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THE DIVERGENCE THEOREM AND PERRON INTEGRATION WITH EXCEPTIONAL SETS

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0. INTRODUCTION

Let λ_α, $0 \leqslant \alpha \leqslant n$, denote normalized α-dimensional outer Hausdorff measure in $\mathbf{R}^n$ ($n \in \mathbf{N}$), which reduces for integral α to the outer Lebesgue measure on $\mathbf{R}^\alpha \subseteq \mathbf{R}^n$ (λ_0 being the counting function). We often write

$$\lambda_n = \mathcal{L}, \quad \lambda_{n-1} = \mathcal{H}, \quad \text{and} \quad \lambda_\alpha(E) = |E|_\alpha \quad \text{for } E \subseteq \mathbf{R}^n.$$

It is convenient to speak of α-null sets, α-finite sets, and σ_α-finite sets if resp. $\lambda_\alpha(E) = 0$ or $\lambda_\alpha(E) < \infty$ or E is a countable union of α-finite sets. As class $\mathcal{A}$ we define the collection of all compact sets $A \subseteq \mathbf{R}^n$, whose boundary ∂A is $(n-1)$-finite. Federer [2: 4.5.6 and 4.5.12] has shown, in particular, the following important facts: For each $A \in \mathcal{A}$ there is a vector function (exterior normal) $\vec{n}_A : \partial A \to \mathbf{R}^n$, which is $\mathcal{H}$-measurable and has (euclidean) norm $\|\vec{n}_A\| \leqslant 1$. Furthermore, for any vector function $\vec{v}$ into $\mathbf{R}^n$, which is Lipschitzian on an open neighborhood of A, the Divergence Theorem

$$(1) \qquad \int_{\partial A} \vec{v} \cdot \vec{n}_A \, \mathrm{d}\mathcal{H} = {}^{*}\!\!\int_A \operatorname{div} \vec{v}$$

holds if the integral on the right is interpreted as the Lebesgue integral $\int_A \operatorname{div} \vec{v} \, \mathrm{d}\mathcal{L}$. The main purpose of this paper is to relax the requirements on $\vec{v}$ for the truth of (1) by allowing certain exceptional sets.

Suppose that $A \in \mathcal{A}$ and that the vector function $\vec{v}$: $A \to \mathbf{R}^n$ is bounded on A. Let $D = D_{\vec{v}}$ (resp. $C = C_{\vec{v}}$) denote the set of points $x \in A$, where $\vec{v}$ is totally differentiable (resp. continuous) relative to A. We also introduce the set $L = L_{\vec{v}}$ of points $x \in A$, where $\vec{v}$ is locally Lipschitzian relative to A, i.e. where

$$\text{(L)} \qquad \|\vec{v}(y) - \vec{v}(x)\| = O(1)\|y - x\| \text{ if } y \to x, \quad y \in A$$

holds (but not necessarily uniformly in x). Clearly $D \subseteq L \subseteq C$, and by Stepanoff's theorem $L \neg D$ ($\neg$ means set difference resp. complement) is always an n-null set, cf. [2: 3.1.9]. Our additional requirement concerns the exceptional sets, viz. that

$$\text{(2)} \qquad A \neg L \text{ is } \sigma_{n-1}\text{-finite and } |A \neg C|_{n-1} = 0.$$

Under these assumptions $\vec{v}$ is $\mathcal{H}$-measurable on A so that the left side of (1) exists. If we now define $\operatorname{div} \vec{v}$ as usual on $A^0 \cap D$ and zero elsewhere we may find that the right side of (1) does not exist as an $\mathcal{L}$-integral. However, we will define in this paper a new type of Perron integration such that the right side of (1) does exist with this interpretation and yields the correct value, so that the divergence theorem (1) becomes true. We formulate this briefly as

Theorem 1. *If $A \in \mathcal{A}$ and the bounded vector function $\vec{v}\colon A \to \mathbb{R}^n$ satisfies* (2) *then* (1) *holds with $*$ indicating a certain Perron type integration.*

This is an improvement over several known results which will be discussed in detail in section 1. In particular, it contains the remarkable result of Besivovitch concerning a continuous function of a complex variable on an open set, which says: if this function is locally Lipschitzian there except for points of a σ_1-finite set then the function is already analytic there provided that the Cauchy-Riemann equations are satisfied almost everywhere, cf. Saks [18: p. 197]. We mention this result especially, because the ideas of Besicovitch, in abstracted form, will be at the very root of the Perron type integration defined in this paper, cf. our Decomposition Theorem (section 4). A result, which is very similar to Theorem 1, has been obtained by Pfeffer-Yang [17], but there exceptional sets do not include the situation of Besicovitch.

The proof of theorem 1 will be based on one hand on Federer's result for test functions, e.g. vector functions of type $C_0^1(\mathbb{R}^n)$ or $C_0^\infty(\mathbb{R}^n)$ or the like, where the index zero refers to the requirement of bounded support. In particular, we shall use (1) for constant and linear vector functions. On the other hand we use a general result about the integral representation of an additive set function which we will describe in the following. Given $A \in \mathcal{A}$ we denote by $\mathcal{A}(A)$ the collection of all sets $B \in \mathcal{A}$ with $B \subseteq A$. If $\vec{v}\colon A \to \mathbb{R}^n$ is bounded and $\mathcal{H}$-measurable then

$$\text{(3)} \qquad F(B) = F(B, \vec{v}) = \int_{\partial B} \vec{v} \cdot \vec{n}_B \, \mathrm{d}\mathcal{H}$$

is defined for all $B \in \mathcal{A}(A)$. It is known that this set function F is additive on $\mathcal{A}(A)$ in the following sense:

$$F(B) = F(B_1) + F(B_2) \quad \text{for } B_1, B_2 \in \mathcal{A}(A), \tag{4}$$

provided that $B = B_1 \cup B_2$ with disjoint B_1^0, B_2^0 (this situation will be abbreviated by $B = B_1 \uplus B_2$; B^0 denotes the interior). Given a set function $F\colon \mathcal{A}(A) \to \mathbb{R}$ and a point $x \in A$, we consider the conditions

$$\left.\begin{array}{ll} (\Lambda) & F(B) = O(1)|\partial B|_{n-1} \\ (C) & F(B) = o(1)|\partial B|_{n-1} \end{array}\right\} \text{ if } B \in \mathcal{A}(A) \text{ with } x \in B \text{ and } d(B) \to 0,$$

where $d(B)$ denotes the diameter of B and the O-constant may depend upon x. The set of all $x \in A$ satisfying (Λ) resp. (C) will be denoted by $\Lambda = \Lambda_F$ resp. $C = C_F$. In case that F is given by (3) it is clear that $\Lambda_F = A$ and $C_F \supseteq C_{\vec{v}}$ (by subtracting from $\vec{v}$ the constant vector $\vec{v}(x)$).

