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ON COMPLEX RADON MEASURES II

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Various types of regular extensions for complex and positive measures on $\mathscr{D}(\mathscr{K}_0)$ are studied and are made use of to characterize μ_θ and M_θ in terms of the restrictions $\mu_\theta|\mathscr{D}(\mathscr{K})$ and $\mu_\theta|\mathscr{D}(\mathscr{K}_0)$, where $\theta \in \mathscr{K}(X)^*$, $\mathscr{D}(\mathscr{K}_0)$, $\mathscr{D}(\mathscr{K})$, $\mathscr{K}(X)$, μ_θ and M_θ being given as in [10]. Several characterizations for $\theta \in \mathscr{K}(X)^*$ to be bounded are given as well as a generalization of Theorem 54.2 of [9] to complex Radon measures is obtained. Finally, $\mathscr{K}(X)^*$, $\mathscr{K}(X,\mathbb{R})^*$ and $\mathscr{K}(X)^*_b$ are identified with certain spaces of complex or real measures on $\mathscr{D}(\mathscr{K}_0)$ and $\mathscr{D}(\mathscr{K})$ and is shown that the space of all $\mathbb{C}$-valued additive set functions of finite variation on a ring of sets is isomorphic to $\mathscr{K}(X)^*$ for a properly chosen locally compact Hausdorff space X.

1. INTRODUCTION

The present paper is a continuation of [10]. We use the same notation and terminology of [10]. The main purpose of the present work is to generalize Theorem 54.2 of McShane [9] to complex Radon measures on a locally compact Hausdorff space X and to characterize μ_θ and M_θ in terms of the restrictions $\mu_\theta|\mathscr{D}(\mathscr{K}_0)$ and $\mu_\theta|\mathscr{D}(\mathscr{K})$, where μ_θ and M_θ are as in [10]. Also are included results concerning regular extensions of positive and complex measures on $\mathscr{D}(\mathscr{K}_0)$ and the study of spatial isomorphisms of $\mathscr{K}(X)^*$, $\mathscr{K}(X,\mathbb{R})^*$ and $\mathscr{K}(X)^*_b = \{\theta \in \mathscr{K}(X)^* : \theta \text{ bounded}\}$. Finally, we show that the space of all $\mathbb{C}$-valued additive set functions of finite (resp., of bounded) variation on a ring of sets is isomorphic to $\mathscr{K}(X)^*$ (resp., isometrically isomorphic to $\mathscr{K}(X)^*_b$) for a suitably chosen totally disconnected locally compact Hausdorff space X.

In this connection, we would like to point out that the isometric isomorphism of $\mathscr{K}(X)^*_b$ to the Banach space of all regular complex measures on $\mathscr{B}(X)$ (vide [6]) was

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used by Thomas in [12] to deduce the Grothendieck's weak compactness criterion (given in [4]) of a subset H of $\mathscr{K}(X)^*_b$ from the compactness criterion established by Bartle, Dunford and Schwartz in [1]. Similarly, our representation theorem for $\mathscr{K}(X)^*$ might be useful to shed more light on the inter-relations between abstract measures and Radon measures.

2. Regular extensions of positive and complex measures

Using the various notions of regularity given in [10] for positive and complex measures defined on a δ-ring $\mathscr{R}$ containing $\mathscr{D}(\mathscr{K})$ or $\mathscr{D}(\mathscr{K}_0)$, we study the regular extensions of positive and complex measures defined on $\mathscr{D}(\mathscr{K}_0)$. The results of this section play a key role in the rest of the paper.

Theorem 2.1. *Let μ_0 be a finite (positive) measure on $\mathscr{D}(\mathscr{K}_0)$. Then there exist unique extensions μ, ν and w of μ_0 to $\mathscr{D}(\mathscr{K})$, $\mathscr{B}_c(X)$ and $\mathscr{B}(X)$, respectively, such that μ is $\mathscr{D}(\mathscr{K})$-regular, ν is $\mathscr{B}_c(X)$-regular and w is Radon-regular. Besides, $\mu = \nu|\mathscr{D}(\mathscr{K}) = w|\mathscr{D}(\mathscr{K})$ and $\nu = w|\mathscr{B}_c(X)$.*

Proof. Let μ_0 be the unique extension of μ_0 to $\mathscr{B}_0(X) = \mathscr{S}(\mathscr{D}(\mathscr{K}))$ as a measure. Then by Theorem 3.7 of [10] μ_0 has a unique extension ν to $\mathscr{B}_c(X)$ (resp., w to $\mathscr{B}(X)$) such that ν is $\mathscr{B}_c(X)$-regular (resp., w is Radon-regular) and $\nu = w|\mathscr{B}_c(X)$. Take $\mu = \nu|\mathscr{D}(\mathscr{K})$. Then μ, ν and w are the extensions of μ_0 with the required properties. The uniqueness of μ follows by Theorems 3.9(i) and 3.7(i) of [10], while ν and w are unique by (i) and (ii) of Theorem 3.7 of [10], respectively. □

Lemma 2.2. *Let μ_1, μ_2 be finite (positive) measures on $\mathscr{D}(\mathscr{K}_0)$ (resp., $\mathscr{D}(\mathscr{K})$-regular measures on $\mathscr{D}(\mathscr{K})$). Then there exist unique extensions w_1, w_2 and w_3 of μ_1, μ_2 and $\mu_1+\mu_2$, respectively, to $\mathscr{D}(\mathscr{K})$ (resp., to $\mathscr{B}(X)$) as $\mathscr{D}(\mathscr{K})$-regular (resp., Radon-regular) measures and $w_3 = w_1 + w_2$.*

Proof. Let μ_1', μ_2' and μ_3' be the unique extensions of μ_1, μ_2 and $\mu_1 + \mu_2$, respectively, to $\mathscr{B}_0(X)$ (resp., to $\mathscr{B}_c(X)$) as measures. Clearly, $\mu_3' = \mu_1' + \mu_2'$. Let

$$\theta_j(f) = \int_X f\,\mathrm{d}\mu_j', \quad f \in C_c(X), \quad j = 1,2,3.$$

Then $\theta_3 = \theta_1 + \theta_2$ and by Proposition 15, §1, Chapter IV of [2], $\check{\mu}_{\theta_3} = \check{\mu}_{\theta_1} + \check{\mu}_{\theta_2}$ on $\mathscr{B}(X)$ (vide Theorem 2.2 of [10] for the notation). Let $w_j = \check{\mu}_{\theta_j}$ if μ_j is $\mathscr{D}(\mathscr{K})$-regular and let $w_j = \check{\mu}_{\theta_j}|\mathscr{D}(\mathscr{K})$ if μ_j is defined on $\mathscr{D}(\mathscr{K}_0)$. Then by Theorem 2.2 of [10] $\check{\mu}_{\theta_j}$ is Radon-regular and $\check{\mu}_{\theta_j}|\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular by Lemma 3.5(i) of

[10]. By Proposition 3.4(i) (resp., by Theorem 3.9(i)) of [10], μ_1', μ_2' and μ_3' are $\mathscr{B}_0(X)$-regular (resp., $\mathscr{B}_c(X)$-regular). As shown in the proof of Theorem 3.7 (resp., of Theorem 3.8) of [10], w_j extends μ_j' and hence μ_j. The uniqueness of w_j on $\mathscr{D}(\mathscr{K})$ (resp., on $\mathscr{B}(X)$) follows from Theorem 2.1 (resp., from Theorem 2.2(vii) and Proposition 3.4(iv) of [10]). Clearly, $w_3 = w_1 + w_2$ in both the cases. □

Lemma 2.3. *(i) If ν is a real $\mathscr{D}(\mathscr{K})$-regular measure, let $\nu_0 = \nu|\mathscr{D}(\mathscr{K}_0)$. Then $\nu_0^+ = \nu^+|\mathscr{D}(\mathscr{K}_0)$ and $\nu_0^- = \nu^-|\mathscr{D}(\mathscr{K}_0)$.*

(ii) If ν_1 and ν_2 are $\mathscr{D}(\mathscr{K})$-regular complex measures on $\mathscr{D}(\mathscr{K})$ and if $\nu_1|\mathscr{D}(\mathscr{K}_0) = \nu_2|\mathscr{D}(\mathscr{K}_0)$, then $\nu_1 = \nu_2$.

Proof. (i) $\nu_0^+(E) \leqslant \nu^+(E)$, $\nu_0^-(E) \leqslant \nu^-(E)$ for $E \in \mathscr{D}(\mathscr{K}_0)$. Let $w_1 = \nu^+|\mathscr{D}(\mathscr{K}_0)$ and $w_2 = \nu^-|\mathscr{D}(\mathscr{K}_0)$. Since $w_1 + \nu_0^- = w_2 + \nu_0^+$ on $\mathscr{D}(\mathscr{K}_0)$, by Lemma 2.2 there exist unique $\mathscr{D}(\mathscr{K})$-regular extensions $\hat{\nu}_0^+$, $\hat{\nu}_0^-$, $\hat{w}_1$, $\hat{w}_2$ of ν_0^+, ν_0^-, w_1 and w_2, respectively, such that $\hat{w}_1 + \hat{\nu}_0^- = \hat{w}_2 + \hat{\nu}_0^+$ and by Theorem 2.1 $\nu^+ = \hat{w}_1$ and $\nu^- = \hat{w}_2$. Thus $\nu = \hat{\nu}_0^+ - \hat{\nu}_0^-$, whence $\nu^+ \leqslant \hat{\nu}_0^+$ and $\nu^- \leqslant \hat{\nu}^-$. Hence (i) holds.

