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ON A CLASS OF FUNCTIONAL BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETER

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In this paper sufficient conditions concerning only operators Q, F are given for the functional differential equation

$$y''(t) - Q[y, y'](t) \cdot y(t) = F[y, y', \mu](t)$$

depending on the parameter μ to admit, for a suitable value of μ, a solution y satisfying functional boundary conditions

$$\alpha_1\big(y(t_1) - y(t)\big|J_1\big) = 0, \quad y(t_2) = 0, \quad \alpha_2\big(y(t_3) - y(t)\big|J_2\big) = 0,$$

where $-\infty < t_1 < t_2 < t_3 < \infty$, α_i are continuous functionals and $y(t)\big|J_i$ denotes the restriction of y to $J_i = \langle t_i, t_{i+1}\rangle$ $(i = 1, 2)$. Next, sufficient conditions are given under which the above equation has, for a suitable value of the parameter μ, a bounded solution y on the halfline $\langle t_1, \infty)$ and $\alpha_1\big(y(t_1) - y(t)\big|J_1\big) = 0$, $y(t_2) = 0$.

1. INTRODUCTION

Let $-\infty < t_1 < t_2 < t_3 < \infty$, $-\infty < a < b < \infty$, $J = \langle t_1, t_3\rangle$, $J_1 = \langle t_1, t_2\rangle$, $J_2 = \langle t_2, t_3\rangle$, $I = \langle a, b\rangle$ and X $(X_1\,;\, X_2)$ be the Banach space of the C^0-functions on J $(J_1; J_2)$ with the norm $\|y\| = \max\big\{|y(t)|;\, t \in J\big\}$ $\big(\|y\|_1 = \max\big\{|y(t)|;\, t \in J_1\big\}$; $\|y\|_2 = \max\big\{|y(t)|;\, t \in J_2\big\}\big)$. Consider the functional differential equation

$$y''(t) - Q[y, y'](t) \cdot y(t) = F[y, y', \mu](t), \tag{1}$$

depending on a parameter μ. Here $Q\colon X \times X \to X$, $F\colon X \times X \times I \to X$ are continuous operators, $Q[y, z](t) > 0$ on J for all $[y, z] \in X \times X$.

Let $\alpha_i\colon X_i \to R$ $(i = 1,2)$ be continuous increasing (i.e. $\alpha_i(x) < \alpha_i(y)$ for all $x, y \in X_i$, $x(t) < y(t)$ for $t \in J_i - \{t_{2i-1}\}$, $x(t_{2i-1}) = y(t_{2i-1}) = 0$) functionals, $\alpha_i(0) = 0$. The purpose of this paper is to obtain using the Schauder linearization technique and the Schauder fixed point theorem, sufficient conditions imposed on the operators Q, F under which equation (1) admits, for a suitable value of the parameter μ, a solution y satisfying the functional boundary conditions

$$(2) \qquad \alpha_1\big(y(t_1) - y(t)\big|J_1\big) = 0, \quad y(t_2) = 0, \quad \alpha_2\big(y(t_3) - y(t)\big|J_2\big) = 0,$$

where $y(t)\big|J_i$ $(i = 1,2)$ denotes the restriction of y to the interval J_i.

In Section 4, we use BVP (1)–(2) to consider bounded solutions of (1) on the halfline $\langle t_1, \infty)$ satisfying the functional boundary conditions

$$\alpha_1\big(y(t_1) - y(t)\big|J_1\big) = 0, \quad y(t_2) = 0.$$

The paper generalizes the author's results in [1]–[3] and, in a special case, also his results in [4]. In [1] the existence of solutions of (1) satisfying for example the boundary conditions $y(t_1) - y(t_2) = y(t_3) = y(t_4) - y(t_5) = 0$ $(-\infty < t_1 < t_2 < t_3 < t_4 < t_5 < \infty)$ was studied.

In [2] sufficient conditions for the existence (and uniqueness) of solutions of the differential equation

$$(3) \qquad y'' - q(t)y = f(t, y, y', \mu)$$

satisfying the boundary conditions

$$(4) \qquad y(t_1) = y(t_2) = y(t_3) = 0$$

$(-\infty < t_1 < t_2 < t_3 < \infty)$ was established.

