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# ON UNBOUNDED NONOSCILLATORY SOLUTIONS OF SYSTEMS OF NEUTRAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Valter Šeda on the occasion of his sixtieth birthday

## 1. INTRODUCTION

In this paper we consider systems of neutral differential equations of the form

$$(1_r)\qquad \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[x_1(t)+(-1)^r a_i(t)x_i\bigl(h_i(t)\bigr)\right]=\sum_{j=1}^{N} p_{ij}(t)f_{ij}\bigl(x_j\bigl(g_{ij}(t)\bigr)\bigr),$$
$$i=1,\dots,N,\ N\geqslant 2,\ n\geqslant 2,\ r\in\{0,1\}.$$

subject to the hypotheses

$$(2)\qquad \begin{aligned}&a_i\colon[t_0,\infty)\to[0,\beta_i],\ t_0\geqslant 0,\ 0<\beta_i<1,\\ &h_i,p_{ij},g_{ij}\colon[t_0,\infty)\to R \text{ and } f_{ij}\colon R\to R,\ 1\leqslant i,j\leqslant N\end{aligned}$$

are continuous functions;

$$(3)\qquad \begin{aligned}&h_i(t)<t \text{ for } t\geqslant t_0,\ \lim_{t\to\infty}h_i(t)=\infty,\ \lim_{t\to\infty}g_{ij}(t)=\infty,\\ &1\leqslant i,j\leqslant N;\end{aligned}$$

$$(4)\qquad f_{ij}(u)u>0 \text{ for } u\neq 0 \text{ and } f_{ij} \text{ are nondecreasing functions, } 1\leqslant i; j\leqslant N;$$

$$(5)\qquad \lim_{t\to\infty}a_i(t)\left(\frac{h_i(t)}{t}\right)^k=\bar a_{ik}a_{ik}\in[0,\beta_i],\ \text{for } 1\leqslant i\leqslant N$$

and every $k\in\{1,\dots,n-1\}$.

Let $t_1 \geqslant t_0$. Denote

$$t_2 = \min\left\{ \inf_{t \geqslant t_1} h_i(t),\ \inf_{t \geqslant t_1} g_{ij}(t),\ 1 \leqslant i,\ j \leqslant N \right\}.$$

A function $X = (x_1, \dots, x_N)$ is a solution of the system $(1_r)$, if there exists a $t_1 \geqslant t_0$ such that $X$ is continuous on $[t_2, \infty)$, $x_i(t) + (-1)^r a_i(t) x_i(h_i(t))$, $1 \leqslant i \leqslant N$ are $n$-times continuously differentiable on $[t_1, \infty)$ and $X$ satisfies $(1_r)$ on $[t_1, \infty)$.

A solution $X = (x_1, \dots, x_N)$ of $(1_r)$ is nonoscillatory if there exists an $a \geqslant t_0$ such that its every component is different from zero for all $t \geqslant a$.

Our aim in this paper is to extend some of the results obtained in [1–4] to the system $(1_r)$. We give conditions for the system $(1_r)$ to possess nonoscillatory solutions $X = (x_1, \dots, x_N)$ with the asymptotic behavior

$$\lim_{t \to \infty} \frac{x_i(t)}{t^{k_i}} = c_i \neq 0,\ \operatorname{sgn} c_i = \operatorname{sgn} c_1$$

or

$$\lim_{t \to \infty} \frac{x_i(t)}{t^{k_i}} = 0,\ \lim_{t \to \infty} \frac{x_i(t)}{t^{k_i - 1}} = \infty$$

for some $k_i \in \{1, \dots, n-1\}$, $1 \leqslant i \leqslant N$.