(ii) Clearly, it suffices to prove the result for ν_1 and ν_2 real. Let $\nu_1|\mathscr{D}(\mathscr{K}_0) = \nu_2|\mathscr{D}(\mathscr{K}_0) = \mu_0$ (say). By Theorem 2.1 there exist unique $\mathscr{D}(\mathscr{K})$-regular extensions $\hat{\mu}_0^+$ and $\hat{\mu}_0^-$ of μ_0^+ and μ_0^- to $\mathscr{D}(\mathscr{K})$. Now, by hypothesis and (i), $\nu_1^+|\mathscr{D}(\mathscr{K}_0) = \nu_2^+|\mathscr{D}(\mathscr{K}_0) = \mu_0^+$ so that by the uniqueness part of Theorem 2.1 we conclude that $\nu_j^+ = \hat{\mu}_0^+$ for $j = 1, 2$. Similarly, $\nu_j^- = \hat{\mu}_0^-$ for $j = 1, 2$. Thus $\nu_1 = \nu_2$. □

Theorem 2.4. *Let μ_0, ν_0 be complex measures on $\mathscr{D}(\mathscr{K}_0)$, μ_0 being of bounded variation on $\mathscr{D}(\mathscr{K}_0)$. Then:*

(i) ν_0 has a unique extension ν to $\mathscr{D}(\mathscr{K})$ as a $\mathscr{D}(\mathscr{K})$-regular complex measure. ν is real (resp., positive) if ν_0 is so.

(ii) The unique extension $\hat{\mu}_0$ to $\mathscr{B}_0(X)$ of μ_0 as a complex measure is $\mathscr{B}_0(X)$-regular.

(iii) μ_0 has a unique extension μ to $\mathscr{B}_c(X)$ (resp., w to $\mathscr{B}(X)$) as a $\mathscr{B}_c(X)$-regular (resp., $\mathscr{B}(X)$-regular) complex measure. μ and w are real (resp., positive) if μ_0 is so.

(iv) $\mu = w|\mathscr{B}_c(X)$ and $\hat{\mu}_0 = \mu|\mathscr{B}_0(X) = w|\mathscr{B}_0(X)$.

(v) Let $M = \sup\{v(\mu_0, \mathscr{D}(\mathscr{K}_0))(E) \colon E \in \mathscr{D}(\mathscr{K}_0)\}$. Then

$$\sup\{v(\eta, \mathscr{R})(E) \colon E \in \mathscr{R}\} = M$$

for $\eta = \hat{\mu}_0$, μ or w and $\mathscr{R} = \mathscr{B}_0(X)$, $\mathscr{B}_c(X)$ or $\mathscr{B}(X)$, respectively.

Proof. (i) Let $\nu_1 = \operatorname{Re}\nu_0$ and $\nu_2 = \operatorname{Im}\nu_0$. By Theorem 2.1 there exist unique $\mathscr{D}(\mathscr{K})$-regular extensions $\hat{\nu}_j^+$ and $\hat{\nu}_j^-$ of ν_j^+ and ν_j^- for $j = 1, 2$ and $\hat{\nu}_j = \hat{\nu}_j^+ - \hat{\nu}_j^-$ is well defined and a $\mathscr{D}(\mathscr{K})$-regular real measure on $\mathscr{D}(\mathscr{K})$. Let $\nu = \hat{\nu}_1 + \mathrm{i}\hat{\nu}_2$. Then ν

is $\mathscr{D}(\mathscr{K})$-regular, $\nu|\mathscr{D}(\mathscr{K}_0) = \nu_0$ and is unique by Lemma 2.3(ii). The rest of (i) is obvious.

Let $|\mu_0| = v(\mu_0, \mathscr{D}(\mathscr{K}_0))$. By Theorem 17.26 of [11] there exists a unique extension $\hat{\mu}_0$ of μ_0 to $\mathscr{B}_0(X)$ as a complex measure of bounded variation. Besides, $|\hat{\mu}_0| = v(\hat{\mu}_0, \mathscr{B}_0(X))$ extends $|\mu_0|$ to $\mathscr{B}_0(X)$ and

$$\sup\{|\hat{\mu}_0|(E) \colon E \in \mathscr{B}_0(X)\} = M. \tag{1}$$

If $\eta_1 = \operatorname{Re}\mu_0$ and $\eta_2 = \operatorname{Im}\mu_0$, then

$$\hat{\mu}_0 = (\hat{\eta}_1^+ - \hat{\eta}_1^-) + \mathrm{i}(\hat{\eta}_2^+ - \hat{\eta}_2^-)$$

on $\mathscr{B}_0(X)$, where $\hat{\eta}_j^+$ and $\hat{\eta}_j^-$ are the unique extensions of η_j^+ and η_j^- to $\mathscr{B}_0(X)$ as measures for $j = 1, 2$.

(ii) By Proposition 3.4(i) of [10], $\hat{\eta}_j^+$ and $\hat{\eta}_j^-$ are $\mathscr{B}_0(X)$-regular for $j = 1, 2$ and hence $\hat{\mu}_0$ is $\mathscr{B}_0(X)$-regular.

(iii) By Theorem 3.7(ii) of [10] there exist unique extensions w_j and w_j' of $\hat{\eta}_j^+$ and $\hat{\eta}_j^-$, respectively, to $\mathscr{B}(X)$ as Radon-regular measures. For $C \in \mathscr{K}$, by Proposition 11, §14 of [3], there exists $C_0 \in \mathscr{K}_0$ with $C \subset C_0$ so that

$$w_j(C) \leqslant w_j(C_0) = \eta_j^+(C_0) \leqslant M$$

by (1) and similarly, $w_j'(C) \leqslant M$. Consequently, $w_j(X) \leqslant M$, and $w_j'(X) \leqslant M$ for $j = 1, 2$. Then by Proposition 3.4(iv) of [10], $w = (w_1 - w_1') + \mathrm{i}(w_2 - w_2')$ is $\mathscr{B}(X)$-regular and extends $\hat{\mu}_0$ and μ_0. Besides, $\operatorname{Re} w|\mathscr{D}(\mathscr{K}_0) = \eta_1$ and $\operatorname{Im} w|\mathscr{D}(\mathscr{K}_0) = \eta_2$; $\operatorname{Re} w$ and $\operatorname{Im} w$ are $\mathscr{B}(X)$-regular. Let w' and w'' be also $\mathscr{B}(X)$-regular scalar extensions of η_1 and η_2, respectively. Then, as $\operatorname{Im} w'|\mathscr{B}_0(X) = 0$, we have

$$\int_X f \,\mathrm{d}(\operatorname{Im} w')^+ = \int_X f \,\mathrm{d}(\operatorname{Im} w')^-, \quad f \in C_c(X)$$

and hence by Theorem 2.2(vii) of [10], $(\operatorname{Im} w')^+ = (\operatorname{Im} w')^-$ so that w' is real. Similarly, w'' is real. Clearly, $\eta_1^+ + w'^-|\mathscr{D}(\mathscr{K}_0) = \eta_1^- + w'^+|\mathscr{D}(\mathscr{K}_0)$. Since w'^+ and w'^- are the unique Radon-regular extensions of their respective restrictions to $\mathscr{D}(\mathscr{K}_0)$, by Lemma 2.3 we have $w_1 + w'^- = w_1' + w'^+$ and hence $\operatorname{Re} w = w'$. Similarly, $\operatorname{Im} w = w''$ and hence w is unique.

Taking $\mu = w|\mathscr{B}_c(X)$, by Lemma 3.5(ii) of [10] we observe that μ is a $\mathscr{B}_c(X)$-regular extension of μ_0 and besides, μ is the unique extension of $\mu|\mathscr{D}(\mathscr{K})$. As $\mu|\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular by Lemma 3.5(i) of [10], the uniqueness of μ follows from Lemma 2.3(ii). Clearly, from the above proof it follows that μ and w are real (resp., positive) if μ_0 is so.

(iv) Follows from the uniqueness of μ and w and from their definition.

(v) By applying the above extensions of $|\hat{\mu}_0|$ to $\mathscr{B}_c(X)$ and $\mathscr{B}(X)$ we deduce the result from (1) and from Definition 3.2(iii) of [10]. □

The proof of the following theorem is similar to that of Theorem 2.4 and hence omitted.

Theorem 2.5. *Let μ be a $\mathscr{D}(\mathscr{K})$-regular complex measure of bounded variation with $\sup\{v(\mu, \mathscr{D}(\mathscr{K}))(E)\colon E \in \mathscr{D}(\mathscr{K})\} = M$. Then:*

(i) *The unique extension $\hat{\mu}$ of μ to $\mathscr{B}_c(X)$ as a complex measure is $\mathscr{B}_c(X)$-regular and is real (resp., positive) if μ is so.*

(ii) *μ has a unique extension w to $\mathscr{B}(X)$ as a $\mathscr{B}(X)$-regular complex measure and w is real (resp., positive) if μ is so. Besides, $\hat{\mu} = w|\mathscr{B}_c(X)$.*

(iii) $\sup\{v(\hat{\mu}, \mathscr{B}_c(X))(E)\colon E \in \mathscr{B}_c(X)\} = \sup\{v(w, \mathscr{B}(X))(E)\colon E \in \mathscr{B}(X)\} = M$.

Corollary 2.6. *Every complex measure μ_0 on $\mathscr{B}_0(X)$ has a unique extension μ to $\mathscr{B}_c(X)$ (resp., w to $\mathscr{B}(X)$) as a $\mathscr{B}_c(X)$-regular (resp., $\mathscr{B}_c(X)$-regular) complex measure and μ (resp., w) is real if μ_0 is real and μ (resp., w) is positive if μ_0 is positive. Besides,*

$$\sup\{v(\eta, \mathscr{R})(E)\colon E \in \mathscr{R}\} = \sup\{v(\mu_0, \mathscr{B}_0(X))(E)\colon E \in \mathscr{B}_0(X)\} < \infty$$

where $\eta = \mu$ or w and $\mathscr{R} = \mathscr{B}_c(X)$ or $\mathscr{B}(X)$, respectively.