A set $A \in \mathcal{A}$ will be called ϱ-regular, $\varrho > 0$, if

$$d(A)^n \leqslant \varrho|A|_n \text{ and } |\partial A|_{n-1} \leqslant \varrho\, d(A)^{n-1} \tag{ϱ}$$

holds. The first condition is the typical regularity condition used for differentiation, while the second condition ensures that the boundary does not oscillate unnecessarily. A very similar condition is used by Pfeffer-Yang [17], and it may be useful in this context to remember the inequalities

$$|A|_n \leqslant c_n d(A)^n, |A|_n \leqslant c_n d(A)|\partial A|_{n-1}, |A|_n^{n-1} \leqslant c_n|\partial A|_{n-1}^n,$$

where c_n denotes certain positive absolute constants. We define $\mathcal{A}_\varrho$ to be the collection of all $A \in \mathcal{A}$ which are ϱ-regular, and $\mathcal{A}_\varrho(A) = \mathcal{A}(A) \cap \mathcal{A}_\varrho$. Given a set function $F\colon \mathcal{A}(A) \to \mathbb{R}$ and a point $x \in A^0 \cap C_F$ we consider the condition

$$F(B) = O(1)|B|_n \text{ if } B \in \mathcal{A}_\varrho(A) \text{ with } x \in B \text{ and } d(B) \to 0, \tag{L_ϱ}$$

where the O-constant may depend upon x and ϱ. The set of all $x \in A^0 \cap C_F$ satisfying the condition (L_ϱ) for all $\varrho > 0$ will be denoted by $L = L_F$. Finally, given a set function $F\colon \mathcal{A}(A) \to \mathbb{R}$ and a point $x \in A^0 \cap C_F$ we consider the condition that there exists a number $f \in \mathbb{R}$ such that

$$F(B) = (f + o(1))|B|_n \text{ if } B \in \mathcal{A}_\varrho(A) \text{ with } x \in B \text{ and } d(B) \to 0 \tag{D_ϱ}$$

holds for all $\varrho > 0$. We call F differentiable at x if this condition is satisfied and denote by $\dot{F}(x)$ the unique value f. The set of all x, where F is differentiable, is denoted by $D = D_F$. At points $x \in A$, where F is not differentiable we always set $\dot{F}(x) = 0$. According to the definitions we have $D_F \subseteq L_F \subseteq A^0 \cap C_F$. In case that F is given by (3) we easily obtain

$$L_F \supseteq A^0 \cap L_{\vec{v}}, D_F \supseteq A^0 \cap D_{\vec{v}} \quad \text{with } \dot{F} = \operatorname{div} \vec{v} \text{ on } A^0 \cap D_{\vec{v}} \tag{5}$$

by subtracting from $\vec{v}(y)$ the constant term $\vec{v}(x)$ resp. the linear term $\vec{v}(x) + \vec{v}'(x)(y-x)$ and using (ϱ), which implies $d(B)|\partial B|_{n-1} = O(1)d(B)^n = O(1)|B|_n$.

If we make the assumptions of theorem 1 the additive set function F given by (3) has the property that its exceptional sets satisfy

$$\begin{cases} |A\neg D|_n = 0, \ A\neg L \text{ is } \sigma_{n-1}\text{-finite} \\ \quad |A\neg C|_{n-1} = 0, \ \Lambda = A, \end{cases} \tag{6}$$

because these statements follow from the corresponding statements about the exceptional sets for $\vec{v}$.

In section 2 we shall introduce $*$-integration so that the Fundamental Theorem of Calculus holds in the following form.

Theorem 2. *If $A \in \mathcal{A}$ and F is an additive set function $\mathcal{A}(A) \to \mathbf{R}$ satisfying (6), then $\dot{F}$ is $*$-integrable over A and $F(A) = {}^*\!\int_A \dot{F}$.*

Since values of the integrand on n-null sets are irrelevant, theorem 1 follows immediately from theorem 2. We remark that theorem 2 is of a general character and not directly related to the divergence theorem. Furthermore, only a small part of Federer's results were used to check the assumptions of theorem 2. Nevertheless we obtain theorem 1 which is a considerable improvement concerning the divergence theorem.

The proof of theorem 2 will be given in section 3. In section 5 we apply theorem 1 to a two-dimensional situation and obtain a form of Green's theorem with exceptional sets.

1. Discussion of known results

It is natural to ask for minimal assumptions on A and $\vec{v}$ which imply the Divergence Theorem (0.1). The conditions for A will be the weakest in some sense if we consider (0.1) only for test functions. Since the left side of (0.1) represents a continuous linear functional on vector functions of type $C(\partial A)$, an equation like (0.1) can only hold for all test functions if the right side can be extended to such a functional. Mařík [11] has completly characterized such sets A and Karták-Mařík [10] have shown that all $A \in \mathcal{A}$ have this property, which also follows from Federer's results. Since no further simple sets are known, which have Mařík's property, we shall be satisfied with the class $\mathcal{A}$ for the moment. This answers the geometric part of our question.

Now we turn to the analytic part of our question, which concerns the conditions on $\vec{v}$ and the (related) type of integration to be used. First we consider the case, where A is an n-dimensional interval (always compact with $A^0 \neq \emptyset$) and $\vec{v}$ is totally differentiable everywhere or at least in a neighborhood of A. Then the left side of (0.1) clearly exists, while $\operatorname{div} \vec{v}$ may not be $\mathcal{L}$-integrable. Mawhin [12] [13] has defined suitable types of Perron integration, so that the integral on the right of (0.1) exists in his sense and the equation (0.1) holds. The same result could have been obtained with the Perron integration defined by Bauer [1] in 1915 by means of a corresponding Fundamental Theorem of Calculus. All these cases share some regularity condition for the intervals used in the definition of the integral, cf. also Jurkat-Knizia [7]. Jarník-Kurzweil-Schwabik [3] introduced another type of Perron integration yielding the same result, where the regularity condition is replaced by bounds for certain sums (n-dimensional control conditions). In summary, in our first case there are several possible interpretations of $*$-integration over intervals which make (0.1) correct without any further assumptions; moreover, if $\operatorname{div} \vec{v}$ is $\mathcal{L}$-integrable they agree with $\mathcal{L}$-integration.

Next we discuss more general sets A while we keep the assumption that $\vec{v}$ be totally differentiable everywhere or at least in a neighborhood of A. Here the problem is to define a suitable Perron integral over such sets. This important step was first made by Jarník-Kurzweil [4] for sets A with a relatively smooth boundary yielding the result that (0.1) holds automatically with this interpretation of $*$-integration. Their later papers [5], [6] are based on an interesting combination of partitions of unity with related n-dimensional control conditions. Another far-reaching appproach was made by Pfeffer [16] and Pfeffer-Yang [17] leading in various steps, including a transfinite induction, to a Perron type integration for all sets $A \in \mathcal{A}$ so that (0.1) holds automatically also with this interpretation of $*$-integration. It is not clear yet how Pfeffer-Yang integration is related to Jarník-Kurzweil integration, but both extend $\mathcal{L}$-integration.