In [4] the author considered the functional differential equation

$$y''(t) - q(t)y(t) = f\big(t, y(t), y\big(h_0(t)\big), y'(t), y'\big(h_1(t)\big), \mu\big)$$

with boundary conditions

$$\sum_{i=1}^{m} \alpha_i y(t_i) = 0, \quad y(c) = 0, \quad \sum_{j=1}^{n} \beta_j y(x_j) = 0$$

$(\alpha_i > 0$, $\beta_j > 0$ constants, $a = t_1 < \ldots < t_m < c < x_n < \ldots < x_1 = b)$.

In [3]—among other—sufficient conditions for the boundedness of solutions of (3) on a halfline $\langle t_1, \infty)$ satisfying the boundary conditions $y(t_1) = y(t_2) = 0$ $(t_2 > t_1)$ were obtained.

A functional boundary value problem depending on one parameter was studied also in [5]. In this paper the retarded functional differential equation

$$y'' - q(t)y = f(t, y_t, \mu)$$

with boundary conditions (4) was considered.

2. Notation, lemmas

Let $\varphi \in C^1(J)$ and let u_φ, v_φ be the solutions of the differential equation

$$y'' = Q[\varphi, \varphi'](t)\, y, \tag{5}$$

$u_\varphi(t_2) = 0$, $u'_\varphi(t_2) = 1$, $v_\varphi(t_2) = 1$, $v'_\varphi(t_2) = 0$. For $(t,s) \in J \times J$ define $r(t,s;\varphi)$ and $r'_1(t,s;\varphi)$ by

$$\begin{aligned} r(t,s;\varphi) &= u_\varphi(t)v_\varphi(s) - u_\varphi(s)v_\varphi(t) \ \big(= -r(s,t;\varphi)\big), \\ r'_1(t,s;\varphi) &= u'_\varphi(t)v_\varphi(s) - u_\varphi(s)v'_\varphi(t) \ \Big(= \frac{\partial}{\partial t} r(t,s;\varphi)\Big). \end{aligned}$$

Then $r(t,s;\varphi) > 0$ for all $t_1 \leqslant s < t \leqslant t_3$, $r(t,s;\varphi) < 0$ for all $t_1 \leqslant t < s \leqslant t_3$, $r'_1(t,s;\varphi) > 1$ for all $(t,s) \in J \times J$ and $t \neq s$, $r'_1(t,t;\varphi) = 1$ for all $t \in J$ (for the proof, see e.g. [2]).

Lemma 1. *Assume $\varphi \in C^1(J)$, $h \in C^0(J \times I)$, $h(t,\cdot)$ is increasing on I for each fixed $t \in J$ and*

$$h(t,a)\, h(t,b) \leqslant 0 \quad \text{for all } t \in J. \tag{6}$$

Then there is a unique $\mu_0 \in I$ such that the differential equation

$$y'' = Q[\varphi, \varphi'](t)\, y + h(t,\mu) \tag{7}$$

with $\mu = \mu_0$ admits a solution y satisfying (2). Moreover, this solution y is unique.

Proof. The function $y(t;\mu,c)$ defined on $J \times I \times R$ by

$$y(t;\mu,c) = c\, u_\varphi(t) + \int_{t_2}^{t} r(t,s;\varphi) h(s,\mu)\, \mathrm{d}s$$

is the general solution of (7) vanishing at the point $t = t_2$. Since

$$\begin{aligned} y(t_1;\mu,c) - y(t;\mu,c) &= c\,(u_\varphi(t_1) - u_\varphi(t)) + \\ &+ \int_{t_2}^{t} \big[r(t_1,s;\varphi) - r(t,s;\varphi)\big] h(s,\mu)\,\mathrm{d}s + \int_{t}^{t_1} r(t_1,s;\varphi)h(s,\mu)\,\mathrm{d}s, \\ y(t_3;\mu,c) - y(t;\mu,c) &= c\,(u_\varphi(t_3) - u_\varphi(t)) + \\ &+ \int_{t_2}^{t} \big[r(t_3,s;\varphi) - r(t,s;\varphi)\big] h(s,\mu)\,\mathrm{d}s + \int_{t}^{t_3} r(t_3,s;\varphi)h(s,\mu)\,\mathrm{d}s \end{aligned}$$