3. Bounded complex Radon measures

In this section we give several characterizations for $\theta \in \mathscr{K}(X)^*$ to be bounded, in the sense that $\sup\{|\theta(f)|\colon f \in \mathscr{K}(X), \|f\|_u \leqslant 1\} < \infty$.

Definition 3.1. A complex Radon measure μ_θ on X is said to be bounded if $\sup\{|\mu_\theta(E)|\colon E \in \mathscr{D}(\mathscr{K})\} < \infty$. We define

$$\|\mu_\theta\| = \sup\{v(\mu_\theta|\mathscr{D}(\mathscr{K}))(E)\colon E \in \mathscr{D}(\mathscr{K})\}$$

for $\theta \in \mathscr{K}(X)^*$.

Lemma 3.2. *Let $\theta \in \mathscr{K}(X)^*$ and $E \in \mathscr{D}(\mathscr{K}_0)$. Then*

$$v(\mu_\theta|\mathscr{D}(\mathscr{K}_0), \mathscr{D}(\mathscr{K}_0))(E) = v(\mu_\theta, M_\theta)(E) = \mu_{|\theta|}(E).$$

In particular,

$$v(\mu_\theta|\mathscr{D}(\mathscr{K}_0),\mathscr{D}(\mathscr{K}_0))(E)=v(\mu_\theta|\mathscr{D}(\mathscr{K}),\mathscr{D}(\mathscr{K}))(E).$$

Proof. Let $\nu=\mu_\theta|\mathscr{D}(\mathscr{K}_0)$ and let $|\nu|=v(\nu,\mathscr{D}(\mathscr{K}_0))$. By Theorem 4.1 of [10] and by Theorem 2.1 there exist a unique extension $|\nu|\hat{}$ of $|\nu|$ to $\mathscr{B}(X)$ as a Radon-regular measure and a positive linear form ψ on $C_c(X)$ such that $|\nu|\hat{}=\check{\mu}_\psi$.

Let $f\in C_c(X)$ with $\operatorname{supp} f\subset K\in\mathscr{K}_0$. Since f is integrable with respect to every Baire measure (vide p. 241 of [5]) and $\mathscr{B}_0(X)\cap K=\mathscr{B}_0(K)$, as in the proof of Theorem 4.7(ix) of [10] we have

$$|\theta(f)|=\Big|\int_K f\,\mathrm{d}(\mu_\theta|\mathscr{B}_0(K))\Big|\leqslant\int_X|f|\,\mathrm{d}\check{\mu}_\psi=\psi(|f|)$$

so that $|\theta|\leqslant\psi$. On the other hand, for $E\in\mathscr{D}(\mathscr{K}_0)$ we have

$$|\nu|(E)\leqslant v(\mu_\theta,M_\theta)(E)=\mu_{|\theta|}(E)$$

by Theorem 4.11 of [10], so that $\check{\mu}_\psi|\mathscr{D}(\mathscr{K}_0)\leqslant\mu_{|\theta|}|\mathscr{D}(\mathscr{K}_0)$. If $w=(\mu_{|\theta|}-\check{\mu}_\psi)|\mathscr{D}(\mathscr{K}_0)$, then by Lemma 2.2 there exists a unique Radon-regular extension $\hat{w}$ of w to $\mathscr{B}(X)$ such that $\hat{w}+\check{\mu}_\psi=\check{\mu}_{|\theta|}$ on $\mathscr{B}(X)$. Thus $\check{\mu}_\psi\leqslant\check{\mu}_{|\theta|}$. Therefore, $\psi=|\theta|$ and then, by Theorem 4.11 of [10], we have

$$|\nu|(E)=\check{\mu}_\psi(E)=\mu_{|\theta|}(E)=v(\mu_\theta,M_\theta)(E),\quad E\in\mathscr{D}(\mathscr{K}_0).$$

□

Theorem 3.3. *Let μ_θ be a complex Radon measure on X. Then the following assertions are equivalent:*

(i) *μ_θ is bounded.*

(ii) *θ is bounded.*

(iii) *$\mathscr{R}\subset M_\theta$, where $\mathscr{R}=\mathscr{B}_0(X)$ or $\mathscr{B}_c(X)$ or $\mathscr{B}(X)$.*

(iv) *M_θ is a σ-algebra in X.*

(v) *$\sup\{|\mu_\theta(E)|\colon E\in\mathscr{R}\}<\infty$, where $\mathscr{R}=\mathscr{B}_0(X)$ or $\mathscr{B}_c(X)$ or $\mathscr{B}(X)$ or M_θ.*

(vi) *$\sup\{v(\mu_\theta|\mathscr{R},\mathscr{R})(E)\colon E\in\mathscr{R}\}<\infty$, where $\mathscr{R}=M_\theta$ or $\mathscr{B}(X)$ or $\mathscr{B}_c(X)$ or $\mathscr{B}_0(X)$ or $\mathscr{D}(\mathscr{K})$ or $\mathscr{D}(\mathscr{K}_0)$.*

(vii) *$\sup\{v(\mu_\theta|\mathscr{D}(\mathscr{K}_0),\mathscr{D}(\mathscr{K}_0))(K)\colon K\in\mathscr{K}_0\}<\infty$.*

(viii) *$\|\mu_\theta\|<\infty$.*

Besides, $\|\theta\|=\|\mu_\theta\|$ for $\theta\in\mathscr{K}(X)^*$. The functional θ is bounded if and only if $M_\theta=M_{\mu^*_{|\theta|}}$ and when θ is bounded, $\|\mu_\theta\|$ is given by the supremum in (vi) *with $\mathscr{R}$*

being anyone of the δ-rings M_θ, $\mathscr{B}(X)$, $\mathscr{B}_c(X)$, $\mathscr{B}_0(X)$, $\mathscr{D}(\mathscr{K})$ *or* $\mathscr{D}(\mathscr{K}_0)$ *and by the supremum in* (vii). *In particular,*

$$\|\mu_\theta\| = v(\mu_\theta|\mathscr{B}(X), \mathscr{B}(X))(X).$$

Proof. (i) ⇒ (ii) Let $f \in C_c(X)$, with $\|f\|_u \leqslant 1$ and let $\operatorname{supp} f = K$. Then

$$\begin{aligned}|\theta(f)| = \Big|\int_K f\,\mathrm{d}(\mu_\theta|\mathscr{B}(K))\Big| &\leqslant \int_K |f|\,\mathrm{d}v(\mu_\theta|\mathscr{B}(K), \mathscr{B}(K))\\ &\leqslant v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))(K)\\ &\leqslant 4\sup\{|\mu_\theta(E)|\colon E \in \mathscr{D}(\mathscr{K})\}\end{aligned}$$

and hence θ is bounded if μ_θ is bounded.

(ii) ⇒ (vii) Let $K \in \mathscr{K}_0$. By Proposition 11, §14 of [3] there exists $U_0 \in \mathscr{U} \cap \mathscr{D}(\mathscr{K}_0)$ such that $K \subset U_0$. Let $f \in C_c^+(X)$ with $\chi_K \leqslant f \leqslant \chi_{U_0}$. Then $\|f\|_u = 1$ and by (ii) we have

$$\begin{aligned}(1)\qquad v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))(K) &\leqslant \int_{\bar{U}_0} |f|\,\mathrm{d}v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))\\ &\leqslant \int_X |f|\,\mathrm{d}\breve{\mu}_{|\theta|}\\ &= |\theta|(f) \leqslant \|\theta\|\end{aligned}$$

since $v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))(E) = \mu_{|\theta|}(E)$ for $E \in \mathscr{D}(\mathscr{K})$ by Theorems 4.7 and 4.11 of [10] and since $\|\theta\| = |\!|\!|\theta|\!|\!|$ by Corollary 1 on p. 58 of [2]. If M_0 is the supremum in (vii), then by (1), $M_0 \leqslant \|\theta\|$.

Since (1) holds also for $K \in \mathscr{K}$, by (iv) and (v) of Theorem 4.7 of [10] we have

$$(2)\qquad \|\mu_\theta\| \leqslant \|\theta\|.$$

Let $w = v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))$ and let $M_{\mathscr{R}} = \sup\{v(\mu_\theta|\mathscr{R}, \mathscr{R})(E)\colon E \in \mathscr{R}\}$, where $\mathscr{R}$ is one of the δ-rings in (vi). By Theorem 4.7 of [10]

$$(3)\qquad w = v(\mu_\theta, M_\theta)|\mathscr{D}(\mathscr{K}).$$

For $E \in \mathscr{D}(\mathscr{K})$, by Proposition 11, §14 of [3] there exists $K \in \mathscr{K}_0$ with $\overline{E} \subset K$ and hence, by Lemma 3.2, we have

$$w(E) \leqslant w(K) = v(\mu_\theta, M_\theta)(K) = v(\mu_\theta|\mathscr{D}(\mathscr{K}_0), \mathscr{D}(\mathscr{K}_0))(K)$$

whence it follows that $M_{\mathscr{D}(\mathscr{K})}$ coincides with the supremum in (vii). By Corollary 2.6, $M_{\mathscr{B}_0(X)} = M_{\mathscr{B}(X)}$ and by Theorem 4.7(vii) of [10], $M_{M_\theta} = M_{\mathscr{B}(X)}$. By (3) and by the fact that $v(\mu_\theta, M_\theta)$ is M_θ-regular by Theorem 4.7 of [10], we have $M_{M_\theta} = M_{\mathscr{D}(\mathscr{K})}$.