The final and most interesting step consists of relaxing further the conditions on $\vec{v}$. For instance, differentiability should be restricted to A^0 while continuity may be relevant along the boundary. Clearly the boundary plays the role of an exceptional set and by dividing A into parts one could permit further exceptional sets even in A^0. While certain exceptional sets were allowed in Pfeffer [15] and Jarník-Kurzweil [6] the greatest progress in this direction was made by Pfeffer [16] and Pfeffer-Yang [17]: They require that $A \neg D$ is contained in a compact set which itself is a countable union of compact sets of finite $\mathcal{H}$-measure and that $A \neg C$ is contained in a compact set of $\mathcal{H}$-measure zero (assuming anyway that $\vec{v}$ be bounded). These conditions are stronger than our condition (0.2), while the relation between the corresponding integration processes is not clear yet. The most general exceptional sets have occured earlier in the work of Besicovitch, cf. Saks [18: p. 193], and our theorem 2 may be viewed as a kind of improvement over theorem 4.4 stated there.

This discussion was based on a thesis by Nonnenmacher [14].

2. Perron integration with exceptional sets

Let $A \in \mathcal{A}$ be given. A partition Π of A consists of finitely many sets $A_k \in \mathcal{A}(A)$ together with points $x_k \in A_k$ such that $A = \biguplus A_k$ (i.e. $A = \cup A_k$ with disjoint A_k^0). A gauge $\delta = \delta(\cdot)$ is any positive function δ on A. The partition Π is δ-fine if always $d(A_k) < \delta(x_k)$ holds. The partition Π is called (ϱ, σ)-regular with $\varrho > 0, \sigma > 0$, if the inequality

$$(1) \qquad \sum_{A_k \notin \mathcal{A}_\varrho} |\partial A_k|_{n-1} \leqslant \sigma$$

holds. This controls the contribution of those parts of Π which are not ϱ-regular. It is an $(n-1)$-dimensional control condition and sort of related to the n-dimensional control conditions of Jarník-Kurzweil-Schwabik [3]. Using condition (1) it is possible to define Perron integration so that (0.1) holds if $\vec{v}$ is differentiable in a neighborhood of A. However, if there are exceptional sets of various kinds we introduce also corresponding control conditions. For simplicity we work with three kinds only. Let $\mathcal{M} = (M_i)$, $\mathcal{N}' = (N_i')$, $\mathcal{N}'' = (N_i'')$ be three countable families of exceptional subsets in $\mathbb{R}^n$ so that all occurring subsets are mutually disjoint and always

$$(2) \qquad |M_i|_{n-1} < \infty, \quad |N_i'|_n = 0, \quad |N_i''|_{n-1} = 0$$

is satisfied. In such a case we speak of an exceptional system $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$. Let $K = (K_i)$, $\Delta' = (\Delta_i)$, $\Delta'' = (\Delta_i'')$ be three corresponding families of positive numbers and consider the following control conditions for Π (to be satisfied for all i):

$$(3) \qquad \sum_{x_k \in M_i} |\partial A_k|_{n-1} \leqslant K_i, \sum_{x_k \in N_i'} |A_k|_n \leqslant \Delta_i', \sum_{x_k \in N_i''} |\partial A_k|_{n-1} \leqslant \Delta_i''.$$

They control the contribution of those parts of Π, which are related to an exceptional set in a way that depends upon the size of the exceptional set. Note that condition (3) for Π depends only on the choice of $(\mathcal{M}, K; \mathcal{N}', \Delta'; \mathcal{N}'', \Delta'')$. A δ-fine partition of A satisfying (1) and (3) will be called admissible and they are characterized by the data $(\mathcal{M}; \mathcal{N}', \mathcal{N}''; \varrho, \sigma, K; \Delta', \Delta'', \delta)$. If we increase ϱ, σ, K_i, Δ_i', Δ_i'', $\delta(\cdot)$ clearly more partitions will become admissible (monotonicity). Intuitively speaking we think of (ϱ, σ, K) as bounded terms and of $(\Delta', \Delta'', \delta)$ as small terms.

Given a partition Π of $A \in \mathcal{A}$ and a function $f\colon A \to \mathbb{R}$ we may form the Riemann sum

$$(4) \qquad S(f, \Pi) = \sum_k f(x_k)|A_k|_n.$$

Definition. Given $A \in \mathcal{A}$ and $\varrho > 0$, a function $f\colon A \to \mathbb{R}$ is called ϱ-integrable over A if there exists an exceptional system $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$ and a number $F \in \mathbb{R}$ qUFh that for each choice of σ, K and $\varepsilon > 0$ there are corresponding Δ', Δ'', δ with the property that

$$(5) \qquad |S(f, \Pi) - F| \leqslant \varepsilon$$

holds for all admissible Π. If this is true for all sufficiently large ϱ the function f is called $*$-integrable over A.

Clearly, the property of being ϱ-integrable over A depends on A and ϱ only; consequently, the property of being $*$-integrable over A depends on A only. The collection of all functions $f\colon A \to \mathbb{R}$ having these properties will be denoted by $P_\varrho(A)$ resp. $P_*(A)$. If the exceptional system is specified in advance we speak of ϱ-integrability relative to $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$. In this connection the following geometric result is essential.

Richness Property (R). *For any $A \in \mathcal{A}$, any $\varrho \geqslant c_n$, and any exceptional system $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$ there are corresponding choices $\sigma = \sigma^*, K = K^*$ (fixed and depending only on ϱ, A, $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$) such that for any choice of Δ', Δ'', δ there will exist admissible partitions of A. (Here, c_n denotes a positive absolute constant.)*

This will be shown in section 4. Because of (R) we shall always assume $\varrho \geqslant c_n$ in the definition above. If we wish we may also assume $\sigma \geqslant \sigma^*$ and all $K_i \geqslant K_i^*$ so that condition (5) is never empty.