Denote

$$\gamma(t_0) = \max\{\sup\{s \geqslant t_0 \colon h_i(s) \leqslant t_0, g_{ij}(s) \leqslant t_0 \text{ for } 1 \leqslant i,\ j \leqslant N\}\},$$

(6) $$H_i(o, t) \equiv t,\ H_i(k, t) = H_i(k-1, h_i(t)),\ 1 \leqslant i \leqslant N,\ k = 1, 2, \dots,$$

(7) $$A_i(0, t) \equiv 1,\ A_i(k, t) = \prod_{j=0}^{k-1} a_i(H_i(j, t)),\ 1 \leqslant i \leqslant N,\ k = 1, 2, \dots,$$

(8) $$\begin{aligned} &(p_{ij})^+_{k_i}(t) = \max\{(-1)^{n-k_i} p_{ij}(t), 0\} \text{ and} \\ &(p_{ij})^-_{k_i}(t) = \max\{-(-1)^{n-k_i} p_{ij}(t), 0\},\ t \geqslant t_0, \\ &1 \leqslant i,\ j \leqslant N, k_i \in \{1, \dots, n-1\}. \end{aligned}$$

Note that

(9) $$\begin{aligned} &(-1)^{n-k_i} p_{ij}(t) = (p_{ij})^+_{k_i}(t) - (p_{ij})^-_{k_i}(t), \\ &|p_{ij}(t)| = (p_{ij})^+_{k_i}(t) + (p_{ij})^-_{k_i}(t),\ 1 \leqslant i,\ j \leqslant N. \end{aligned}$$

## 2. Main results

**Theorem 1.** *Let the assumptions (2)–(5) hold and let $k_i \in \{1,2,\dots,n-1\}$, $1 \leqslant i \leqslant N$. If*

$$\int_{\gamma(t_0)}^{\infty} t^{n-k_i-1}\sum_{j=1}^{N}|p_{ij}(t)|f_{ij}\left(b_j\left(g_{ij}(t)\right)^{k_j}\right)\,dt < \infty \tag{10}$$

*for some constants $b_j > 0$, $1 \leqslant i \leqslant N$, then for any $(\bar{c}_1,\dots,\bar{c}_N)$, $(\bar{c}_i > 0,\ 1 \leqslant i \leqslant N)$ there exists a positive solution $X = (x_1,\dots,x_N)$ of the system $(1_r)$ such that*

$$\lim_{t\to\infty}\frac{x_i(t)}{t^{k_i}} = \bar{c}_i > 0,\ 1 \leqslant i \leqslant N. \tag{11}$$

Proof. Let $c_i > 0$, $1 \leqslant i \leqslant N$ be some arbitrary, but fixed constants and let $k_i \in \{1,\dots,n-1\}$, $1 \leqslant i \leqslant N$. We put $b_i = c_i + d_i$, where

$$0 < d_i < c_i\,\frac{1-\beta_i}{1+\beta_i}. \tag{12}$$

Let $T \geqslant t_0$ be such that

$$T_0 = \min\left\{\inf_{t\geqslant T} h_i(t),\ \inf_{t\geqslant T} g_{ij}(t);\ l \leqslant i,j \leqslant N\right\} \geqslant t_0 \tag{13}$$

and

$$\int_{T}^{\infty} t^{n-k_i-1}\sum_{j=1}^{N}|p_{ij}(t)|f_{ij}\left(b_j\left(g_{ij}(t)\right)^{k_j}\right)\,dt < d_j,\ 1 \leqslant i \leqslant N$$

We denote by $C[T_0,\infty)$ the locally convex space of all continuous vector functions $X = (x_i,\dots,x_n)$ defined on $[T_0,\infty)$ which are constant on $[T_0,T]$, with the topology of uniform convergence on any compact subinterval of $[T_0,\infty)$. Thus $C[T_0,\infty)$ is a Frechet space.