The proof of the equivalence of the remaining assertions is easy and hence omitted to the reader.

From the proof of (i) $\Rightarrow$ (ii) and from (2) it follows that $\|\theta\| = \|\mu_\theta\|$ for $\theta \in \mathscr{K}(X)^*$ (even though θ is not bounded).

If θ is bounded, by (iv), $X \in M_\theta$ and hence by Theorem 4.11 of [10], $M_\theta = M_{|\theta|} = M_{\mu^*_{|\theta|}}$. The condition is also sufficient since $M_{\mu^*_{|\theta|}}$ is a σ-algebra in X. □

4. Characterizations of complex Radon measures

Using the properties of complex Radon measures established in [10], we characterize these measures in terms of complex measures on $\mathscr{D}(\mathscr{K}_0)$ and those on $\mathscr{D}(\mathscr{K})$, which are besides $\mathscr{D}(\mathscr{K})$-regular. Also we include another characterization of these measures in terms of $\mathscr{D}$-regular complex measures μ defined on a δ-ring $\mathscr{D}$ containing $\mathscr{D}(\mathscr{K})$ and this result precisely generalizes the result of McShane [9], mentioned in the introduction, to complex Radon measures.

Lemma 4.1. *Let $\theta \in \mathscr{K}(X)^*$, $\theta_1 = \operatorname{Re}\theta$ and $\theta_2 = \operatorname{Im}\theta$. Let $\nu = v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))$. Then:*

(i) ν is $\mathscr{D}(\mathscr{K})$-regular and if $\hat{\nu}$ is the unique Radon-regular extension of ν to $\mathscr{B}(X)$, then $\hat{\nu} = \check{\mu}_{|\theta|}$.

(ii) $\mathscr{B}(X) \cap M_\theta = \{E \in \mathscr{B}(X) \colon \hat{\nu}(E) < \infty\}$.

(iii) $M_\theta = \{E \subset X\colon$ there exist $A, B \in \mathscr{B}(X) \cap M_\theta$ with $A \subset E \subset B$ and $\hat{\nu}(B \setminus A) = 0\}$.

(iv) If $\mu_j = \mu_{\theta_j}|\mathscr{D}(\mathscr{K})$ and $\hat{\mu}_j^+$ and $\hat{\mu}_j^-$ are the unique Radon-regular extensions to $\mathscr{B}(X)$ of μ_j^+ and μ_j^-, respectively, for $j = 1, 2$, then

$$\mu_\theta(E) = (\hat{\mu}_1^+ - \hat{\mu}_1^-)(E) + \mathrm{i}(\hat{\mu}_2^+ - \hat{\mu}_2^-)(E)$$

for $E \in \mathscr{B}(X) \cap M_\theta$.

(v) If $\mathscr{R} = \{E \in \mathscr{B}(X) \colon \hat{\nu}(E) < \infty\}$, then μ_θ is the Lebesgue completion of $\mu_\theta|\mathscr{R}$ with respect to $\mathscr{R}$.

In short, M_θ and μ_θ are uniquely determined by $\mu_\theta|\mathscr{D}(\mathscr{K})$.

Proof. (i) Follows from (v), (viii) and (ix) of Theorem 4.7 and from Theorem 4.11 of [10].

(ii) Let $\mathscr{R}$ be as in (v). Then by (i), $\mathscr{R} = M_{|\theta|} \cap \mathscr{B}(X)$ and therefore, $\mathscr{R} = M_\theta \cap \mathscr{B}(X)$ by Theorem 4.11 of [10].

(iii) By Theorem 4.7 (iv), $\nu = |\mu_\theta||\mathscr{D}(\mathscr{K})$. Since $|\mu_\theta|$ is M_θ-regular by Theorem 4.7(iii) of [10]

$$\begin{aligned}|\mu_\theta|(E) &= \sup\{|\mu_\theta|(K)\colon K \subset E, K \in \mathscr{K}\}\\ &= \sup\{\nu(K)\colon K \subset E, K \in \mathscr{K}\}\\ &= \hat{\nu}(E)\end{aligned}$$

for $E \in \mathscr{B}(X) \cap M_\theta$, since $\hat{\nu}$ is Radon-regular and $\hat{\nu}(E) < \infty$ so that Proposition 3.4(iv) of [10] applies here. Now (iii) is immediate from Theorem 4.7 (vii) of [10].

(iv) By (i) and (v) of Theorem 4.6 of [10], $\hat{\mu}_j^+ = \mu_{\theta_j^+}$ and $\hat{\mu}_j^- = \mu_{\theta_j^-}$ in $\mathscr{B}(X) \cap M_{\theta_j}$ for $j = 1, 2$. Since $M_\theta = M_{\theta_1} \cap M_{\theta_2}$, $\hat{\mu}_j^+$ and $\hat{\mu}_j^-$ are Radon-regular in $\mathscr{B}(X)$ and $\tilde{\mu}_{\theta_j^+}|M_{\theta_j}$ and $\tilde{\mu}_{\theta_j^-}|M_{\theta_j}$ are M_{θ_j}-regular, by Proposition 3.4(iv) of [10] we have

$$\begin{aligned}\hat{\mu}_j^+(E) &= \sup\{\hat{\mu}_j^+(K)\colon K \subset E, K \in \mathscr{K}\}\\ &= \sup\{\tilde{\mu}_{\theta_j^+}(K)\colon K \subset E, K \in \mathscr{K}\}\\ &= \tilde{\mu}_{\theta_j^+}(E)\end{aligned}$$

for $j = 1, 2$. Similarly, $\hat{\mu}_j^-(E) = \tilde{\mu}_{\theta_j^-}(E)$ for $E \in \mathscr{B}(X) \cap M_\theta$ and for $j = 1, 2$.

(v) By (iv) and (vi) of Theorem 4.7 of [10] we have

$$v(\mu_\theta|\mathscr{R}, \mathscr{R})(E) = v(\mu_\theta, M_\theta)(E), \quad E \in \mathscr{R}$$

and consequently, by (i) and by Theorem 4.11 of [10]

$$v(\mu_\theta|\mathscr{R}, \mathscr{R})(E) = \hat{\nu}(E), \quad E \in \mathscr{R}.$$

Now the result is immediate from (iii) and (iv). □

The following lemma is an easy consequence of Lemma 2.2.

Lemma 4.2. *If θ_1 and θ_2 are positive linear functionals on $C_c(X)$, then $\mu_{(\theta_1-\theta_2)} = \mu_{\theta_1} - \mu_{\theta_2}$ on $\mathscr{D}(\mathscr{K})$.*

Lemma 4.3. *Let μ be a real measure on $\mathscr{D}(\mathscr{K})$ and let μ be $\mathscr{D}(\mathscr{K})$-regular. Let $|\mu| = v(\mu, \mathscr{D}(\mathscr{K}))$. If $|\mu|^\wedge$ is the unique extension of $|\mu|$ to $\mathscr{B}(X)$ as a Radon-regular measure, let $\mathscr{R} = \{E \in \mathscr{B}(X)\colon |\mu|^\wedge(E) < \infty\}$. Then:*

(i) *There exists a unique $\theta \in \mathscr{K}(X)^*$, θ real, such that $\mu_\theta|\mathscr{D}(\mathscr{K}) = \mu$. Besides,*

$\mu^+ = \mu_{\theta^+}|\mathscr{D}(\mathscr{K})$ and $\mu^- = \mu_{\theta^-}|\mathscr{D}(\mathscr{K})$.

(ii) $|\mu| = \mu_{|\theta|}|\mathscr{D}(\mathscr{K})$.

(iii) $|\mu|^{\wedge} = \check{\mu}_{|\theta|}$.

(iv) $\mathscr{R} = \mathscr{B}(X) \cap M_\theta$.

(v) *M_θ is the Lebesgue completion of $\mathscr{R}$ with respect to $|\mu|^{\wedge}|\mathscr{R}$.*

(vi) *If ν_1 and ν_2 are the unique Radon-regular extensions to $\mathscr{B}(X)$ of μ^+ and μ^-, respectively, then*

$$\mu_\theta(E) = \nu_1(E) - \nu_2(E), \quad E \in \mathscr{R}.$$

Consequently, given $F \in M_\theta$, $\mu_\theta(F) = \mu_\theta(A) = (\nu_1 - \nu_2)(A)$ where $A \subset F \subset B$, $A, B \in \mathscr{R}$ with $|\mu|^{\wedge}(B \setminus A) = 0$.

Proof. Since μ is $\mathscr{D}(\mathscr{K})$-regular, by the inequality mentioned in the proof of Theorem 4.7(iii) of [10] obviously $|\mu|$ and hence μ^+ and μ^- are $\mathscr{D}(\mathscr{K})$-regular. Thus by Theorem 3.9(ii) of [10] such extensions $|\mu|^{\wedge}$, ν_1 and ν_2 exist uniquely on $\mathscr{B}(X)$.

(i) Since ν_1 and ν_2 are Radon-regular, by Theorem 4.1 of [10] we have positive linear functionals θ_i on $C_c(X)$ such that $\nu_i = \check{\mu}_{\theta_i}$, $i = 1, 2$. Let $\theta = \theta_1 - \theta_2$. Then, by Lemma 4.2, $\mu_\theta|\mathscr{D}(\mathscr{K}) = \mu$. Consequently, by Theorems 4.5(v) and 4.6(v) of [10] we have $\mu^+ = \mu_{\theta^+}|\mathscr{D}(\mathscr{K})$ and $\mu^- = \mu_{\theta^-}|\mathscr{D}(\mathscr{K})$.