Now we see that ϱ-integrability over A relative to the exceptional system $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$ is equivalent to the Cauchy condition: For each σ, K and $\varepsilon > 0$ there are corresponding Δ', Δ'', δ such that

$$|S(f, \Pi_1) - S(f, \Pi_2)| \leqslant \varepsilon \tag{6}$$

holds for all admissible Π_1, Π_2. If we introduce upper and lower integrals by means of (always assuming $\varrho \geqslant c_n$, $\sigma \geqslant \sigma^*$, all $K_i \geqslant K_i^*$)

$$\tag{7} \begin{cases} S_\varrho^+(f, A; \mathcal{M}, \mathcal{N}', \mathcal{N}'') = \sup\limits_{(\sigma, K)} \inf\limits_{(\Delta', \Delta'', \delta)} \sup\limits_{\Pi} S(f, \Pi) \\ \\ S_\varrho^-(f, A; \mathcal{M}, \mathcal{N}', \mathcal{N}'') = \inf\limits_{(\sigma, K)} \sup\limits_{(\Delta', \Delta'', \delta)} \inf\limits_{\Pi} S(f, \Pi) \end{cases}$$

it is immediate that ϱ-integrability relative to $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$ is equivalent to

$$S_\varrho^- = S_\varrho^+ (= F) \in \mathbb{R}, \tag{8}$$

while $S_\varrho^- \leqslant S_\varrho^+$ holds in general. The sufficiency of (6) follows via (8) and makes use of (R). Let us denote the common value of $S_\varrho^\pm$ by $S_\varrho = S_\varrho(f, A; \mathcal{M}, \mathcal{N}', \mathcal{N}'')$, if it exists as a finite number. The following construction will show that this number is independent of $\mathcal{M}, \mathcal{N}', \mathcal{N}''$ and may, therefore, be denoted by ${}^\varrho\!\!\int_A f$.

Given $\varrho > 0$ and two exceptional systems $(\mathcal{M}_1; \mathcal{N}_1', \mathcal{N}_1'')$, $(\mathcal{M}_2; \mathcal{N}_2', \mathcal{N}_2'')$, there is a third exceptional system $(\mathcal{M}; \mathcal{N}', \mathcal{N}'')$ with the following property: if we prescribe for these systems resp. the data $({}_1\Delta', {}_1\Delta'', \delta_1(\cdot))$, $({}_2\Delta', {}_2\Delta'', \delta_2(\cdot))$, (σ, K) we can find resp. data ${}_1K$, ${}_2K$, $(\Delta', \Delta'', \delta(\cdot))$, such that all partitions of A which are admissible with respect to $(\mathcal{M}; \mathcal{N}', \mathcal{N}''; \varrho, \sigma, K; \Delta', \Delta'', \delta(\cdot))$ will also be admissible with respect to $(\mathcal{M}_1; \mathcal{N}_1', \mathcal{N}_1''; \varrho, \sigma, {}_1K; {}_1\Delta', {}_1\Delta'', \delta_1(\cdot))$ and with respect to $(\mathcal{M}_2; \mathcal{N}_2', \mathcal{N}_2''; \varrho, \sigma, {}_2K;\ {}_2\Delta', {}_2\Delta'', \delta_2(\cdot))$; moreover the choice of ${}_1K$ and ${}_2K$ can be based on K alone independently of the other prescribed data.

To find the new system we form the family $\widetilde{\mathcal{M}}$ by uniting the families $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e. $\widetilde{\mathcal{M}} = \mathcal{M}_1 \cup \mathcal{M}_2$ and similarly $\widetilde{\mathcal{N}}' = \mathcal{N}_1' \cup \mathcal{N}_2'$, $\widetilde{\mathcal{N}}'' = \mathcal{N}_1'' \cup \mathcal{N}_2''$.

If the exceptional sets were all disjoint the remaining data would be obvious. Suppose that these controlling data are $\widetilde{K}$, $\widetilde{\Delta}'$, $\widetilde{\Delta}''$, $\widetilde{\delta}(\cdot)$. Now we want to construct the new system $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$ in such a way that the controlling data K, Δ', Δ'',

$\delta(\cdot)$ imply the situation before except that $\widetilde{K}$ will be replaced by $K'(K)$. In $\mathcal{M}$ we separate the $\widetilde{M_i}$ by subtracting the $\widetilde{M_j}$ with $j < i$. This changes $\widetilde{K}$ in the permitted way. In $\mathcal{N}'$ we subtract, e.g., from N'_{1i} all N'_{2j}, but we also introduce the intersections $N'_{1i} \cap N'_{2j}$ as new sets with new controls Δ'_{ij}. Similarly we treat $\widetilde{\mathcal{N}}''$. Now the sets in the changed system $\widetilde{\mathcal{M}}$, $\widetilde{\mathcal{N}}'$, $\widetilde{\mathcal{N}}''$ are disjoint in each group. Next we subtract the sets of $\widetilde{\mathcal{N}}''$ from those in $\widetilde{\mathcal{M}}$. This changes $\widetilde{K}$ again in a permitted way. Finally we subtract the sets of $\widetilde{\mathcal{N}}''$ or $\widetilde{\mathcal{M}}$ from those in $\widetilde{\mathcal{N}}'$. Here is it important to notice that, e.g.,

$$\sum_{x_k \in \widetilde{N}'_i \cap \widetilde{M}_j} |A_k|_n \leqslant c_n \sum_{x_k \in \widetilde{N}'_i \cap \widetilde{M}_j} \delta(x_k) |\partial A_k|_{n-1} \leqslant \Delta'_{ij}$$

can be arranged by bounding $\delta(\cdot)$ suitably on $\widetilde{N}'_i \cap \widetilde{M_j}$, since these sets are disjoint. After all the sets are separated we have arrived at the desired system $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$, and we know that controls with arbitrary K and suitable Δ', Δ'', $\delta(\cdot)$ will imply the original controls with $\widetilde{K} = K'(K)$ and the prescribed $\widetilde{\Delta}'$, $\widetilde{\Delta}''$, $\widetilde{\delta}(\cdot)$. $\square$

Now suppose that f is ϱ-integrable relative to $(\mathcal{M}_1; \mathcal{N}'_1, \mathcal{N}''_1)$ and $(\mathcal{M}_2; \mathcal{N}'_2, \mathcal{N}''_2)$ with corresponding values F_1 and F_2. Then the construction implies that f is also ϱ-integrable relative to $\mathcal{M}; \mathcal{N}', \mathcal{N}''$ with corresponding $F = F_1$ and $F = F_2$, hence $F_1 = F_2$. This shows that ${}^{\varrho}\!\int_A f$ *is uniquely determined by* f *and* A *for each* $f \in P_\varrho(A)$. In case that $f \in P_*(A)$ it is easy to see that ${}^{\varrho}\!\int_A f$ is the same number for all $\varrho \geqslant c_n$. *This common value will be denoted by* ${}^{*}\!\int_A f$. The same construction also proves: if $f_1 \in P_*(A)$ and $f_2 \in P_*(A)$, then $f_1 + f_2 \in P_*(A)$ and ${}^{*}\!\int_A (f_1 + f_2) = {}^{*}\!\int_A f_1 + {}^{*}\!\int_A f_2$.
Trivially, if $f \in P_*(A)$ and $a \in \mathbb{R}$ then $af \in P_*(A)$ and ${}^{*}\!\int_A af = a\, {}^{*}\!\int_A f$.