and $u_\varphi(t_1) - u_\varphi(t) < 0$ on $(t_1, t_3\rangle$, $u_\varphi(t_3) - u_\varphi(t) > 0$ on $\langle t_1, t_3)$, $r(t_1,s;\varphi) - r(t,s;\varphi) = r_1'(\xi,s;\varphi)(t_1 - t) < 0$ for $(t,s) \in J \times J$, $t \neq t_1$ (where ξ lies between t_1 and t), $r(t_3,s;\varphi) - r(t,s;\varphi) = r_1'(\eta,s;\varphi)(t_3 - t) > 0$ for $(t,s) \in J \times J$, $t \neq t_3$ (where η lies between t_3 and t), we see that the functions $p_i\colon I \times R \to R$, $p_i(\mu,c) = \alpha_i\big(y(t_{2i-1};\mu,c) - y(t;\mu,c)\big|J_i\big)$ $(i = 1,2)$ are continuous on $I \times R$, $p_i(\cdot,c)$ are increasing on I for each fixed $c \in R$, $p_1(\mu,\cdot)$ $\big(p_2(\mu,\cdot)\big)$ is decreasing (increasing) on R for each fixed $\mu \in I$. Finally, one can check that $\lim\limits_{c\to-\infty} p_1(\mu,c) > 0$, $\lim\limits_{c\to\infty} p_1(\mu,c) < 0$, $\lim\limits_{c\to-\infty} p_2(\mu,c) < 0$, $\lim\limits_{c\to\infty} p_2(\mu,c) > 0$ for each fixed $\mu \in I$. Hence there are unique functions $c_i\colon I \to R$ $(i = 1,2)$ such that

$$p_i\big(\mu, c_i(\mu)\big) = 0 \qquad \text{for all } \mu \in I \text{ and } i = 1,2,$$

and $c_1(\mu)$ $(c_2(\mu))$ is increasing (decreasing) on I.

To prove that c_i $(i = 1,2)$ are continuous functions on I we suppose there are sequences $\{\mu_n'\}$, $\{\mu_n''\}$ from I such that $\lim\limits_{n\to\infty} \mu_n' = \lim\limits_{n\to\infty} \mu_n'' = \mu_0$ and $\lim\limits_{n\to\infty} c_i(\mu_n') = \lambda_1$, $\lim\limits_{n\to\infty} c_i(\mu_n'') = \lambda_2$, $\lambda_1 < \lambda_2$, for some $i \in \{1,2\}$. Then $0 = \lim\limits_{n\to\infty} p_i(\mu_n', c_i(\mu_n')) = p_i(\mu_0,\lambda_1)$, $0 = \lim\limits_{n\to\infty} p_i(\mu_n'', c_i(\mu_n'')) = p_i(\mu_0,\lambda_2)$, which is a contradiction to $p_i(\mu_0,\lambda_1) \neq p_i(\mu_0,\lambda_2)$.

It remains to prove the existence of a unique $\mu_0 \in I$ such that $c_1(\mu_0) = c_2(\mu_0)$. Since $h(t,a) \leqslant 0$, $h(t,b) \geqslant 0$ on J (cf. (6)) we have $y(t_1;a,0) - y(t;a,0) \leqslant 0$, $y(t_1;b,0) - y(t;b,0) \geqslant 0$ for $t \in \langle t_1,t_2\rangle$, $y(t_3;a,0) - y(t;a,0) \leqslant 0$, $y(t_3;b,0) - y(t;b,0) \geqslant 0$ for $t \in \langle t_2,t_3\rangle$, and then $p_i(a,0) \leqslant 0$, $p_i(b,0) \geqslant 0$ $(i = 1,2)$. Using the fact that $p_1(a,\cdot)$, $p_1(b,\cdot)$ $(p_2(a,\cdot), p_2(b,\cdot))$ are decreasing (increasing) on R and $p_i\big(a,c_i(a)\big) = 0$, $p_i(b,c_i(b)) = 0$ $(i = 1,2)$, we get $c_1(a) \leqslant 0$, $c_1(b) \geqslant 0$, $c_2(a) \geqslant 0$, $c_2(b) \leqslant 0$, therefore $c_1(a) - c_2(a) \leqslant 0$, $c_1(b) - c_2(b) \geqslant 0$. Since $c_1(\mu) - c_2(\mu)$ is continuous increasing on I, the equation $c_1(\mu) - c_2(\mu) = 0$ has a unique solution on I. □