To prove the uniqueness of θ, if possible, let $w \in \mathscr{K}(X)^*$ such that $\mu = \mu_w|\mathscr{D}(\mathscr{K})$. Let $w_1 = \operatorname{Re} w$ and $w_2 = \operatorname{Im} w$. Since $\mu_{w_2}|\mathscr{D}(\mathscr{K}) = 0$, $\mu_{w_2^+}|\mathscr{D}(\mathscr{K}) = \mu_{w_2^-}|\mathscr{D}(\mathscr{K})$. Then by the uniqueness part of Theorem 3.9(ii) of [10], we have $\check{\mu}_{w_2^+} = \check{\mu}_{w_2^-}$ so that $w_2^+(f) = \int_X f \,\mathrm{d}\check{\mu}_{w_2^+} = \int_X f \,\mathrm{d}\check{\mu}_{w_2^-} = w_2^-(f)$ for $f \in C_c(X)$. Thus $w_2 = 0$. Hence w is real and

$$\mu = \mu_w|\mathscr{D}(\mathscr{K}) = (\mu_{w^+} - \mu_{w^-})|\mathscr{D}(\mathscr{K}) = (\mu_{\theta^+} - \mu_{\theta^-})|\mathscr{D}(\mathscr{K}).$$

Therefore,

$$(\mu_{w^+} + \mu_{\theta^-})|\mathscr{D}(\mathscr{K}) = (\mu_{\theta^+} + \mu_{w^-})|\mathscr{D}(\mathscr{K})$$

and consequently, by Proposition 15, §1, Chapter IV of [2] we have

$$\check{\mu}_{w^+ + \theta^-}|\mathscr{D}(\mathscr{K}) = \check{\mu}_{\theta^+ + w^-}|\mathscr{D}(\mathscr{K}).$$

Thus, by the uniqueness part of Theorem 3.9(ii) of [10] we have that $\check{\mu}_{w^+ + \theta^-} = \check{\mu}_{\theta^+ + w^-}$ so that $(w^+ + \theta^-)(f) = (\theta^+ + w^-)(f)$ for $f \in C_c(X)$. Hence $\theta = w$ and thus θ is unique.

Since $|\mu| = v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))$ by (i), (ii) and (iii) hold by Lemma 4.1(i), whereas (iv) follows from Lemma 4.1(ii). Similarly, (v) is immediate from Lemma 4.1(iii).

Finally, (vi) follows from (iv) and (v) of Lemma 4.1. □

As a consequence of the above lemmas we shall give the following theorem, which, among other things, characterizes a complex Radon measure in terms of its restriction to $\mathscr{D}(\mathscr{K})$.

Theorem 4.4. (i) *A complex measure μ on $\mathscr{D}(\mathscr{K})$ is the restriction of a complex Radon measure μ_θ if and only if μ is $\mathscr{D}(\mathscr{K})$-regular. In such case, θ is unique and is called the functional determined by μ.*

(ii) *Let μ be a $\mathscr{D}(\mathscr{K})$-regular complex measure on $\mathscr{D}(\mathscr{K})$ and let $|\mu| = v(\mu, \mathscr{D}(\mathscr{K}))$. Let $|\mu|\hat{}$ be the unique extension of $|\mu|$ to $\mathscr{B}(X)$ as a Radon-regular measure and let $\mathscr{R} = \{E \in \mathscr{B}(X)\colon |\mu|\hat{}(E) < \infty\}$. Let $\mu_1 = \operatorname{Re}\mu$, $\mu_2 = \operatorname{Im}\mu$, and $\hat{\mu}_i^+$ and $\hat{\mu}_i^-$ be the Radon-regular extensions to $\mathscr{B}(X)$ of μ_i^+ and μ_i^-, respectively. Besides, by (i) let θ be the functional determined by μ. Then:*

(a) $\mathscr{R} = \mathscr{B}(X) \cap M_\theta$.

(b) *M_θ is the Lebesgue completion of $\mathscr{R}$ with respect to $|\mu|\hat{}\,|\mathscr{R}$.*

(c) *Given $E \in M_\theta$, there exist $A, B \in \mathscr{R}$ with $A \subset E \subset B$ and $|\mu|\hat{}(B \setminus A) = 0$. Besides, $\mu_\theta(E) = \{(\hat{\mu}_1^+ - \hat{\mu}_1^-) + \mathrm{i}(\hat{\mu}_2^+ - \hat{\mu}_2^-)\}(A)$.*

(d) *$|\mu| = \mu_{|\theta|}|\mathscr{D}(\mathscr{K})$, so that $|\mu|$ determines $|\theta|$.*

(e) *θ is real if and only if μ is real; θ is positive if and only if μ is positive. When μ is real, μ^+ and μ^- determine θ^+ and θ^-, respectively.*

Proof. (i) By Theorem 4.7(v) of [10] the condition in necessary. Conversely, let μ be $\mathscr{D}(\mathscr{K})$-regular, with $\mu_1 = \operatorname{Re}\mu$ and $\mu_2 = \operatorname{Im}\mu$. Since $\mathscr{D}(\mathscr{K})$ is a δ-ring, $\mu_j = \mu_j^+ - \mu_j^-$ and μ_j^+ and μ_j^- are $\mathscr{D}(\mathscr{K})$-regular for $j = 1, 2$. Consequently, by Lemma 4.3 there exist $\theta_j \in \mathscr{K}(X)^*$, θ_j real, such that $\mu_j = \mu_{\theta_j}|\mathscr{D}(\mathscr{K})$ for $j = 1, 2$. Let $\theta = \theta_1 + \mathrm{i}\theta_2$. Then $\theta \in \mathscr{K}(X)^*$ and $\mu = \mu_\theta|\mathscr{D}(\mathscr{K})$. Clearly, θ is unique by the uniqueness part of Lemma 4.3(i).

(ii) (a) As $\mu = \mu_\theta|\mathscr{D}(\mathscr{K})$, the result holds by Lemma 4.1(ii).

(b) This is the same as Lemma 4.1(iii).

(c) Follows from (iv) and (v) of Lemma 4.1.

(d) By Theorems 4.7(iv) and 4.11 of [10], $|\mu| = v(\mu_\theta, M_\theta)|\mathscr{D}(\mathscr{K}) = \mu_{|\theta|}|\mathscr{D}(\mathscr{K})$ and hence (d) holds.

(e) This is immediate from Lemma 4.3(i). □

The following result generalizes Theorem 54.2 of [9] to complex Radon measures. The hypothesis that $\mathscr{B}(X) \cap \mathscr{D} = \{E \in \mathscr{B}(\mathbb{R}^n)\colon |\nu|\hat{}(E) < \infty\}$ is the same as $\mathscr{B}(X) \cap \mathscr{D} = \{E \in \mathscr{B}(\mathbb{R}^n)\colon |\nu|^*(E) < \infty\}$ in the case of $\mathbb{R}^n$ and $\mathscr{D}$-regularity of μ is redundant in this case.

Theorem 4.5. *Let $\mathscr{D}$ be a δ-ring containing $\mathscr{D}(\mathscr{K})$ and let μ be a $\mathscr{D}$-regular complex measure on $\mathscr{D}$. Let $\nu = \mu|\mathscr{D}(\mathscr{K})$ and $|\nu| = v(\nu, \mathscr{D}(\mathscr{K}))$. Suppose $\mathscr{R} =$*

$\mathscr{B}(X)\cap\mathscr{D}=\{E\in\mathscr{B}(X)\colon |\nu|^{\hat{}}(E)<\infty\}$, where $|\nu|^{\hat{}}$ is the unique Radon-regular extension of $|\nu|$ to $\mathscr{B}(X)$. If μ and $\mathscr{D}$ are the Lebesgue completions of $\mu|\mathscr{R}$ and $\mathscr{R}$, respectively, with respect to $\mathscr{R}$ and $\mu|\mathscr{R}$, then there exists a unique $\theta\in\mathscr{K}(X)^*$ such that $\mu=\mu_\theta$ and $\mathscr{D}=M_\theta$. Besides, θ is real (respectively, θ is positive) if μ is real (respectively, μ is positive).

Proof. Let $|\mu|=v(\mu,\mathscr{D})$. Then, by the inequality mentioned in the proof of Theorem 4.7(iii) of [10], $|\mu|$ is $\mathscr{D}$-regular and as $\mathscr{D}(\mathscr{K})\subset\mathscr{D}$, the argument given in the proof of Theorem 4.5(vi) of [10] holds here verbatim to show that $|\mu||\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular, if we replace there $|\mu_\theta|$ by $|\mu|$ and M_θ by $\mathscr{D}$. Consequently, $\nu=\mu|\mathscr{D}(\mathscr{K})$ is also $\mathscr{D}(\mathscr{K})$-regular.

Again, since μ is $\mathscr{D}$-regular, by an argument similar to that given in the proof of Theorem 4.5(vii) (c) of [10] one can show that $v(\mu,\mathscr{D}(\mathscr{K}),\mathscr{D})(E)=|\mu|(E)$ for $E\in\mathscr{D}$. Consequently,

$$|\nu|=v(\mu|\mathscr{D}(\mathscr{K}),\mathscr{D}(\mathscr{K}))=|\mu||\mathscr{D}(\mathscr{K}). \tag{1}$$

Since $\mathscr{B}(X)\cap\mathscr{D}=\{E\colon |\nu|^{\hat{}}(E)<\infty\}$, $|\nu|^{\hat{}}$ is Radon-regular and $|\mu|$ is $\mathscr{D}$-regular, by (1) and by Proposition 3.4(iv) of [10]

$$\begin{aligned}|\nu|^{\hat{}}(E)&=\sup\{|\nu|(K)\colon K\subset E,K\in\mathscr{K}\}\\&=\sup\{|\mu|(K)\colon K\subset E,K\in\mathscr{K}\}\\&=|\mu|(E)\end{aligned}$$

for $E\in\mathscr{B}(X)\cap\mathscr{D}$. That is, $|\nu|^{\hat{}}|\mathscr{R}=|\mu||\mathscr{R}$.