(I) Let $r = 0$. We consider the closed convex subset $S_0$ of $C[T_0,\infty)$ defined by

$$S_0 = \left\{Y = (y_1,\dots,y_N) \in C[T_0,\infty);\, y_i(t) = c_i\,\frac{T^{k_i}}{k_i!} \text{ for } t \in [T_0,T],\ \frac{1}{k_i!}(c_i-d_i)t^{k_i} \leqslant y_i(t) \leqslant \frac{1}{k_i!}b_i t^{k_i} \text{ for } t \geqslant T,\ 1 \leqslant i \leqslant N\right\}. \tag{15}$$

For each $Y \in S_0$ we define functions $x_i (1 \leqslant i \leqslant N)$ by

$$x_i(t) = \begin{cases} \dfrac{y_i(T)}{1+a_i(T)},\ t \in [T_0,T], \\ \displaystyle\sum_{k=0}^{n_i(t)-1}(-1)^k A_i(k,t)y_i\left(H_i(k,t)\right) \\ \qquad +(-1)^{n_i(t)}A_i\left(n_i(t),t\right)\dfrac{y_i(T)}{1+a_i(T)},\ t \leqslant T, \end{cases} \tag{16}$$

where $n_i(t)$, $1 \leqslant i \leqslant N$ are the least positive integers such that $T_0 < H_i(n_i(t), t) \leqslant T$. The functions in (16) are adaptions of the function introduced in [1,5].

We easily verify that $x_i(t) \in C[T_0, \infty)$, $1 \leqslant i \leqslant N$, and they satisfy the functional equations

$$x_i(t) + a_i(t)x_i(h_i(t)) = y_i(t), \ t \geqslant T, \ 1 \leqslant i \leqslant N. \tag{17}$$

Let $n_i(t) = 2m_i + 1$ or $n_i(t) = 2m_i + 2$, $m = 0, 1, \ldots$, $1 \leqslant i \leqslant N$. Then (16) together with $Y \in S_0$, (2) and (3) implies

$$\begin{aligned} x_i(t) \geqslant \frac{1}{k_i!}\Big((c_i - d_i)t^{k_i} - a_i(t)b_i(h_i(t))^{k_i} \\ + A_i(2,t)\big[(c_i - d_i)(H_i(2,t)) - a_i(H_i(2,t))b_i(H_i(3,t))^{k_i}\big] + \ldots \\ + A_i(2m_i,t)\big[(c_i - d_i)(H_i(2m_i,t))^{k_i} - a_i(H_i(2m_i,t))b_i(h_i(2m_i+1,t))^{k_i}\big]\Big) \\ \geqslant \frac{1}{k_i!}\big[(c_i - d_i) - \beta_i b_i\big]\big[t^{k_i} + A_i(2,t)(H_i(2,t))^{k_i} + \ldots + A_i(2m_i,t)(H_i(2m_i,t))^{k_i}\big] \\ \geqslant \frac{1}{k_i!}\big[c_i(1-\beta_i) - d_i(1+\beta_i)\big]t^{k_i} > 0, \quad t \geqslant T, \ 1 \leqslant i \leqslant N. \end{aligned}$$

Taking into account $Y \in S_0$ and the last inequality we obtain from (17)

$$0 < \frac{1}{k_i!}\big[c_i(1-\beta_i) - d_i(1+\beta_i)\big]t^{k_i} \leqslant x_i(t) \leqslant y_i(t) \leqslant \frac{1}{k_i!}\, b_i t^{k_i}. \tag{18}$$

We define an operator $F = (F_1, \ldots, F_N)\colon S_0 \to C[T_0, \infty)$ by

$$F_iY(t) = \begin{cases} \dfrac{c_i T^{k_i}}{k_i!}, \ t \in [T_0, T], \\ \dfrac{c_i t^{k_i}}{k_i!} + (-1)^{n-k_i} \displaystyle\int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\ \quad \times \displaystyle\sum_{j=1}^N p_{ij}(u) f_{ij}(x_j(g_{ij}(u)))\, du\, ds, \ t \geqslant T, \ 1 \leqslant i \leqslant N. \end{cases} \tag{19}$$

We shall show that the operator $F$ is continuous and maps $S_0$ into a compact subset of $S_0$.