Next, we will suppose that there exist positive constants r_0, r_1 such that the operators Q, F satisfy the following assumptions:

(H_1) $|F[y,y',\mu](t)| \leqslant r_0 \cdot Q[y,y'](t)$ for all $t \in J$ and $[y,y',\mu] \in D \times I$, where $D = \{[y,y'];\, y \in C^1(J), \|y^{(i)}\| \leqslant r_i \text{ for } i = 0,1\}$;

(H_2) $F[y,y',\mu_1](t) < F[y,y',\mu_2](t)$ for all $t \in J$ and $[y,y'] \in D, \mu_1,\mu_2 \in I, \mu_1 < \mu_2$;

(H_3) $F[y,y',a](t) \cdot F[y,y',b](t) \leqslant 0$ for all $t \in J$ and $[y,y'] \in D$;

(H_4) $\min\{(A + r_0 B)\tau, 2\sqrt{r_0}\sqrt{A + r_0 B}\} \leqslant r_1$, where $A = \sup\{\|F[y,y',\mu]\|;\, [y,y',\mu] \in D \times I\}$, $B = \sup\{\|Q[y,y']\|;\, [y,y'] \in D\}$, $\tau = \max\{t_2 - t_1, t_3 - t_2\}$.

Lemma 2. *Let assumptions* (H_1)–(H_4) *be fulfilled for positive constants* r_0, r_1 *and let* $\varphi \in C^1(J)$, $\|\varphi^{(i)}\| \leqslant r_i$ $(i = 0,1)$. *Then there exists a unique* $\mu_0 \in I$ *such that the equation*

$$y'' = Q[\varphi,\varphi'](t)\, y + F[\varphi,\varphi',\mu](t) \tag{8}$$

with $\mu = \mu_0$ *admits a (then unique) solution* y *satisfying* (2) *and, moreover,*

$$\|y^{(i)}\| \leqslant r_i \quad \text{for } i = 0,1. \tag{9}$$

Proof. Setting $h(t,\mu) = F[\varphi,\varphi',\mu](t)$ for $(t,\mu) \in J \times I$, the function h fulfils the assumptions of Lemma 1 and hence there is a unique $\mu_0 \in I$ such that equation (8) with $\mu = \mu_0$ admits a (then unique) solution y satisfying (2).

Now we prove $\|y\| \leqslant r_0$. Let $|y(\xi)| = \|y\| > r_0$ for some $\xi \in J$. If $\xi \in (t_1,t_3)$ then the function $y \cdot \operatorname{sign} y(\xi)$ has a local maximum at the point $t = \xi$, which contradicts $y''(\xi) \cdot \operatorname{sign} y(\xi) > 0$. The last inequality follows from assumption (H_1). Hence $\xi \in \{t_1,t_3\}$. If $\xi = t_1$ ($\xi = t_3$) then due to $y(t_2) = 0$ and assumption (H_1) we have $(y(t_1) - y(t)) \operatorname{sign} y(t_1) > 0$ for all $t \in (t_1,t_2\rangle$ $((y(t_3) - y(t)) \cdot \operatorname{sign} y(t_3) > 0$ for all $t \in \langle t_2,t_3))$, which contradicts $\alpha_1(y(t_1) - y(t)\big|J_1) = 0$ $(\alpha_2(y(t_3) - y(t)\big|J_2) = 0)$. Thus $\|y\| \leqslant r_0$.

Since $\alpha_i(y(t_{2i-1}) - y(t)\big|J_i) = 0$, α_i are increasing functionals and $\alpha_i(0) = 0$ $(i = 1,2)$, there exist $\xi_1 \in (t_1,t_2\rangle$, $\xi_2 \in \langle t_2,t_3)$ such that $y(t_{2i-1}) - y(\xi_i) = 0$ and therefore $y'(\eta_i) = 0$ for some $\eta_1 \in (t_1,\xi_1)$, $\eta_2 \in (\xi_2,t_3)$. For the next part of the proof of the inequality $\|y'\| \leqslant r_1$ see e.g. [2] and [4]. □

3. Existence theorem

Theorem 1. *Assume assumptions* (H_1)–(H_4) *are fulfilled for positive constants* r_0 *and* r_1. *Then there exists* $\mu_0 \in I$ *such that equation* (1) *with* $\mu = \mu_0$ *admits a solution* y *satisfying* (2) *and* (9).

