numbers such that either the left or the right ideal generated by $x_i - \lambda_i$ $(i = 1, \ldots, n)$ is proper. Actually, we can take any other joint spectrum instead of the Harte spectrum (see the remark bellow).

Let $n \geqslant 0$, $k \geqslant 0$ be integers. An arbitrary polynomial of degree $\leqslant k$ in n variables may be written in the form

$$p(z_1, \ldots, z_n) = \sum_{|\mu| \leqslant k} a_\mu(p) z^\mu$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is an n-tuple of non-negative integers, $|\mu| = \sum_{j=1}^{n} \mu_j$, the coefficients $a_\mu(p)$ are complex numbers, $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and $z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$.

The set of all polynomials of degree $\leqslant k$ in n variables will be denoted by $\mathscr{P}_k(n)$. Denote further $\mathscr{P}_k^1(n)$ the set of all polynomials $p(z) = \sum_{|\mu|\leqslant k} a_\mu(p)z^\mu \in \mathscr{P}_k(n)$ with $\sum_{|\mu|=k} |a_\mu(p)| = 1$. These polynomials were called monic in [9].

Let $x_1, \ldots, x_n$ be mutually commuting elements of a Banach algebra $\mathbf{A}$. The joint capacity of $(x_1, \ldots, x_n)$ was defined in [9] by

$$\operatorname{cap}(x_1, \ldots, x_n) = \liminf_{k\to\infty} \operatorname{cap}_k(x_1, \ldots, x_n)^{1/k}$$

where

$$\operatorname{cap}_k(x_1, \ldots, x_n) = \inf\{\|p(x_1, \ldots, x_n)\| : p \in \mathscr{P}_k^1(n)\}.$$

For a compact subset $K \subset \mathbf{C}^n$ define the corresponding capacity by

$$\operatorname{cap} K = \liminf_{k\to\infty}(\operatorname{cap}_k K)^{1/k}$$

where

$$\operatorname{cap}_k K = \inf\{\|p\|_K : p \in \mathscr{P}_k^1(n)\} \qquad \text{and} \qquad \|p\|_K = \sup\{|p(z)| : z \in K\}.$$

This capacity was studied in [10] and called the *homogeneous Tshebyshev constant* of a compact set K.

By Siciak [4], the capacity can be expressed in another, more convenient way. Denote by $Q_k(n)$ the set of all polynomials $p(z) = \sum_{|\mu|\leqslant k} z^\mu \in \mathscr{P}_k(n)$ such that

$$\sup\left\{\left|\sum_{|\nu|=k} a_\mu(p)z^\nu\right| : z \in T\right\} = 1$$

where $T = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : |z_i| = 1 \quad (i = 1, \ldots, n)\}$ is the n-dimensional torus.

Theorem 1. *Let $x_1, \ldots, x_n$ be mutually commuting elements of a Banach algebra A. Then*

(a) $\operatorname{cap}(x_1, \ldots, x_n) = \lim_{k\to\infty} \operatorname{cap}_k(x_1, \ldots, x_n)^{1/k} = \inf_k \inf\{\|p(x)\|^{1/k} : p \in Q_k(n)\}$,

(b) $\operatorname{cap}(x_1, \ldots, x_n) = \inf_k \inf\{(\operatorname{cap} p(x_1, \ldots, x_n))^{1/k} : p \in Q_k(n)\}$,

(c) $\operatorname{cap}(x_1, \ldots, x_n) = \operatorname{cap} \sigma(x_1, \ldots, x_n)$.

P r o o f. (a) We use the argument of [4], Remark 9.5.

Let $p = \sum_{|\nu|\leqslant k} a_\nu(p)z^\nu \in \mathscr{P}_k(n)$. By Cauchy formulas we have for every μ with $|\mu| = k$

$$|a_\mu(p)| \leqslant \max\left\{\left|\sum_{|\nu|=k} a_\nu(p)z^\nu\right| : z \in T\right\} = \left\|\sum_{|\nu|=k} a_\nu(p)z^\nu\right\|_T.$$

Further

$$\left\|\sum_{|\nu|=k} a_\nu(p)z^\nu\right\|_T \leqslant \sum_{|\mu|=k} |a_\mu(p)| \leqslant \binom{k+n-1}{n-1}\left\|\sum_{|\nu|=k} a_\nu(p)z^\nu\right\|_T,$$

where $\binom{k+n-1}{n-1}$ is the number of coefficients $a_\mu(p)$ with $|\mu| = k$. Denote by

$$\alpha_k = \inf\{\|p(x_1,\ldots,x_n)\| : p \in Q_k(n)\}.$$

Then

$$\text{(1)} \qquad \operatorname{cap}_k(x_1,\ldots,x_n) \leqslant \alpha_k \leqslant \binom{k+n-1}{n-1}\operatorname{cap}_k(x_1,\ldots,x_n).$$

Let $p \in Q_k(n)$ and let m, s be non-negative integers, $0 \leqslant s \leqslant k-1$. Then $q = p^m \cdot z_1^s \in Q_{mk+s}(n)$. Thus $\alpha_{mk+s} \leqslant \alpha_k^m\|x_1\|^s$, $\alpha_{mk+s}^{1/mk+s} \leqslant \alpha_k^{\frac{m}{mk+s}}\max\{1,\|x_1\|^{k-1}\}^{1/mk+s}$ and $\limsup_{r\to\infty}\alpha_r^{1/r} \leqslant \alpha_k^{1/k}$. So the limit $\lim_{k\to\infty}\alpha_k^{1/k}$ exists and is equal to $\inf_k \alpha_k^{1/k}$.