Next suppose that $A, B, C \in \mathcal{A}$ and $A = B \uplus C$. Observe that a partition Π_1 of B and a partition Π_2 of C together give a partition Π of A with bounds $\sigma(A) = \sigma(B) + \sigma(C)$, $K(A) = K(B) + K(C)$, $\Delta'(A) = \Delta'(B) + \Delta'(C)$, $\Delta''(A) = \Delta''(B) + \Delta''(C)$. If $f \in P_\varrho(A)$ and a possible exceptional system has been selected we determine for given σ, K, ε corresponding Δ', Δ'', δ according to the definition of ϱ-integrability. If we now consider decompositions Π_1 of B and Π_2 of C with bounds $\frac{1}{2}\sigma$, $\frac{1}{2}K$, $\frac{1}{2}\Delta'$, $\frac{1}{2}\Delta''$ we obtain decompositions Π of A which satisfy (5). Using (R) it is easy to see that the Cauchy conditions for B and C are satisfied so that $F_1 = {}^{\varrho}\!\int_B f$ and $F_2 = {}^{\varrho}\!\int_C f$ exist, and it follows

$$(9) \qquad {}^{\varrho}\!\int_A f = {}^{\varrho}\!\int_B f + {}^{\varrho}\!\int_C f .$$

This implies the following result: *If* $f \in P_*(A)$ *then* $F(B) = {}^{\varrho}\!\int_B f$ *exists for all* $B \in \mathcal{A}(A)$ *and defines an additive set function* $\mathcal{A}(A) \to \mathbb{R}$.

Finally suppose that $A \in \mathcal{A}$ and $f\colon A \to \mathbb{R}$ is zero except on a set N with $|N|_n = 0$. We can decompose N into countably many sets N_i' so that f is bounded on each N_i'. Then using $\mathcal{N}' = (N_i')$ alone as exceptional system for all ϱ the Riemann sums will be small for suitable choices of Δ'. This shows: $f \in P_*(A)$ and ${}^*\!\int_A f = 0$. As a consequence, $*$-integrability and the value of the integral do not change if we alter the integrand on an n-null set.

Further general properties of $*$-integration will be developped in a later paper, in particular, the converse of (9) (additivity) and that $*$-integration extends $\mathcal{L}$-integration.

3. Proof of the Fundamental Theorem

We know that the integral ${}^*\!\int_B f$ represents an additive set function. A central question is which additive set functions can be represented in this way. Theorem 2, i.e. the Fundamental Theorem of Calculus, gives a partial answer to this question. There are other types of Perron integration for which the answer is completely known, cf. Jurkat-Knizia [7], [8].

P r o o f of Theorem 2. We assume that F is additive on $\mathcal{A}(A)$ and satisfies (0.6). Let $L \neg D = N'$, $C \neg L = M$, $A \neg C = N''$, so that A is the disjoint union $D \cup N' \cup M \cup N''$ and

$$|N'|_n = 0, M \text{ is } \sigma_{n-1}\text{-finite}, \qquad |N''|_{n-1} = 0 \tag{1}$$

holds as a consequence of (0.6). We fix $\varrho \geqslant c_n$ arbitrarily and select the exceptional system as follows: Because (L_ϱ) holds for $x \in N'$ there are positive functions $K'(\cdot)$ and $\delta_1(\cdot)$ on N' such that

$$|F(B)| \leqslant K'(x)|B|_n \text{ if } B \in \mathcal{A}_\varrho(A),\ x \in B,\ d(B) < \delta_1(x)$$

is true. Hence we can decompose N' into countably many n-null sets N_i' where $K'(\cdot)$ is bounded, say by $K_i' > 0$. then we have

$$|F(B)| \leqslant K_i'|B|_n \text{ if } B \in \mathcal{A}_\varrho(A),\ x \in B,\ d(B) < \delta_1(x),\ x \in N_i'. \tag{2}$$

Next we decompose M into countably many $(n-1)$-finite sets M_i and remark that (C) holds whenever $x \in M_i$. So, if we select any positive numbers ε_i, ε', there is a positive function $\delta_2(\cdot)$ on $C \supseteq M = \cup M_i$ such that

$$\begin{cases} |F(B)| \leqslant \varepsilon_i |\partial B|_{n-1} \text{ if } B \in \mathcal{A}(A),\ x \in B,\ d(B) < \delta_2(x),\ x \in M_i \\ |F(B)| \leqslant \varepsilon' |\partial B|_{n-1} \text{ if } B \in \mathcal{A}(A),\ x \in B,\ d(B) < \delta_2(x),\ x \in C \end{cases} \tag{3}$$

is true. Because (A) holds for $x \in N''$ there are positive functions $K''(\cdot)$ and $\delta_3(\cdot)$ on N'' such that

$$|F(B)| \leqslant K''(x)|\partial B|_{n-1} \text{ if } B \in \mathcal{A}(A),\ x \in B,\ d(B) < \delta_3(x)$$

is true. Hence we can decompose N'' into countably many $(n-1)$-null sets N_i'' where $K''(\cdot)$ is bounded, say by $K_i'' > 0$. Then we have

$$(4) \qquad |F(B)| \leqslant K_i''|\partial B|_{n-1} \text{ if } B \in \mathcal{A}(A),\ x \in B,\ d(B) < \delta_3(x),\ x \in N_i''.$$

Since ϱ is fixed also the exceptional system $\mathcal{M} = (M_i)$, $\mathcal{N}' = (N_i')$, $\mathcal{N}'' = (N_i'')$ is fixed as well as $\delta_1(\cdot)$, $K' = (K_i')$, $\delta_3(\cdot)$, $K'' = (K_i'')$, while the choice of $\delta_2(\cdot)$ depends on additional numbers ε_i, ε'. Finally, there is a positive function $\delta_4(\cdot)$ on D such that

$$(5) \qquad \left|F(B) - \dot{F}(x)|B|_n\right| \leqslant \varepsilon'|B|_n \text{ if } B \in \mathcal{A}_\varrho(A),\ x \in B,\ d(B) < \delta_4(x),\ x \in D$$

is true.