Since ν is $\mathscr{D}(\mathscr{K})$-regular, by (i) of Theorem 4.4 there exists a unique $\theta\in\mathscr{K}(X)^*$ such that $\nu=\mu_\theta|\mathscr{D}(\mathscr{K})$ and by (ii) (e) of the same theorem, the functional θ is real (resp., positive) if ν is real (resp., ν is positive). Besides, by (ii) (d) of the said theorem, $|\nu|=\mu_{|\theta|}|\mathscr{D}(\mathscr{K})$. Thus by hypothesis, by (a) and (b) of Theorem 4.4.(ii) and by the fact that $|\nu|^{\hat{}}|\mathscr{R}=|\mu||\mathscr{R}$, we conclude that $\mathscr{D}=M_\theta$.

Since $\operatorname{Re}\mu$ and $\operatorname{Im}\mu$ are $\mathscr{D}$-regular, by following an argument similar to that in the proof of (v) of Theorem 4.5 of [10] we note that $(\operatorname{Re}\mu)^+|\mathscr{D}(\mathscr{K})=(\operatorname{Re}\nu)^+$, $(\operatorname{Re}\mu)^-|\mathscr{D}(\mathscr{K})=(\operatorname{Re}\nu)^-$, $(\operatorname{Im}\mu)^+|\mathscr{D}(\mathscr{K})=(\operatorname{Im}\nu)^+$ and $(\operatorname{Im}\mu)^-|\mathscr{D}(\mathscr{K})=(\operatorname{Im}\nu)^-$ and $(\operatorname{Re}\nu)^+$, $(\operatorname{Re}\nu)^-$, $(\operatorname{Im}\nu)^+$ and $(\operatorname{Im}\nu)^-$ are $\mathscr{D}(\mathscr{K})$-regular. Thus, if $\nu_1=\operatorname{Re}\nu$ and $\nu_2=\operatorname{Im}\nu$ and $\hat{\nu}_j^+$ and $\hat{\nu}_j^-$ are the unique Radon-regular extensions to $\mathscr{B}(X)$ of ν_j^+ and ν_j^-, respectively, for $j=1,2$, then by Proposition 3.4(iv) of [10] we have

$$\begin{aligned}\hat{\nu}_j^+(E)&=\sup\{\nu_j^+(K)\colon K\subset E,K\in\mathscr{K}\}\\&=\sup\{\mu_j^+(K)\colon K\subset E,K\in\mathscr{K}\}\\&=\mu_j^+(E)\end{aligned}$$

for $E \in \mathscr{R}$ and for $j = 1, 2$, where $\mu_1 = \operatorname{Re}\mu$ and $\mu_2 = \operatorname{Im}\mu$. Similarly, $\hat{\nu}_j^-(E) = \mu_j^-(E)$ for $E \in \mathscr{R}$ and for $j = 1, 2$.

Consequently, by Theorem 4.4(ii) (c), given $E \in \mathscr{D} = M_\theta$ there exist $A, B \in \mathscr{R}$ with $A \subset E \subset B$, $|\mu|(B \setminus A) = 0$ and

$$\begin{aligned} \mu_\theta(E) &= (\hat{\nu}_1^+ - \hat{\nu}_1^-)(A) + \mathrm{i}(\hat{\nu}_2^+ - \hat{\nu}_2^-)(A) \\ &= (\mu_1^+ - \mu_1^-)(A) + \mathrm{i}(\mu_2^+ - \mu_2^-)(A) \\ &= \mu(A) \\ &= \mu(E) \end{aligned}$$

since by hypothesis μ is the Lebesgue completion of $\mu|\mathscr{R}$ with respect to $\mathscr{R}$. Hence $M_\theta = \mathscr{D}$ and $\mu_\theta = \mu$. □

In the next theorem we consider the restriction of μ_θ in $\mathscr{D}(\mathscr{K}_0)$ and study its properties.

Theorem 4.6. (i) *A complex measure ν on $\mathscr{D}(\mathscr{K}_0)$ is the restriction of a complex Radon measure μ_θ and such θ is unique. We say that θ is determined by ν if $\nu = \mu_\theta|\mathscr{D}(\mathscr{K}_0)$; θ is real (resp., positive) if ν is real (resp., positive).*

(ii) *If ν is a complex measure on $\mathscr{D}(\mathscr{K}_0)$ and μ is the unique extension of ν to $\mathscr{D}(\mathscr{K})$ as a $\mathscr{D}(\mathscr{K})$-regular complex measure (vide Theorem 2.4(i)), then μ and ν determine the same functional $\theta \in \mathscr{K}(X)^*$. That is, $\nu = \mu_\theta|\mathscr{D}(\mathscr{K}_0)$ and $\mu = \mu_\theta|\mathscr{D}(\mathscr{K})$.*

In the following, let ν, μ, θ be as in (ii). *Let $\nu_1 = \operatorname{Re}\nu$, $\nu_2 = \operatorname{Im}\nu$, $\mu_1 = \operatorname{Re}\mu$, $\mu_2 = \operatorname{Im}\mu$.*

(iii) *The unique $\mathscr{D}(\mathscr{K})$-regular extensions $\hat{\nu}_j^+$ and $\hat{\nu}_j^-$ of ν_j^+ and ν_j^-, respectively, to $\mathscr{D}(\mathscr{K})$ (vide Theorem 2.1), are given by $\hat{\nu}_j^+ = \mu_j^+$ and $\hat{\nu}_j^- = \mu_j^-$, $j = 1, 2$.*

(iv) *If ν is real and $|\nu| = v(\nu, \mathscr{D}(\mathscr{K}_0))$, then $\nu^+ = \mu^+|\mathscr{D}(\mathscr{K}_0)$, $\nu^- = \mu^-|\mathscr{D}(\mathscr{K}_0)$ and $|\nu| = |\mu||\mathscr{D}(\mathscr{K}_0)$, where $|\mu| = v(\mu, \mathscr{D}(\mathscr{K}))$. Consequently, $\nu^+ = \mu_{\theta^+}|\mathscr{D}(\mathscr{K}_0)$, $\nu^- = \mu_{\theta^-}|\mathscr{D}(\mathscr{K}_0)$ and $|\nu| = \mu_{|\theta|}|\mathscr{D}(\mathscr{K}_0)$. Thus ν^+, ν^- and $|\nu|$ determine θ^+, θ^- and $|\theta|$, respectively.*

(v) *If ν is complex, then $|\nu| = |\mu||\mathscr{D}(\mathscr{K}_0) = \mu_{|\theta|}|\mathscr{D}(\mathscr{K}_0)$ so that $|\nu|$ determines $|\theta|$.*

Proof. By Theorem 2.4(i) there exists a unique $\mathscr{D}(\mathscr{K})$-regular complex measure μ on $\mathscr{D}(\mathscr{K})$ such that $\mu|\mathscr{D}(\mathscr{K}) = \nu$ and such μ is real (resp., positive) if ν is real (resp., ν is positive). Then by Theorem 4.4(i) there exists a unique $\theta \in \mathscr{K}(X)^*$ such that $\mu = \mu_\theta|\mathscr{D}(\mathscr{K})$ and hence, $\nu = \mu_\theta|\mathscr{D}(\mathscr{K}_0)$. This functional is real (resp., positive) if ν (and hence μ) is real (resp., ν is positive) by Theorem 4.4(ii) (e). Since ν determines μ uniquely, it follows that ν determines θ uniquely. Thus we have proved (i) and (ii).

(iii) By Lemma 2.3(i) $\nu_j^+ = \mu_j^+|\mathscr{D}(\mathscr{K}_0)$ and $\nu_j^- = \mu_j^-|\mathscr{D}(\mathscr{K}_0)$ and as μ_j^+ and μ_j^- are $\mathscr{D}(\mathscr{K})$-regular, by the uniqueness part of Theorem 2.1 we conclude that $\hat{\nu}_j^+ = \mu_j^+$ and $\hat{\nu}_j^- = \mu_j^-$ for $j = 1, 2$.

(iv) Since μ is real when ν is real, by Lemma 2.3(i) $\nu^+ = \mu^+|\mathscr{D}(\mathscr{K}_0)$ and $\nu^- = \mu^-|\mathscr{D}(\mathscr{K}_0)$. Consequently, by Theorem 4.4(ii) (e) the result holds.

(v) This is immediate from Lemma 3.2. □

5. Isomorphic representations of $\mathscr{K}(X)^*$, $\mathscr{K}(X,\mathbb{R})^*$, $\mathscr{K}(X)_b^*$ and $\mathscr{K}(X,\mathbb{R})_b^*$

Making use of the results of the earlier section we show that $\mathscr{K}(X)^*$ is isomorphic to the space of all complex measures on $\mathscr{D}(\mathscr{K}_0)$ and to the space of all $\mathscr{D}(\mathscr{K})$-regular complex measures on $\mathscr{D}(\mathscr{K})$. The same isomorphism, when restricted to $\mathscr{K}(X,\mathbb{R})^*$, is order preserving and maps $\mathscr{K}(X,\mathbb{R})^*$ onto the real vector space of all real measures on $\mathscr{D}(\mathscr{K}_0)$ and the space of all $\mathscr{D}(\mathscr{K})$-regular real measures on $\mathscr{D}(\mathscr{K})$. Also we show that $\mathscr{K}(X)_b^*$ (resp., $\mathscr{K}(X,\mathbb{R})_b^*$) is isometrically isomorphic to the Banach space of all bounded complex (resp., real) measures on $\mathscr{D}(\mathscr{K}_0)$ and to the Banach space of all bounded $\mathscr{D}(\mathscr{K})$-regular complex (resp., real) measures on $\mathscr{D}(\mathscr{K})$. Finally, the vector space of all $\mathbb{C}$-valued additive set functions of finite (resp., of bounded) variation on a ring of sets is shown to be isomorphic (resp., isometrically isomorphic) to $\mathscr{K}(X)^*$ (resp., to $\mathscr{K}(X)_b^*$) for a suitably chosen totally disconnected locally compact, Hausdorff space X.