Proof. Let Y be the Banach space of the C^1-functions on J with the norm $\|y\|_Y = \|y\| + \|y'\|$ for $y \in Y$ and $K = \{y;\, y \in Y, \|y^{(i)}\| \leqslant r_i \text{ for } i = 0, 1\}$. K is a bounded convex closed subset of Y. Let $\varphi \in K$. By Lemma 2 there is a unique $\mu_0 \in I$ such that equation (8) with $\mu = \mu_0$ admits a (then unique) solution y satisfying (2) and $y \in K$. Setting $T(\varphi) = y$ we obtain an operator $T\colon K \to K$. To prove Theorem 1 it is sufficient to show that T has a fixed point.

First we prove that T is a continuous operator. Let $\{y_n\} \subset K$ be a convergent sequence, $\lim\limits_{n\to\infty} y_n = y$ and let $z_n = T(y_n)$, $z = T(y)$. Then there are sequences $\{\mu_n\} \subset I$, $\{c_n\} \subset R$ and $\mu_0 \in I$, $c_0 \in R$ such that we have (see the proof of Lemma 1)

$$z_n(t) = c_n u_{y_n}(t) + \int_{t_2}^{t} r(t,s;\, y_n) F[y_n, y_n', \mu_n](s)\,\mathrm{d}s \text{ for all } t \in J \text{ and } n \in N,$$
$$z(t) = c_0 u_y(t) + \int_{t_2}^{t} r(t,s;\, y) F[y, y', \mu_0](s)\,\mathrm{d}s \text{ for all } t \in J,$$

and

$$\alpha_1\big(z_n(t_1) - z_n(t)\big|J_1\big) - 0, \quad z_n(t_2) = 0, \quad \alpha_2\big(z_n(t_3) - z_n(t)\big|J_2\big) = 0 \text{ for all } n \in N,$$
$$\alpha_1\big(z(t_1) - z(t)\big|J_1\big) = 0, \quad z(t_2) = 0, \quad \alpha_2\big(z(t_3) - z(t)\big|J_2\big) = 0.$$

The sequence $\{c_n\}$ is bounded since $\lim\limits_{n\to\infty} y_n = y$ and $\|z_n\| \leqslant r_0$ for all $n \in N$. If $\{c_n\}$ is not convergent there are convergent subsequences $\{c_{k_n}\}$, $\{c_{r_n}\}$ and convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$ of $\{\mu_n\}$ such that $\lim\limits_{n\to\infty} c_{k_n} = c^{(1)}$, $\lim\limits_{n\to\infty} c_{r_n} = c^{(2)}$, $\lim\limits_{n\to\infty} \mu_{k_n} = \mu^{(1)}$, $\lim\limits_{n\to\infty} \mu_{r_n} = \mu^{(2)}$, $c^{(1)} < c^{(2)}$ and $\mu^{(1)}$, $\mu^{(2)}$ are either equal or not. Then

$$(k_1(t) :=) \lim_{n\to\infty} z_{k_n}(t) = c^{(1)} u_y(t) + \int_{t_2}^{t} r(t,s;\, y), F[y, y', \mu^{(1)}](s)\,\mathrm{d}s,$$
$$(k_2(t) :=) \lim_{n\to\infty} z_{r_n}(t) = c^{(2)} u_y(t) + \int_{t_2}^{t} r(t,s;\, y) F[y, y', \mu^{(2)}](s)\,\mathrm{d}s$$

uniformly on J and

$$\begin{aligned} &\alpha_1(k_i(t_1) - k_i(t)\big|J_1) = 0, \quad k_i(t_2) = 0, \\ &\alpha_2(k_i(t_3) - k_i(t)\big|J_2) = 0 \quad \text{for } i = 1, 2. \end{aligned} \tag{10}$$