(i) We prove that $F(S_0) \subset S_0$. From (19) in view of (9), (4), (15), (12) and (14) we conclude that

$$(20)\quad F_iY(t) \leqslant \frac{c_i t^{k_i}}{k_i!} + \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_T^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^N (p_{ij})^+_{k_i}(u) f_{ij}\big(b_j\big(g_{ij}(u)\big)^{k_j}\big)\,du\,ds \leqslant$$

$$\leqslant \frac{c_i t^{k_i}}{k_i!} + d_i \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!}\,ds \leqslant \frac{1}{k_i!} b_i t^{k_i},\ t \geqslant T,\ 1 \leqslant i \leqslant N,$$

$$F_iY(t) \geqslant \frac{c_i t^{k_i}}{k_i!} - \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_T^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^N (p_{ij})^-_{k_i}(u) f_{ij}\big(b_j\big(g_{ij}(u)\big)^{k_j}\big)\,du\,ds \geqslant$$

$$\geqslant \frac{c_i t^{k_i}}{k_i!} + d_i \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!}\,ds \geqslant \frac{(c_i - d_i)}{k_i!} t^{k_i},\ t \geqslant T,\ 1 \leqslant i \leqslant N.$$

(ii) We prove that the operator $F$ is continuous. Let $Y_m = (y_{1m}, \dots, y_{Nm}) \in S_0$, $m = 1, 2, \dots$ and $y_{im} \to y_i$ for $m \to \infty$, $1 \leqslant i \leqslant N$ in the space $C[T_0, \infty)$.

Denote

$$x_{im}(t) = \sum_{k=0}^{n_i(t)-1} (-1)^k A_i(k,t) y_{im}\big(H_1(k,t)\big) + (-1)^{n_i(t)} A_i\big(n_i(t), t\big) \times \frac{y_{im}(T)}{1 + a_i(T)},\ t \geqslant T,\ 1 \leqslant i \leqslant N,\ m = 1, 2, \dots.$$

Using (19) we obtain

$$(21)\quad \big|F_iY_m(t) - F_iY(t)\big| \leqslant \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_T^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^N |p_{ij}(u)| \big|f_{ij}\big(x_{jm}(g_{ij}(u))\big) - f_{ij}\big(x_j(g_{ij}(u))\big)\big|\,du\,ds \leqslant$$

$$\leqslant \frac{(t-T)^{k_i}}{k_i!} \int_T^\infty G_i^m(u)\,du, \qquad 1 \leqslant i \leqslant N,$$

where

$$G_i^m(t) = t^{n-k_i-1}\sum_{j=1}^{N}|p_{ij}(u)|\left|f_{ij}\left(x_{jm}(t)\right) - f_{ij}\left(x_j\left(g_{ij}(t)\right)\right)\right|, \ t \geqslant T, 1 \leqslant i \leqslant N.$$

We easily see that

$$\lim_{m\to\infty} G_i^m(t) = 0 \text{ for } t \geqslant T \text{ and } g_i^m(t) \leqslant M_i(t), \text{ where}$$

$$M_i(t) = 2t^{n-k_i-1}\sum_{j=1}^{N}|p_{ij}(u)|f_{ij}\left(b_j\left(g_{ij}(t)\right)^{k_j}\right), \ t \geqslant T, 1 \leqslant i \leqslant N.$$

By virtue of (10) and the Lebesgue dominated convergence theorem we conclude that $F_iY_m(t) \to F_iY(t)$ in $C[T_0,\infty)$ for $m \to \infty$, $1 \leqslant i \leqslant N$. This implies the continuity of $F$.