Now we want to show that $\dot{F}$ is ϱ-integrable over A relative to $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$ with $F = F(A)$. Suppose that σ, K, and $\varepsilon > 0$ are arbitrarily selected, let Δ', Δ'', $\delta(\cdot)$ be undetermined for the moment, and consider all admissible partitions Π of A. We have, of course,

$$(6) \qquad F(A) = \sum_k F(A_k)$$

and break the sum on the right into five pieces according to the following summation conditions.

$$\sum_1 : x_k \in N''; \quad \sum_2 : x_k \in M; \quad \sum_3 : x_k \in L,\ A_k \notin \mathcal{A}_\varrho;$$
$$\sum_4 : x_k \in N',\ A_k \in \mathcal{A}_\varrho; \quad \sum_5 : x_k \in D,\ A_k \in \mathcal{A}_\varrho.$$

This is a complete decomposition and we have accordingly

$$(7) \qquad F(A) = \sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5.$$

If we assume $\delta(\cdot) \leqslant \delta_1(\cdot)$ on N', $\delta(\cdot) \leqslant \delta_2(\cdot)$ on M, $\delta(\cdot) \leqslant \delta_3(\cdot)$ on N'', and $\delta(\cdot) \leqslant \delta_4(\cdot)$ on D we obtain the following estimates using (4), (3), (3), (2), (5) respectively with

$B = A_k$, $x = x_k$ in conjunction with the control conditions:

$$|\sum_1| \leqslant \sum_i \sum_{x_k \in N_i''} K_i'' |\partial A_k|_{n-1} \leqslant \sum_i K_i'' \Delta_i'', \tag{8}$$

$$|\sum_2| \leqslant \sum_i \sum_{x_k \in M_i} \varepsilon_i |\partial A_k|_{n-1} \leqslant \sum_i \varepsilon_i K_i, \tag{9}$$

$$|\sum_3| \leqslant \sum_{x_k \in C, A_k \notin \mathcal{A}_\varrho} \varepsilon' |\partial A_k|_{n-1} \leqslant \varepsilon' \sigma, \tag{10}$$

$$|\sum_4| \leqslant \sum_i \sum_{x_k \in N_i', A_k \in \mathcal{A}_\varrho} K_i' |A_k|_n \leqslant \sum_i K_i' \Delta_i', \tag{11}$$

$$\left| \sum_{x_k \in D, A_k \in \mathcal{A}_\varrho} (F(A_k) - \dot{F}(x_k) |A_k|_n) \right| \leqslant \sum \varepsilon' |A_k|_n \leqslant \varepsilon' |A|_n. \tag{12}$$

If we restrict $\delta(\cdot)$ further so that $|\dot{F}(x)|\delta(x) \leqslant \varepsilon'$ holds on A, we also obtain

$$\begin{cases} \left| \displaystyle\sum_{x_k \in D, A_k \notin \mathcal{A}_\varrho} \dot{F}(x_k) |A_k|_n \right| \leqslant \displaystyle\sum_{A_k \notin \mathcal{A}_\varrho} |\dot{F}(x_k)| \, c_n \, d(A_k) \, |\partial A_k|_{n-1} \\ \leqslant c_n \displaystyle\sum_{A_k \notin \mathcal{A}_\varrho} |\dot{F}(x_k)| \delta(x_k) \, |\partial A_k|_{n-1} \leqslant c_n \, \varepsilon' \, \sigma. \end{cases} \tag{13}$$

Finally we select ε_i, ε', $\Delta' = (\Delta_i')$, $\Delta'' = (\Delta_i'')$ so that the last term in each of the six inequalities (8)–(13) is $\leqslant \varepsilon/6$. Then $\delta_2(\cdot)$, $\delta_4(\cdot)$ are determined and $\delta(\cdot)$ can be found as required. Since

$$(14) \qquad \begin{cases} S(\dot{F}, \Pi) & = \displaystyle\sum_{x_k \in D} \dot{F}(x_k) \, |A_k|_n \\ & = \displaystyle\sum_{x_k \in D, A_k \in \mathcal{A}_\varrho} + \displaystyle\sum_{x_k \in D, A_k \notin \mathcal{A}_\varrho}, \end{cases} \tag{14}$$

we infer from our six inequalities

$$|S(\dot{F}, \Pi) - F(A)| \leqslant \varepsilon \tag{15}$$

for all admissible Π. This proves ${}^{\varrho}\!\int_A \dot{F} = F(A)$ for all $\varrho \geqslant c_n$ as desired. □

R e m a r k s. The proof above shows very clearly how the given properties of F (in terms of exceptional sets) are matched with control conditions for Π so that the corresponding parts of the Riemann sums behave as desired. And exactly these restricted Riemann sums have been used to define our Perron type integration. Obviously this procedure can be generalized to cover other 'singularities' of F by means of corresponding Perron type integrals. Further results in this direction will be developped in a later paper.

4. The Decomposition Theorem

Here we give our abstraction of the ideas of Besicovitch as presented in Saks [18: pp. 192–195] which leads to a geometric result not unsimilar to Vitali's covering theorem.

Let I be a fixed cube (compact with $I^0 \neq \emptyset$) in $\mathbb{R}^n$ which we decompose into 2^n subcubes of half the size, called the dyadic subcubes of level 1. Repeating this process with each of these subcubes we obtain the dyadic subcubes of level 2 etc., in general of level $h \in \mathbb{N}_0$, where $h = 0$ corresponds to I itself. We denote by $\mathcal{J}_h$ the collection of all dyadic subcubes of level h or higher. Now suppose that E is a subset of I with $|E|_\alpha < \infty$, where $0 \leqslant \alpha \leqslant n$. According to Lemma 4.1 in Saks the following is true. For each h and each $\varepsilon > 0$ there is a countable family (J_i) in $\mathcal{J}_h$ which satisfies

$$\sum_i d(J_i)^\alpha \leqslant c_n \left(|E|_\alpha + \varepsilon\right) \tag{1}$$

with an absolute constant c_n, and which covers E completely in the following sense: For each $x \in E$ there is a level $h(x)$ such that all dyadic subcubes of level $h(x)$, which contain x, actually belong to (J_i). This result will be strengthened by using an arbitrary positive function $\delta(\cdot)$ on I. We denote by $\mathcal{J}_\delta^* = \mathcal{J}_\delta^*(E)$ the collection of all dyadic subcubes J for which there is an $x \in E \cap J$ with $\delta(x) > d(J)$.

Lemma. *Suppose $E \subseteq I$, $|E|_\alpha < \infty$ $(0 \leqslant \alpha \leqslant n)$. Then, for each positive $\delta(\cdot)$ on I and each $\varepsilon > 0$ there is a countable family (J_i) in $\mathcal{J}_\delta^*$ which covers E completely and satisfies* (1).

P r o o f. Let $G_h = G_h(E, \delta)$ be the set of all points $x \in I$ with the property that any $J \in \mathcal{J}_h$ belongs to $\mathcal{J}_\delta^*$ if $x \in J$. We observe that

$$I \neg G_h \;=\; \cup\{J : J \in \mathcal{J}_h\, J \notin \mathcal{J}_\delta^*\}$$

is a Borel set and so is G_h. It is immediate that

$$G_h \nearrow \; G = \cup\, G_h \supseteq E (h \nearrow)$$

holds. Hence, if we form $E_0 = E \cap G_0$, $E_h = E \cap (G_h \neg G_{h-1})$ for $h \geqslant 1$, it follows

$$E = \bigcup_{h=0}^{\infty} E_h \quad \text{and} \quad |E|_\alpha = \sum_{h=0}^{\infty} |E_h|_\alpha .$$

Furthermore, any $J \in \mathcal{J}_h$ which meets E_h belongs to $\mathcal{J}_\delta^*$ by our definitions. By the result above we have for each h a countable family (J_{hj}) in $\mathcal{J}_h$ which covers E_h completely and satisfies

$$\sum_i d(J_{hj})^\alpha \leqslant c_n \left(|E_h|_\alpha + \frac{\varepsilon}{2^{h+1}}\right). \tag{2}$$

If we drop all J_{hj} which do not meet E_h these statements are still true; moreover, the remaining J_{hj} belong to $\mathcal{J}_\delta^*$. If we now unite our families we obtain a new countable family (J_i) which meets the requirements of the lemma. □

Now we formulate our Decomposition Theorem.