By (1) the limit $\lim_{k\to\infty}\operatorname{cap}_k(x_1,\ldots,x_n)^{1/k}$ also exists and

$$\begin{aligned}\operatorname{cap}(x_1,\ldots,x_n) &= \lim_{k\to\infty}\operatorname{cap}_k(x_1,\ldots,x_n)^{1/k} = \lim_{k\to\infty}\alpha_k^{1/k}\\ &= \inf_k \alpha_k^{1/k} = \inf_k \inf\{\|p(x_1,\ldots,x_n)\|^{1/k} : p \in Q_k(n)\}.\end{aligned}$$

(b) Let $p \in Q_k(n)$ and let $q = z^s + \sum_{i=0}^{s-1} a_i(q)z^i \in \mathscr{P}_s^1(1) = Q_s(1)$. Then $q \circ p \in Q_{sk}(n)$ so that

$$\text{(2)} \qquad \operatorname{cap}(x_1,\ldots,x_n) \leqslant \|(q\circ p)(x_1,\ldots,x_n)\|^{1/sk} \qquad (q \in Q_s(1)).$$

Hence

$$\begin{aligned}\operatorname{cap}(x_1,\ldots,x_n) &\leqslant \inf_s \inf\{\|q(p(x_1,\ldots,x_n))\|^{1/sk} : q \in Q_s(1)\}\\ &= (\operatorname{cap} p(x_1,\ldots,x_n))^{1/k}\end{aligned}$$

and

$$\operatorname{cap}(x_1,\ldots,x_n) \leqslant \inf_k \inf\{(\operatorname{cap} p(x_1,\ldots,x_n))^{1/k} : p \in Q_k(n)\}.$$

On the other hand $\operatorname{cap} p(x_1, \ldots, x_n) \leqslant \|p(x_1, \ldots, x_n)\|$ for every $p \in Q_k(n)$. Together with (a) this gives $\operatorname{cap}(x_1, \ldots, x_n) = \inf_k \inf\{(\operatorname{cap} p(x_1, \ldots, x_n))^{1/k} : p \in Q_k(n)\}$.

(c) By (2) we have $\operatorname{cap}(x_1, \ldots, x_n) \leqslant \|p(x_1, \ldots, x_n)^s\|^{1/sk}$ for every $p \in Q_k(n)$ and positive integer s. So

$$\operatorname{cap}(x_1, \ldots, x_n) \leqslant \inf_s\{\|p(x_1, \ldots, x_n)^s\|^{1/sk}\} = |p(x_1, \ldots, x_n)|_\sigma^{1/k}.$$

By the spectral mapping theorem for commuting elements $x_1, \ldots, x_n \in A$ (see [2]) we have

$$|p(x_1, \ldots, x_n)|_\sigma^{1/k} = \max\{|p(z)| : z \in \sigma(x_1, \ldots, x_n)\}^{1/k}.$$

So

$$\begin{aligned} \operatorname{cap}(x_1, \ldots, x_n) &\leqslant \inf_k \inf\{\|p\|_{\sigma(x_1, \ldots, x_n)}^{1/k} : p \in Q_k(n)\} \\ &\leqslant \inf_k \binom{k+n-1}{n-1}^{1/k} (\operatorname{cap}_k \sigma(x_1, \ldots, x_n))^{1/k}. \end{aligned}$$

Hence $\operatorname{cap}(x_1, \ldots, x_n) \leqslant \operatorname{cap} \sigma(x_1, \ldots, x_n)$.

On the other hand,

$$\|p(x_1, \ldots, x_n)\| \geqslant |p(x_1, \ldots, x_n)|_\sigma = \|p\|_{\sigma(x_1, \ldots, x_n)}$$

for every polynomial $p \in \mathscr{P}_k(n)$, so that

$$\operatorname{cap}_k(x_1, \ldots, x_n) \geqslant \operatorname{cap}_k \sigma(x_1, \ldots, x_n)$$

and

$$\operatorname{cap}(x_1, \ldots, x_n) \geqslant \operatorname{cap} \sigma(x_1, \ldots, x_n).$$

□

Following the concept of Żelazko [11], a subspectrum $\tilde{\sigma}$ is a set-valued function which assignes to every n-tuple of commuting elements $x_1, \ldots, x_n$ of a Banach algebra A a non-empty compact subset $\tilde{\sigma}(x_1, \ldots, x_n) \subset \mathbf{C}^n$ such that 1) $\tilde{\sigma}(x_1, \ldots, x_n) \subset \prod_{i=1}^n \sigma(x_i)$ and 2) $\tilde{\sigma}(p(x_1, \ldots, x_n)) = p(\tilde{\sigma}(x_1, \ldots, x_n))$ for every m-tuple $p = (p_1, \ldots, p_m)$ of polynomials in n variables.