The equalities ($i = 1, 2$)

$$k_i(t_1) - k_i(t) = c^{(i)}(u_y(t_1) - u_y(t)) + \int_{t_2}^{t} (r(t_1, s; y) - r(t, s; y)) \times F[y, y', \mu^{(i)}](s)\,\mathrm{d}s + \int_{t}^{t_1} r(t_1, s; y) F[y, y', \mu^{(i)}](s)\,\mathrm{d}s,$$

$$k_i(t_3) - k_i(t) = c^{(i)}(u_y(t_3) - u_y(t)) + \int_{t_2}^{t} (r(t_3, s; y) - r(t, s; y)) \times F[y, y', \mu^{(i)}](s)\,\mathrm{d}s + \int_{t}^{t_3} r(t_3, s; y) F[y, y', \mu^{(i)}](s)\,\mathrm{d}s$$

imply (see the proof of Lemma 1)

$$k_1(t_1) - k_1(t) > k_2(t_1) - k_2(t) \quad \text{for } t \in (t_1, t_2\rangle \text{ and } \mu^{(1)} \geqslant \mu^{(2)},$$
$$k_2(t_3) - k_2(t) > k_1(t_3) - k_1(t) \quad \text{for } t \in \langle t_2, t_3) \text{ and } \mu^{(1)} \leqslant \mu^{(2)},$$

which contradicts (10). Hence $\{c_n\}$ is convergent, and let $\lim_{n\to\infty} c_n = c^*$. If $\{\mu_n\}$ is not convergent there are convergent subsequences $\{\mu_{j_n}\}$, $\{\mu_{i_n}\}$, $\lim_{n\to\infty} \mu_{j_n} = \lambda^{(1)}$, $\lim_{n\to\infty} \mu_{i_n} = \lambda^{(2)}$, $\lambda^{(1)} < \lambda^{(2)}$. Then

$$(p_1(t) :=) \lim_{n\to\infty} z_{j_n}(t) = c^* u_y(t) + \int_{t_2}^{t} r(t, s; y) F[y, y', \lambda^{(1)}](s)\,\mathrm{d}s,$$
$$(p_2(t) :=) \lim_{n\to\infty} z_{i_n}(t) = c^* u_y(t) + \int_{t_2}^{t} r(t, s; y) F[y, y', \lambda^{(2)}](s)\,\mathrm{d}s$$

uniformly on J and

$$\begin{aligned} &\alpha_1\big(p_i(t_1) - p_i(t)\big|J_1\big) = 0, \quad p_i(t_2) = 0, \\ &\alpha_2\big(p_i(t_3) - p_i(t)\big|J_2\big) = 0 \quad \text{for } i = 1, 2. \end{aligned} \tag{11}$$

As above we may verify

$$p_2(t_1) - p_2(t) > p_1(t_1) - p_1(t) \quad \text{for all } t \in (t_1, t_2\rangle\,,$$
$$p_2(t_3) - p_2(t) > p_1(t_3) - p_1(t) \quad \text{for all } t \in \langle t_2, t_3)\,,$$

which contradicts (11). Hence $\{\mu_n\}$ is convergent, and let $\lim_{n\to\infty} \mu_n = \mu^*$. Then

$$(z^*(t) :=) \lim_{n\to\infty} z_n(t) = c^* u_y(t) + \int_{t_2}^{t} r(t, s; y) F[y, y', \mu^*](s)\,\mathrm{d}s$$

uniformly on J, and consequently, z^* is a solution of the differential equation

$$w'' = Q[y, y'](t)\, w + F[y, y', \mu^*](t)$$

and

$$\alpha_1\big(z^*(t_1) - z^*(t)\big|J_1\big) = 0, \quad z^*(t_2) = 0, \quad \alpha_2\big(z^*(t_3) - z^*(t)\big|J_2\big) = 0.$$

By Lemma 2 it is necessary that $z = z^*$ and $\mu_0 = \mu^*$. Since $\lim_{n\to\infty} z_n^{(i)}(t) = z^{(i)}(t)$ uniformly on J for $i = 0, 1$, we have $z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} T(y_n) = T(y)$ and therefore T is a continuous operator. Let $\varphi \in K$ and $T(\varphi) = y$. Then the equality

$$y''(t) = Q[\varphi, \varphi'](t)\, y(t) + F[\varphi, \varphi', \mu_0](t)$$

holds on J for some $\mu_0 \in I$, thus $\|y''\| \leqslant A + r_0 B$ $(:= r_2)$ and $K \subset L = \{y;\ y \in C^2(J), \|y^{(i)}\| \leqslant r_i \text{ for } i = 0, 1, 2\}$. Since L is a compact subset of Y, K is a relative compact subset of Y.