Theorem 3. *Suppose that the cube I is the disjoint union of countably many subsets E_m with $|E_m|_{\alpha_m} < \infty$ $(0 \leqslant \alpha_m \leqslant n)$, and that a function $\delta(\cdot) > 0$ on I and numbers $\varepsilon_m > 0$ have been selected arbitrarily. Then there are finitely many dyadic subcubes I_k and points x_k satisfying $I = \biguplus I_k$, $x_k \in I_k$, $d(I_k) < \delta(x_k)$, and*

$$\sum_{x_k \in E_m} d(I_k)^{\alpha_m} \leqslant c_n(|E_m|_{\alpha_m} + \varepsilon_m) \quad \text{for all } m, \tag{3}$$

where c_n denotes a positive absolute constant.

P r o o f. We apply our Lemma to each E_m and obtain, thereby, for each m a countable family (J_{mj}) in $\mathcal{J}_\delta^*(E_m)$ which covers E_m completely and satisfies

$$\sum_j d(J_{mj})^{\alpha_m} \leqslant c_n\,(|E_m|_{\alpha_m} + \varepsilon_m). \tag{4}$$

By uniting our families we obtain a new countable family (J_i) of dyadic subcubes which completely covers I (in our special sense). Any family with this property contains a finite subfamily (I_k) such that $I = \biguplus I_k$. This can be seen by contradiction: If I cannot be represented in that form, the same is true for a subcube of the first level, which contains such a subcube of the second level etc. So there is a nested sequence of such subcubes going through all levels which converges to some $x \in I$. But in view of the complete covering there is a level $h(x)$ at which all cubes of that level, which contain x, are in our family. Hence, when the nested sequence reaches the level $h(x)$ our subcube must occur in (J_i) and has a trivial representation contradicting the construction.

We now select a finite subfamily (I_k) such that $I = \biguplus I_k$. Each I_k is a J_{mj} with $m = m_k$ and since this $J_{mj} \in \mathcal{J}_\delta^*$ (E_m), we many select an $x_k \in E_{m_k} \cap I_k$ with $\delta(x_k) > d(I_k)$. Because the sets E_m are disjoint we can recover m_k from the condition $x_k \in E_m$. Hence all I_k with $x_k \in E_m$ must be certain cubes J_{mj}, and condition (3) follows from (4), while the other requirements are already satisfied. □

In our application we use a covering version of the decomposition theorem.

Theorem 3'. *Let A be a compact set and (E_j) be a countable family of disjoint sets with $|E_j|_{\alpha_j} < \infty$ $(0 \leqslant \alpha_j \leqslant n)$. Then, for each function $\delta(\cdot) > 0$ on A and any numbers $\varepsilon_j > 0$ there are finitely many cubes J_k and points x_k satisfying $A \subseteq \biguplus J_k$, $x_k \in A \cap J_k$, $d(J_k) < \delta(x_k)$ and*

$$\sum_{x_k \in E_j} d(J_k)^{\alpha_j} \leqslant c_n(|E_j|_{\alpha_j} + \varepsilon_j) \qquad \textit{for all } j.$$

P r o o f. Without loss of generality we may assume $E_j \subseteq A$. Then we select a cube $I \supseteq A$ and set $E = I \neg \cup E_j$, $|E|_n < \infty$, so that I is the disjoint union of E and the E_j. Finally we define $\delta(x)$ on $I \neg A$ as the distance from x to A. Now we apply Theorem 3 and keep only those I_k with $x_k \in A$. Since the I_k with $x_k \notin A$ do not meet A the result follows. □

Our next aim is a proof of the richness property (R). We begin with a remark: *Suppose that E is an $(n-1)$-null set and M is an $(n-1)$-finite set. Then, for each $\varepsilon > 0$ there is an open set $G \supseteq E$ with $|G \cap M|_{n-1} \leqslant \varepsilon$.* This can be seen as follows. According to Saks [18: p. 53] there is a sequence of open sets $G_m \searrow G' \supseteq E$ with $|G'|_{n-1} = 0$. But then $|G_m \cap M|_{n-1} \to |G' \cap M|_{n-1} = 0$, which implies the result.

A trivial observation is that all cubes are ϱ-regular, if $\varrho \geqslant c_n$ with a suitable absolute constant $c_n > 0$. From now on $\varrho \geqslant c_n$ will be assumed.

P r o o f o f (R). We form (E_j) by uniting (M_i), (N_i'), (N_i'') and set $\alpha_j = n-1$ if E_j is M_i or N_i'', and $\alpha_j = n$ if E_j is N_i'. The function $\delta(\cdot) > 0$ on A and the numbers $\varepsilon_j > 0$ (representing ε_i^*, ε_i', ε_i'') will be determined later.

Now we apply theorem 3' and set $A_k = A \cap J_k \in \mathcal{A}(A)$. Clearly $A = \biguplus A_k$, $x_k \in A_k$, $d(A_k) < \delta(x_k)$, and for all i

$$\sum_{x_k \in M_i} \mathrm{d}(J_k)^{n-1} \leqslant c_n(|M_i|_{n-1} + \varepsilon_i^*) = K_i' \text{ (choose } \varepsilon_i^* = 1), \tag{5}$$

$$\sum_{x_k \in N_i'} d(J_k)^n \leqslant c_n(|N_i'|_n + \varepsilon_i') = c_n\, \varepsilon_i', \tag{6}$$

$$\sum_{x_k \in N_i''} d(J_k)^{n-1} \leqslant c_n(|N_i''|_{n-1} + \varepsilon_i'') = c_n\, \varepsilon_i''. \tag{7}$$

Thus we have obtained a δ-fine partition of A, and we are now concerned with the control conditions (2.1) and (2.3).