Before stating the relevant theorems we fix the notation for various spaces of real and complex measures.

N o t a t i o n 6.1. $\mathscr{M}_0(X)$ (resp., $\mathscr{M}_c(X)$) denotes the vector space of all complex (resp., $\mathscr{D}(\mathscr{K})$-regular complex) measures on $\mathscr{D}(\mathscr{K}_0)$ (resp., on $\mathscr{D}(\mathscr{K})$), with operations of addition and scalar multiplication being defined setwise. Let $\hat{\mathscr{M}}(X)$ (resp., $\hat{\mathscr{M}}_c(X)$) be the vector space of all complex measures on $\mathscr{B}(X)$ (resp., on $\mathscr{B}_c(X)$), which are $\mathscr{B}(X)$-regular (resp., $\mathscr{B}_c(X)$-regular) and let $\hat{\mathscr{M}}_0(X)$ be that of all complex measures on $\mathscr{B}_0(X)$. Let $\mathscr{M}_0(X)_b$ (resp., $\mathscr{M}_c(X)_b$) be the vector space of all bounded complex measures on $\mathscr{D}(\mathscr{K}_0)$ (resp., $\mathscr{D}(\mathscr{K})$-regular complex measures on $\mathscr{D}(\mathscr{K})$). The spaces $\mathscr{M}_0^r(X)$, $\mathscr{M}_c^r(X)$, $\hat{\mathscr{M}}(X,\mathbb{R})$, $\hat{\mathscr{M}}_c(X,\mathbb{R})$, $\hat{\mathscr{M}}_0(X,\mathbb{R})$, $\mathscr{M}_0^{(r)}(X)_b$ and $\mathscr{M}_c^{(r)}(X)_b$ are the spaces of corresponding real measures in $\mathscr{M}_0(X)$, $\mathscr{M}_c(X)$, etc., respectively.

Theorem 5.2. *Let $T: \mathscr{M}_c(X) \to \mathscr{K}(X)^*$ be given by $T\mu = \theta$ if $\mu = \mu_\theta|\mathscr{D}(\mathscr{K})$ and $T_0: \mathscr{M}_0(X) \to \mathscr{K}(X)^*$ given by $T_0\nu = \theta$ if $\nu = \mu_\theta|\mathscr{D}(\mathscr{K}_0)$. Then:*

(i) T and T_0 are well defined and are linear isomorphisms onto $\mathscr{K}(X)^*$.

(ii) *If* $T\mu = \theta$ *(*$T_0\nu = \theta$*, resp.) then* $T(v(\mu, \mathscr{D}(\mathscr{K}))) = |\theta|$ *(*$T_0(v(\nu, \mathscr{D}(\mathscr{K}_0))) = |\theta|$*, resp.).*

(iii) *Let* $T^{(r)} = T|\mathscr{M}_c^{(r)}(X)$ *and* $T_0^{(r)} = T_0|\mathscr{M}_0^{(r)}(X)$*. Then* $T^{(r)}$ *(resp.,* $T_0^{(r)}$*) is an order preserving linear isomorphism of* $\mathscr{M}_c^{(r)}(X)$ *(resp., of* $\mathscr{M}_0^{(r)}(X)$*) onto* $\mathscr{K}(X,\mathbb{R})^*$*, where* $\mu_1 \leqslant \mu_2$ *if* $\mu_1(E) \leqslant \mu_2(E)$ *for all* $E \in \mathscr{D}(\mathscr{K})$ *(resp.,* $E \in \mathscr{D}(\mathscr{K}_0)$*),* $\mu_1, \mu_2 \in \mathscr{M}_c^{(r)}(X)$ *(resp.,* $\mu_1, \mu_2 \in \mathscr{M}_0^{(r)}(X)$*). In particular, if* $T^{(r)}\mu_i = \theta_i$*,* $i = 1,2$*, then* $T^{(r)}(\mu_1 \vee \mu_2) = \theta_1 \vee \theta_2$*, where*

$$\mu_{\theta_1\vee\theta_2}(E) = \sup_{\substack{F\subset E\\ F\in\mathscr{D}(\mathscr{K})}} \{\mu_{\theta_1}(F) + \mu_{\theta_2}(E\setminus F)\}$$

and $T^{(r)}(\mu_1 \wedge \mu_2) = \theta_1 \wedge \theta_2$*, where*

$$\mu_{\theta_1\wedge\theta_2}(E) = \inf_{\substack{F\subset E\\ F\in\mathscr{D}(\mathscr{K})}} \{\mu_{\theta_1}(F) + \mu_{\theta_2}(E\setminus F)\}$$

for $E \in \mathscr{D}(\mathscr{K})$*. A similar result holds if* μ_1 *and* μ_2 *belong to* $\mathscr{M}_0^{(r)}(X)$*.*

(iv) $\mathscr{M}_c(X)$ *(resp.,* $\mathscr{M}_0(X)$*) is the dual of* $\mathscr{K}(X)$ *and*

$$\theta(f) = \int_K f\,\mathrm{d}(\mu_\theta|\mathscr{B}(K)) \tag{1}$$

for $f \in C_c(X)$ *with* $\operatorname{supp} f = K$*, where* $T\mu = \theta$ *(resp.,* $T_0\mu = \theta$*).*

(v) $\mathscr{M}_c^{(r)}(X)$ *(resp.,* $\mathscr{M}_0^{(r)}(X)$*) is the dual of* $\mathscr{K}(X,\mathbb{R})$ *and an expression similar to* (1) *holds if* $T^{(r)}\mu = \theta$ *(resp.,* $T_0^{(r)}\mu = \theta$*).*

Proof. (i) By Theorems 4.4(i) and 4.6(i), T and T_0 are well defined. If $T\mu_1 = T\mu_2 = \theta$, then $\mu_1 = \mu_\theta|\mathscr{D}(\mathscr{K}) = \mu_2$ and hence T and similarly, T_0 are injective. Making use of Proposition 15, §1, Chapter IV of [2], it can be shown that T and T_0 are linear. The details are left to the reader. T is an onto mapping by Theorem 4.7(v) of [10], while T_0 is evidently an onto mapping.

(ii) Follows from Theorem 4.4(ii) (d) for T and from Theorem 4.6(v) for T_0.

(iii) $T^{(r)}$ is order preserving by Theorem 4.4(ii) (e) and $T_0^{(r)}$ is order preserving by Theorem 4.6(i). The rest of (iii) is an immediate consequence of the order preserving property of these isomorphisms $T^{(r)}$ and $T_0^{(r)}$.

(iv) As in the proof of Theorem 4.6(iv) of [10] we have

$$\begin{aligned}\theta(f) &= \int_K f\,\mathrm{d}(\mu_{\theta_1}|\mathscr{B}(K)) + \mathrm{i}\int_K f\,\mathrm{d}(\mu_{\theta_2}|\mathscr{B}(K))\\ &= \int_K f\,\mathrm{d}(\mu_\theta|\mathscr{B}(K))\end{aligned}$$

for $f \in C_c(X)$ with $\operatorname{supp} f = K$. Since T (resp., T_0) is an isomorphism, the result follows.

(v) The proof is similar to that of (iv). □

Theorem 5.3. *Let $\mathscr{S} = \mathscr{B}_c(X)$ (resp., $\mathscr{B}(X)$, $\mathscr{B}_0(X)$, $\mathscr{D}(\mathscr{K})$, $\mathscr{D}(\mathscr{K}_0)$) and let $\mathscr{M}(\mathscr{S}) = \hat{\mathscr{M}}_c(X)$ (resp., $\hat{\mathscr{M}}(X)$, $\hat{\mathscr{M}}_0(X)$, $\mathscr{M}_c(X)_b$, $\mathscr{M}_0(X)_b$). For $\mu \in \mathscr{M}(\mathscr{S})$, let $\|\mu\| = \sup\{v(\mu,\mathscr{S})(E) \colon E \in \mathscr{S}\}$. Then:*

(i) The map $\Phi_{\mathscr{S}} \colon \mathscr{M}(\mathscr{S}) \to \mathscr{K}(X)_b^*$ given by $\Phi_{\mathscr{S}}\mu = \theta$ if $\mu = \mu_\theta|\mathscr{S}$ is well defined, and is an isomorphism onto $\mathscr{K}(X)_b^*$ and $\|\Phi_{\mathscr{S}}\mu\| = \|\mu\|$ for $\mu \in \mathscr{M}(\mathscr{S})$ so that $\Phi_{\mathscr{S}}$ is an isometric isomorphism.

(ii) Each one of the spaces $(\mathscr{M}(\mathscr{S}), \|.\|)$ is the dual of $(C_c(X), \|.\|_u)$ and consequently, $(\mathscr{M}(\mathscr{S}), \|.\|)$ are Banach spaces.

(iii) Results similar to (i) and (ii) hold if $\mathscr{K}(X)_b^*$ and $\mathscr{M}(\mathscr{S})$ are replaced by $\mathscr{K}(X,\mathbb{R})_b^*$ and $\mathscr{M}^r(\mathscr{S})$, respectively, where $\mathscr{M}^r(\mathscr{S}) = \{\mu \in \mathscr{M}(\mathscr{S}) \colon \mu \text{ real}\}$.