By the Schauder fixed point theorem there is a fixed point of T. This completes the proof. □

Remark 1. If $\alpha_1(z) = \alpha_2(z) = z(t_2)$, then Theorem 1 in [2] and Theorem 1 in [4] (where $m = n = 1$) follow from Theorem 1.

Let $t_1 < x_1 < t_2 < x_2 < t_3$. If $\alpha_1(z) = z(x_1)$, $\alpha_2(z) = z(x_2)$, then Theorem 1 in [1] follows from Theorem 1.

Example 1. Consider the functional differential equation

$$y''(t) = y(t) \exp\big\{|y(y'(t))|\big\} + \tfrac{1}{2}\cos\big(t + y'(y(t))\big) + \mu \tag{12}$$

on the interval $J = \langle 0, t_3\rangle$, where $t_3 \geqslant 2\sqrt{1+\mathrm{e}}$. Let $t_2 \in (0, t_3)$. Assumptions (H_1)–(H_4) are fulfilled with $r_0 = 1$, $r_1 = 2\sqrt{1+\mathrm{e}}$ and $I = \langle -\frac{1}{2}, \frac{1}{2}\rangle$. Let $\alpha_1(z) = \int_0^{t_2} z^3(s)\,\mathrm{d}s$ for $z \in C^0(\langle 0, t_2\rangle)$ and $\alpha_2(z) = \max\{z(t);\ t \in \langle t_2, \frac{1}{2}(t_2 + t_3)\rangle\}$ for $z \in C^0(\langle t_2, t_3\rangle)$. Then by Theorem 1 there is $\mu_0 \in \langle -\frac{1}{2}, \frac{1}{2}\rangle$ such that equation (12) with $\mu = \mu_0$ admits a solution y satisfying

$$\int_0^{t_2} \big(y(t_1) - y(s)\big)^3 \mathrm{d}s = 0, y(t_2) = 0, \max\big\{y(t_3) - y(t);\ t \in \langle t_2, \tfrac{1}{2}(t_2 + t_3)\rangle\big\} = 0$$

and

$$\|y\| \leqslant 1, \qquad \|y'\| \leqslant 2\sqrt{1+\mathrm{e}}.$$

4. Bounded solutions on a halfline

In this section BVP (1)–(2) is applied to the investigation of bounded solutions of a functional differential equation of type (1) with functional boundary conditions

$$\alpha_1\big(y(t_1) - y(t)\big|J_1\big) = 0, y(t_2) = 0. \tag{13}$$

Let Y be the space of bounded C^0-functions on the halfline $\langle t_1, \infty)$ with the topology of uniform convergence on compact subintervals of $\langle t_1, \infty)$. Consider the functional differential equation

$$y''(t) - U[y, y'](t)\, y(t) = V[y, y', \mu](t), \tag{14}$$

where $U : Y \times Y \longrightarrow Y$, $V : Y \times Y \times I \longrightarrow Y$ are continuous operators, $U[y, z](t) > 0$ for all $t \geqslant t_1$ and $[y, z] \in Y \times Y$. Further we shall assume that there exists an increasing sequence $\{x_n\} \subset R$, $x_1 > t_2$, $\lim_{n\to\infty} x_n = \infty$ such that the functions $U[y, z](t)$, $V[y, z, \mu](t)$ are defined on $\langle t_1, x_n\rangle$ only by the restrictions of y, z to the interval $\langle t_1, x_n\rangle$ ($n = 1, 2, \ldots$), that is

$$U : Y_n \times Y_n \longrightarrow Y_n, \quad V : Y_n \times Y_n \times I \longrightarrow Y_n \qquad (n = 1, 2, \ldots),$$

where Y_n is the Banach space of the C^0-functions on $\langle t_1, x_n\rangle$ with the sup norm. The differential equation $y'' - q(t, y, y')\,y = f(t, y, y', \mu)$, where $q \in C^0\big(\langle t_1, \infty) \times R^2\big)$, $f \in C^0(\langle t_1, \infty) \times R^2 \times I)$, is a special case of (14).