By [7] (cf. also [6]), $\operatorname{cap} \tilde{\sigma}(x_1, \ldots, x_n) = \operatorname{cap} \sigma(x_1, \ldots, x_n)$ for every subspectrum satisfying

$$\max\{|\lambda| : \lambda \in \tilde{\sigma}(x_1)\} = \max\{|\lambda| : \lambda \in \sigma(x_1)\} \qquad (x_1 \in A).$$

This includes e.g. the approximate point spectrum, the left, right,defect and Taylor spectra. Condition (b) of the previous theorem enables to extend this result to the subspectra satisfying $\operatorname{cap}\tilde{\sigma}(x_1) = \operatorname{cap}\sigma(x_1)$ $(x_1 \in A)$. An important example of such a subspectrum is the essential spectrum of operators in a Banach space.

Corollary. *Let A be a Banach algebra and let $\tilde{\sigma}$ be a subspectrum satisfying $\operatorname{cap}\tilde{\sigma}(x_1) = \operatorname{cap}\sigma(x_1)$ $(x_1 \in A)$. Then*

$$\operatorname{cap}\tilde{\sigma}(x_1, \ldots, x_n) = \operatorname{cap}\sigma(x_1, \ldots, x_n) = \operatorname{cap}(x_1, \ldots, x_n)$$

for every n-tuple $x_1, \ldots, x_n$ of mutually commuting elements of A.

Proof. Let $x_1, \ldots, x_n$ be mutually commuting elements of A. Consider the algebra $C(K)$ of all continuous functions on the compact set $K = \tilde{\sigma}(x_1, \ldots, x_n) \subset \mathbf{C}^n$ with the supnorm on K and let $z_1, \ldots, z_n$ be the independent variables.

As $\|q\|_K = \|q(z_1, \ldots, z_n)\|_{C(K)}$ for every polynomial q it is easy to see that $\operatorname{cap} K = \operatorname{cap}(z_1, \ldots, z_n)$ and $\operatorname{cap} p(K) = \operatorname{cap} p(z_1, \ldots, z_n)$ for every polynomial p. Thus

$$\begin{aligned}
\operatorname{cap}(x_1, \ldots, x_n) &= \inf_k \inf\{(\operatorname{cap} p(x_1, \ldots, x_n))^{1/k} : p \in Q_k(n)\} \\
&= \inf_k \inf\{(\operatorname{cap}\sigma(p(x_1, \ldots, x_n)))^{1/k} : p \in Q_k(n)\} \\
&= \inf_k \inf\{(\operatorname{cap}\tilde{\sigma}(p(x_1, \ldots, x_n)))^{1/k} : p \in Q_k(n)\} \\
&= \inf_k \inf\{(\operatorname{cap} p(\tilde{\sigma}(x_1, \ldots, x_n)))^{1/k} : p \in Q_k(n)\} \\
&= \inf_k \inf\{\operatorname{cap} p(z_1, \ldots, z_n) : p \in Q_k(n)\} \\
&= \operatorname{cap}(z_1, \ldots, z_n) = \operatorname{cap}\tilde{\sigma}(x_1, \ldots, x_n).
\end{aligned}$$

□

Let X be a Banach space. Denote by $B(X)$ the algebra of all bounded operators on X and by $K(X)$ the ideal of all compact operators on X. Denote by π the cannonical projection from $B(X)$ onto the Calkin algebra $B(X)|K(X)$. Let $T_1, \ldots, T_n$ be mutually commuting operators on X. Denote by $\sigma_e(T_1, \ldots, T_n)$ the spectrum of the commuting n-tuple $(\pi(T_1), \ldots, \pi(T_n))$ in the algebra $B(X)|K(X)$.

Let $S \in B(X)$. As $\sigma(S)$ contains only countably many points in the unbounded component of $\mathbf{C} - \sigma_e(S)$ we have $\operatorname{cap}\sigma_e(S) = \operatorname{cap}\sigma(S)$ (cf. [8]). Hence

$$\operatorname{cap}\sigma_e(T_1, \ldots, T_n) = \operatorname{cap}\sigma(T_1, \ldots, T_n) = \operatorname{cap}(T_1, \ldots, T_n)$$

for every mutually commuting operators $T_1, \ldots, T_n \in B(X)$.

Another example when the previous corollary can be used is the essential Taylor spectrum (for the definition see e.g. [3]).

Acknowledgement. This paper was written during the author's stay at the University of Saarbrücken. The research was supported by the Alexander von Humboldt Foundation.

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