Since $|A_k|_n \leqslant |J_k|_n \leqslant c_n\, d(J_k)^n$, we can guarantee

$$\sum_{x_k \in N_i'} |A_k|_n \leqslant \Delta_i' \tag{8}$$

by choosing ε_i' small enough in (6). Next we observe that

$$\partial A_k = \partial(J_k \cap A) \subseteq (\partial J_k) \cup (J_k^0 \cap \partial A),$$

which implies

$$|\partial A_k|_{n-1} \leqslant c_n d(J_k)^{n-1} + |J_k^0 \cap \partial A|_{n-1}. \tag{9}$$

Using the same arguments as in section 2 in connection with joining two exceptional systems we might have assumed at the beginning (without loss of generality) that some of the sets N_i'' together with M_1 form exactly ∂A. Furthermore, we may require that $\delta(x)$ is less than the distance from x to the exterior of A for all $x \in A^0$. This implies that $J_k \subseteq A$ if $x_k \in A^0$, hence

$$\sum_{A_k \notin \mathcal{A}_\varrho} |\partial A_k|_{n-1} \leqslant \sum_{x_k \in \partial A} |\partial A_k|_{n-1} \leqslant c_n \sum_{x_k \in \partial A} d(J_k)^{n-1} + |\partial A|_{n-1}.$$

Since we may assume that all $\varepsilon_i'' \leqslant 1/2^i$ we have

$$\begin{aligned}\sum_{x_k \in \partial A} d(J_k)^{n-1} &\leqslant \sum_{x_k \in M_1} d(J_k)^{n-1} + \sum_i \sum_{x_k \in N_i''} d(J_k)^{n-1} \\ &\leqslant K_1' + \sum_i c_n \varepsilon_i'' \leqslant K_1' + 2c_n,\end{aligned}$$

and therefore

$$\sum_{A_k \notin \mathcal{A}_\varrho} |\partial A_k|_{n-1} \leqslant c_n(K_1' + 2c_n) + |\partial A|_{n-1} = \sigma^*. \tag{10}$$

Now the sets M_i are disjoint from ∂A for $i > 1$, so we have for $i > 1$

$$\left\{\begin{aligned}\sum_{x_k \in M_i} |\partial A_k|_{n-1} = \sum_{x_k \in M_i} |\partial J_k|_{n-1} \leqslant \sum_{x_k \in M_i} c_n\, d(J_k)^{n-1} \\ \leqslant c_n\, K_i' = K_i^*\end{aligned}\right. \tag{11}$$

and for $i = 1$ as above

$$\left\{\begin{aligned}\sum_{x_k \in M_1} |\partial A_k|_{n-1} \leqslant c_n \sum_{x_k \in M_1} d(J_k)^{n-1} + |\partial A|_{n-1} \\ \leqslant c_n K_1' + |\partial A|_{n-1} = K_1^*.\end{aligned}\right. \tag{12}$$

For those N_i'' which are disjoint from ∂A we have

$$(13) \qquad \sum_{x_k \in N_i''} |\partial A_k|_{n-1} = \sum_{x_k \in N_i''} |\partial J_k|_{n-1} \leqslant c_n \sum_{x_k \in N_i''} d(J_k)^{n-1} \leqslant \Delta_i''$$

by choosing ε_i'' small enough in (7). For the other N_i'', which are then contained in ∂A, we have

$$(14) \qquad \begin{cases} \sum_{x_k \in N_i''} |\partial A_k|_{n-1} \leqslant c_n \sum_{x_k \in N_i''} d(J_k)^{n-1} + |U_i \cap \partial A|_{n-1} \\ \text{where} \qquad U_i = \bigcup_{x_k \in N_i''} J_k^0. \end{cases}$$

By the remark before the beginning of the proof there exists an open set $G_i \supseteq N_i''$ such that $|G_i \cap \partial A|_{n-1} \leqslant \frac{1}{2}\Delta_i''$. By restricting $\delta(\cdot)$ on N_i'' we can guarantee that $U_i \subseteq G_i$; and by choosing ε_i'' small enough in (7) we are certain that (14) implies

$$(15) \qquad \sum_{x_k \in N_i''} |\partial A_k|_{n-1} \leqslant \Delta_i''$$

also for these remaining N_i''. Now all conditions of (R) are satisfied with σ^* and K^* dependent only on A, $\mathcal{M}$, $\mathcal{N}'$, $\mathcal{N}''$. □

It is obvious that our decomposition theorem can be used to construct partitions which satisfy further control conditions which in turn enables us to treat further singularities. This will be developped in a later paper.

5. Green's Theorem with exceptional sets

Let γ be a rectifiable simple closed continuous curve in $\mathbb{R}^2$ and G the encircled region according to the Jordan curve theorem. We assume that γ is so oriented that G is encircled in the positive sense. Let A be the set G united with all points of γ. Then $A \in \mathcal{A}$, $A^0 = G$, and ∂A consists of the points of γ.

For test functions $\vec{v} = (f, g)$, e.g. of type $C_0^1(\mathbb{R}^2)$, Green's theorem is known in the form

$$(1) \qquad \int_\gamma (f\,\mathrm{d}y - g\,\mathrm{d}x) = \int_A \operatorname{div} \vec{v}\,\mathrm{d}\mathcal{L}, \operatorname{div} \vec{v} = f_x + g_y;$$

more than that has been shown, e.g., by Shapiro [19]. Comparing with (0.1) in theorem 1 we see that

$$(2) \qquad \int_\gamma (f\,\mathrm{d}y - g\,\mathrm{d}x) = \int_{\partial A} \vec{v} \cdot \vec{n}_A\,\mathrm{d}\mathcal{H}$$

holds for test functions, and since both sides represent continuous linear functionals the identity (2) extends to all functions f, g which are bounded and $\mathcal{H}$-measurable on ∂A. (Parameterize γ by arc-length s which generates the $\mathcal{H}$-measure along ∂A in the present case, cf. Saks [18: pp. 123–125].)

Now we can use theorem 1 to generalize (1) as follows using the notation of the introduction.

Theorem 1'. *Suppose that γ and A are as described above and that $\vec{v} = (f, g)$: $A \to \mathbb{R}^2$ is bounded and its exceptional sets satisfy: $A \neg L$ is σ_1-finite, $|A \neg C|_1 = 0$. Then $\vec{v}$ is $\mathcal{H}$-measurable on A, $\operatorname{div} \vec{v}$ is $*$-integrable over A, and we have the identity*

$$\int_\gamma (f \,\mathrm{d}y - g \,\mathrm{d}x) = {}^*\!\!\int_A \operatorname{div} \vec{v}. \tag{3}$$

With somewhat greater effort we can use theorem 1 to deduce a general form of Green's theorem, where γ need not be simple and the winding number with respect to γ occurs on the right side of (3), cf. Jurkat-Nonnenmacher [9]. There are analogues of this result in higher dimensions which will be discussed in a later paper. Theorem 1' contains, as a special case, a form of Cauchy's integral theorem with exceptional sets and this contains the result of Besicovitch mentioned in the introduction (via Morera's theorem).

Postscript. After the paper was finished I received the following preprints from J. Kurzweil and J. Jarník: "The PU-Integral: Its definition and some basic properties", "The PU-Integral and its properties".

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