Proof. (i) Let $\mu_1, \mu_2 \in \mathscr{M}(\mathscr{S})$ and $\alpha, \beta \in \mathbb{C}$. Clearly, $\alpha\mu_1 + \beta\mu_2 \in \mathscr{M}(\mathscr{S})$ and $(\alpha\mu_1 + \beta\mu_2)|\mathscr{D}(\mathscr{K}_0) = \alpha \cdot \mu_1(\mathscr{D}(\mathscr{K}_0)) + \beta \cdot \mu_2|\mathscr{D}(\mathscr{K}_0)$. Thus, by the uniqueness part of the various assertions in Theorem 2.4 we conclude that $\mathscr{M}(\mathscr{D}(\mathscr{K}_0))$ is the image under a linear onto isomorphism $\Gamma_{\mathscr{S}} \colon \mathscr{M}(\mathscr{S}) \to \mathscr{M}(\mathscr{D}(\mathscr{K}_0))$ given by

$$\Gamma_{\mathscr{S}}(\mu) = \mu|\mathscr{D}(\mathscr{K}_0), \quad \mu \in \mathscr{M}(\mathscr{S}).$$

Let $\Phi_{\mathscr{S}}(\mu) = (T_0 \circ \Gamma_{\mathscr{S}})(\mu)$ for $\mu \in \mathscr{M}(\mathscr{S})$, where T_0 is as in Theorem 5.2. Clearly, $\Phi_{\mathscr{S}}$ is a linear isomorphism of $\mathscr{M}(\mathscr{S})$ onto its image in $\mathscr{K}(X)^*$. If $\Phi_{\mathscr{S}}(\mu) = \theta$, then $T_0(\mu|\mathscr{D}(\mathscr{K}_0)) = \theta$ and by hypothesis,

$$\sup\{|\mu(E)| \colon E \in \mathscr{D}(\mathscr{K}_0)\} < \infty.$$

Consequently, by the equivalence of (ii) and (vi) of Theorem 3.3 we have θ bounded and hence $\Phi_{\mathscr{S}}(\mathscr{M}(\mathscr{S})) \subset \mathscr{K}(X)_b^*$. Conversely, if $\theta \in \mathscr{K}(X)_b^*$ then by Theorem 3.3, μ_θ is bounded in M_θ and $M_\theta \supset \mathscr{B}(X)$. Consequently, $\mu_\theta|\mathscr{S}$ belongs to $\mathscr{M}(\mathscr{S})$ and $\Phi_{\mathscr{S}}(\mu_\theta) = \theta$, so that $\Phi(\mathscr{M}(\mathscr{S})) = \mathscr{K}(X)_b^*$.

(ii) This is immediate from (i) and from the last part of Theorem 3.3.

(iii) The proof is similar to the earlier parts. □

The following theorem can be compared with Theorem 7 of [7] and Theorem 14 of [8].

Theorem 5.4. *Let Ω be a non-void set and let $\mathscr{R}$ be a ring of subsets of Ω. Let $\mathscr{M}$ (resp., $\mathscr{M}_b$) be the vector space of all complex valued finitely additive set functions of finite (resp., of bounded) variation on $\mathscr{R}$ and let $\|\mu\| = \sup\{v(\mu,\mathscr{R})(E)\colon$*

$E \in \mathscr{R}\}$ *for* $\mu \in \mathscr{M}_b$. *Let* $\mathscr{M}^{(r)}$ *(resp.,* $\mathscr{M}_b^{(r)}$*) be the space of corresponding real valued set functions in* $\mathscr{M}$ *(resp., in* $\mathscr{M}_b$*). Then there exists a totally disconnected locally compact Hausdorff space* X *such that* $\mathscr{M}$ *is isomorphic to* $\mathscr{K}(X)^*$; $\mathscr{M}^{(r)}$ *is order isomorphic to* $\mathscr{K}(X,\mathbb{R})^*$ *and* $\mathscr{M}_b$ *(resp.,* $\mathscr{M}_b^{(r)}$*) is isometrically isomorphic (resp., isometrically order isomorphic) to* $\mathscr{K}(X)_b^*$ *(resp., to* $\mathscr{K}(X,\mathbb{R})_b^*$*). When* $\mathscr{R}$ *is an algebra, the space* X *can further be assumed to be compact.*

Proof. By Stone's representation theorem for Boolean rings (vide Theorem 1, §18 of [3]), there exists a totally disconnected locally compact Hausdorff space X such that $\mathscr{R}$ is ring-isomorphic to the ring $\mathscr{C}$ of all compact-open subsets of X. Let Φ be such an isomorphism from $\mathscr{R}$ onto $\mathscr{C}$.

Let $K \in \mathscr{K}_0$ of X. Then by Proposition 1, §14 of [3] there exists $U_n \in \mathscr{U} \cap \mathscr{D}(\mathscr{K}_0)$ with $K = \bigcap_1^\infty U_n$. Since the members of $\mathscr{C}$ form a base for the topology of X, each U_n is of the form $U_n = \bigcup_j \Phi(A_{nj})$, $A_{nj} \in \mathscr{R}$. As K is compact, there exist A_{nj_i}, $i = 1, 2, \ldots, k$ in $\mathscr{R}$ such that $K \subset \bigcup_1^k \Phi(A_{nj_i})$. If $A_n = \bigcup_{i=1}^k A_{nj_i}$, then $K = \bigcap_1^\infty \Phi(A_n)$ so that $K \in \mathscr{D}(\mathscr{C})$. Since $\mathscr{C} \subset \mathscr{K}_0$, it follows that $\mathscr{D}(\mathscr{K}_0) = \mathscr{D}(\mathscr{C})$.

For $\mu \in \mathscr{M}$, let $\psi(\mu)(E) = \mu(\Phi^{-1}(E))$ for $E \in \mathscr{C}$. Since $\Phi^{-1}(\emptyset) = \emptyset$ and since each countable disjoint union $\{E_n\}_1^\infty$ in $\mathscr{C}$ with $\bigcup_1^\infty E_n = E \in \mathscr{C}$ has $E_n = \emptyset$ for all but a finite number of n, it follows that $\nu = \psi(\mu)$ is a complex measure on $\mathscr{C}$. Besides, as μ is of finite variation, ν is also of finite variation on $\mathscr{C}$ and hence admits a unique extension $\hat{\nu}$ to $\mathscr{D}(\mathscr{C}) = \mathscr{D}(\mathscr{K}_0)$ as a complex measure. Conversely, given a complex measure ν on $\mathscr{D}(\mathscr{K}_0)$, let $\mu(\Phi^{-1}(E)) = \nu(E)$ for $E \in \mathscr{C}$. Clearly, μ is well defined on $\mathscr{R}$ and is a complex valued finitely additive set function. Since $\nu|\mathscr{C}$ is of finite variation, $\mu \in \mathscr{M}$. Besides, $\psi(\mu) = \nu|\mathscr{C}$. Also the mapping $\mu \to \{\psi(\mu)\}\hat{}$ is linear and bijective so that $\mathscr{M}$ is isomorphic to $\mathscr{M}_0(X)$. Consequently, by Theorem 5.2(iv) $\mathscr{M}$ is isomorphic to $\mathscr{K}(X)^*$. The other results follow on similar lines. □

Since every non-void open set in the space X of Theorem 5.4 contains a compact-open subset whose characteristic function belongs to $C_c^+(X)$ and since as complex valued additive set function μ on the ring $\mathscr{R}$ is a complex measure if and only if $\lim_n \mu(E_n) = 0$ whenever $E_n \downarrow \emptyset$, $E_n \in \mathscr{R}$, the following corollary is immediate from the above theorem.

Corollary 5.5. *Let* F *(resp.,* G*) be the isomorphism from* $\mathscr{M}$ *onto* $\mathscr{M}_0(X)$ *(resp., onto* $\mathscr{K}(X)^*$*) in Theorem 5.4. Let* $\mathscr{M}_0^s(X) = \{\nu \in \mathscr{M}_0(X)\colon \nu(K) = 0$ *for* $K \in \mathscr{K}_0$ *with* $\operatorname{int} K = \emptyset\}$ *and* $\mathscr{M}_b^s(X) = \mathscr{M}_b(X) \cap \mathscr{M}_0^s$; $\mathscr{K}(X)_s^* = \{\theta \in \mathscr{K}(X)^*\colon \lim_n \theta(f_n) = 0$ *whenever* $f_n \downarrow \chi_K$, $f_n \in C_c^+(X)$, $K \in \mathscr{K}_0$ *and* $\bigwedge_1^\infty f_n = 0$ *in* $C_c^+(X)\}$ *and* $\mathscr{K}(X)_{bs}^* =$

$\mathscr{K}(X)_b^* \cap \mathscr{K}(X)_s^*$. *Let* $\mathscr{M}_{ca} = \{\mu \in \mathscr{M} : \mu$ *countably additive*$\}$ *and* $\mathscr{M}_{bca} = \mathscr{M}_{ca} \cap \mathscr{M}_b$. *Then* $F|\mathscr{M}_{ca}$ *and* $G|\mathscr{M}_{ca}$ *(resp.,* $F|\mathscr{M}_{bca}$ *and* $G|\mathscr{M}_{bca}$*) are isomorphic onto* $\mathscr{M}_0^s(X)$ *and* $\mathscr{K}(X)_s^*$ *(resp., isometrically isomorphic onto* $\mathscr{M}_b^s(X)$ *and* $\mathscr{K}(X)_{bs}^*$*) respectively. The restrictions of these isomorphisms on the corresponding subspaces of real measures are further order preserving.*

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