Suppose there are positive constants r_0, r_1 such that the operators U, V satisfy the following assumptions:

- (C_1) $|V[y, y', \mu](t)| \leqslant r_0 U[y, y'](t)$ for all $t \geqslant t_1$ and $[y, y', \mu] \in H \times I$, where $H = \{[y, y']; y \in C^1(\langle t_1, \infty)), |y^{(i)}(t)| \leqslant r_i \text{ for } t \geqslant t_1, i = 0, 1\}$;
- (C_2) $V[y, y', \mu_1](t) < V[y, y', \mu_2](t)$ for all $t \geqslant t_1$, $[y, y'] \in H$ and μ_1, $\mu_2 \in I$, $\mu_1 < \mu_2$;
- (C_3) $V[y, y', a](t)\, V[y, y', b](t) \leqslant 0$ for all $t \geqslant t_1$ and $[y, y'] \in H$;
- (C_4) $2\sqrt{r_0}\sqrt{A + r_0 B} \leqslant r_1$, where $A = \sup\{\sup_{t\geqslant t_1} |V[y, y', \mu](t)|; [y, y', \mu] \in H \times I\}$, $B = \sup\{\sup_{t\geqslant t_1} |U[y, y'](t)|; [y, y'] \in H\}$.

Lemma 3. *Assume assumptions (C_1)–(C_4) are fulfilled with positive constants r_0, r_1. Then for any x_n ($n = 1, 2, \ldots$) there exists a $\mu_n \in I$ such that equation (14) with $\mu = \mu_n$ admits a solution y_n defined on the interval $\langle t_1, x_n\rangle$ and satisfying the boundary conditions*

$$\alpha_1\big(y_n(t_1) - y_n(t)\big|J_1\big) = 0, \quad y_n(t_2) = 0, \quad y_n(x_n) = 0 \qquad (n = 1, 2, \ldots), \tag{15}$$

and, moreover,

$$
(16) \qquad \begin{aligned} &|y_n(t)| \leqslant r_0, \quad |y_n'(t)| \leqslant r_1, \\ &|y_n''(t)| \leqslant A + r_0 B \qquad \text{for } t \in \langle t_1, x_n \rangle, \qquad (n = 1, 2, \ldots). \end{aligned}
$$

Proof. The proof follows immediately from Theorem 1 if we set $t_3 = x_n$ and $\alpha_2(z) = z(t_2)$. The last inequality in (16) is evident. □

Theorem 2. *Assume assumptions* (C_1)–(C_4) *are fulfilled with positive constants* r_0, r_1. *Then there exists a* $\mu_0 \in I$ *such that equation* (14) *with* $\mu = \mu_0$ *admits a solution* y *satisfying* (13) *and*

$$
(17) \qquad |y(t)| \leqslant r_0, \quad |y'(t)| \leqslant r_1 \quad \text{for} \quad t \geqslant t_1.
$$

Proof. According to Lemma 3 there exists a sequence $\{y_n\}$ of solutions of equation (14) with $\mu = \mu_n (\in I)$ on the intervals $\langle t_1, x_n \rangle$ satisfying (15) and (16). Using the Ascoli-Arzela theorem, a diagonal process of Cantor and the fact that $\{\mu_n\}$ is a bounded sequence, we may assume without loss of generality that $\{y_n(t)\}$ and $\{y_n'(t)\}$ are locally uniformly convergent on $\langle t_1, \infty)$ and $\{\mu_n\}$ is convergent. Setting $\lim\limits_{n\to\infty} y_n(t) = y(t)$ for $t \in \langle t_1, \infty)$ and $\lim\limits_{n\to\infty} \mu_n = \mu_0$, then y is a solution of equation (14) with $\mu = \mu_0$ satisfying (13) and (17). □

Example 2. Consider the functional differential equation

$$
(18) \quad y''(t) = 6\pi y(t) \exp\left\{|y(t + (\sin t)^2)|\right\} + \ln\left(e + |y'(\sqrt{t})|\right) \arctan t + \left(1 + y^2(t)\right)\mu.
$$

The assumptions of Theorem 2 are satisfied with $t_1 \geqslant 1$, $r_0 = 1$, $r_1 = e^3$ and $I = \langle -2\pi, 0 \rangle$. Therefore there exists a $\mu_0 \in \langle -2\pi, 0 \rangle$ such that equation (18) with $\mu = \mu_0$ has a solution y defined on $\langle t_1, \infty)$, and (13) and $|y(t)| \leqslant 1$, $|y'(t)| \leqslant e^3$ for $t \geqslant t_1$ hold.

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