(iii) $F(S_0)$ is relatively compact. This follows from the Arzela-Ascoli theorem and the observation that for $(y_1,\dots,y_N) \in S_0$, $\left((F_1Y(t))',\dots,(F_NY(t))'\right)$ is given by

$$\begin{aligned}\left|(F_iY(t))'\right| &\leqslant \frac{c_it^{k_i-1}}{(k_i-1)!}\\ &+ \frac{t^{k_i-1}}{(k_i-1)!}\int_T^{\infty}\frac{u^{n-k_i-1}}{(n-k_i-1)!}\sum_{j=1}^{N}|p_{ij}(s)|f_{ij}\left(b_j\left(g_{ij}(u)\right)^{k_j}\right)\mathrm{d}u\\ &\leqslant \frac{b_it^{k_i-1}}{(k_i-1)!}, \ t \geqslant T, \ K \in \{1,\dots,n-1\}\ 1 \leqslant i \leqslant N.\end{aligned}$$

Then by the Schauder-Tychonov fixed point theorem there exists a $\bar{Y} = (\bar{y}_1,\dots,\bar{y}_N) \in S_0$ such that $(F_1\bar{Y},\dots,F_N\bar{Y}) = (\bar{y}_1,\dots,\bar{y}_N)$. The components of $\bar{X} = (\bar{x}_1,\dots,\bar{x}_N)$ safisfy the system

$$\begin{aligned}(22)\qquad \bar{y}_i(t) = \frac{c_it^{k_i}}{k_i!} &+ (-1)^{n-k_i}\int_T^t\frac{(t-s)^{k_i-1}}{(k_i-1)!}\int_s^{\infty}\frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!}\\ &\times\sum_{j=1}^{N}p_{ij}(u)f_{ij}\left(\bar{x}_j\left(g_{ij}(u)\right)\right)\mathrm{d}u\,\mathrm{d}s, \quad t \geqslant T, \ 1 \leqslant i \leqslant N.\end{aligned}$$

where $\bar{y}_i(t) = \bar{x}_i(t) + a_i(t)\bar{x}_i\left(h_i(t)\right)$, $1 \leqslant i \leqslant N$, and $(\bar{x}_i,\dots,\bar{x}_N)$ is a solution of $(1_0)$ on $[T_0,\infty)$.

Differentiating (22) $k_i$-times we obtain

$$\begin{aligned}\bar{y}_i^{(k_i)}(t) = c_i + (-1)^{n-k_i}\int_t^{\infty}\frac{(u-t)^{n-k_i-1}}{(n-k_i-1)!}\sum_{j=1}^{N}P_{ij}(u)f_{ij}\left(\bar{x}_j\left(g_{ij}(u)\right)\right)\mathrm{d}u,\\ t \geqslant T, \ 1 \leqslant i \leqslant N.\end{aligned}$$

which implies that $\lim_{t\to\infty} \bar{y}_i^{(k_i)}(t) = c_i > 0$, $1 \leqslant i \leqslant N$. The last relation is equivalent to

$$\lim_{t\to\infty} \frac{\bar{y}_i(t)}{t^{k_i}} = c_i\ (>0),\ 1 \leqslant i \leqslant N. \tag{23}$$

Then (23) together with (5) implies (11), where $\bar{c}_i = c_i/(1+\bar{a}_{ik_i})$, $1 \leqslant i \leqslant N$.

(II) Let $r = 1$. We consider the closed convex subset $S_1$ of $C[T_0,\infty)$ defined by

$$S_1 = \Big\{Z + (z_1,\dots,z_N) \in C[T_0,\infty);\ z_i(t) = \frac{c_i T^{k_i}}{k_i!} \text{ for } t \in [T_0,T],$$

$$\frac{1}{k_i!}(c_i - d_i)t^{k_i} \leqslant z_i(t) \leqslant \frac{1}{k_i!} b_i t^{k_i} \text{ for } t \geqslant T, 1 \leqslant i \leqslant N\Big\}. \tag{24}$$

For each $Z \in S_1$ we define

$$x_i(t) = \begin{cases} \dfrac{z_i(T)}{1-a_i(T)},\ t \in [T_0,T], \\ \displaystyle\sum_{k=0}^{n_i(t)-1} A_i(k,t) z_i\big(H_i(k,t)\big) + A_i\big(n_i(t),t\big)\frac{z_i(T)}{1-a_i(T)},\ t \geqslant T, \end{cases} \tag{25}$$

where $n_i(t)$, $1 \leqslant i \leqslant N$ are the same as in the case (I).

We easily verify that $x_i(t) \in C[T_0,\infty)$, $1 \leqslant i \leqslant N$ and they satisfy the functional equations

$$x_i(t) - a_i(t)x_i\big(h_i(t)\big) = z_i(t),\ t \geqslant T, 1 \leqslant i \leqslant N. \tag{26}$$

From (25) taking into account (26), $Z \in S_1$, the assumptions (2) and (3) we obtain

$$\frac{1}{k_i!}(c_i - d_i)t^{k_i} \leqslant z_i(t) \leqslant x_i(t) \leqslant \frac{1}{k_i!} b_i \big[t^{k_i} + \beta_i\big(h_i(t)\big)^{k_i} + \dots + \beta_i^{n_i(t)}\big(H_i(n_i(t),t)\big)^{k_i}\big] \leqslant \frac{b_i t^{k_i}}{k_i!(1-\beta_i)}. \tag{27}$$

We define an operator $\bar{F} = (\bar{F}_1,\dots,\bar{F}_N)\colon S_1 \to C[T_0,\infty)$ by (19) in which we replace $y_i(t)$ by $z_i(t)$, $1 \leqslant i \leqslant N$.

Proceeding similarly as in the case (I) we prove that the operator $F$ is continuous and maps $S_1$ into a compact subset of $S_1$. Then by the Schauder-Tychonov theorem there exists a fixed point $\bar{Z} = (\bar{z}_1,\dots,\bar{z}_N) \in S_1$ such that the components of $(\bar{x}_1,\dots,\bar{x}_N)$ are solutions of the system $(1_1)$ on $[T_0,\infty)$ with the property

$$\lim_{t\to\infty} \frac{\bar{z}_i(t)}{t^{k_i}} = c_i > 0,\ 1 \leqslant i \leqslant N \tag{28}$$

where $\bar{z}_i(t) = \bar{x}_i(t) - a_i(t)\bar{x}_i\big(h_i(t)\big)$, $1 \leqslant i \leqslant N$. Then (28) together (27) and (5) implies

$$\lim_{t\to\infty} \frac{\bar{x}(t)}{t^{k_i}} = \frac{c_i}{1-\bar{a}_{ik_i}} = \bar{c}_i > 0,\ 1 \leqslant i \leqslant N.$$

The proof of Theorem 1 is complete. □

**Theorem 2.** *Let the assumptions (2)–(4) hold and $k_i \in \{1, 2, \dots, n-1\}$, $1 \leqslant i \leqslant N$. If*

$$(29) \qquad \int\limits_{\gamma(t_0)}^{\infty} t^{n-k_i-1} \sum_{j=1}^{N} (p_{ij})^{+}_{k_i}(t) f_{ij}\left(a_{ij}\left(g_{ij}(t)\right)^{k_j}\right) \mathrm{d}t < \infty$$

*for some constants $a_{ij} > 0$, $1 \leqslant i, j \leqslant N$,*

$$(30) \qquad \int\limits_{\gamma(t_0)}^{\infty} t^{n-k_i} \sum_{j=1}^{N} (p_{ij})^{-}_{k_i}(t) f_{ij}\left(b_{ij}\left(g_{ij}(t)\right)^{k_j}\right) \mathrm{d}t < \infty$$

*for some constants $b_{ij} > 0$, $1 \leqslant i, j \leqslant N$,*

$$(31) \qquad \int\limits_{\gamma(t_0)}^{\infty} t^{n-k_i} (p_{ih})^{+}_{k_i}(t) f_{ih}\left(c_{ih}\left(g_{ih}(t)\right)^{k_h-1}\right) \mathrm{d}t = \infty$$

*for some $h \in \{1, \dots, N\}$ and all constants $c_{ih} > 0$, $1 \leqslant i, j \leqslant N$,*

*then there exists a positive solution of the system $(1_r)$ with the property*

$$(32) \qquad \lim_{t\to\infty} \frac{x_i(t)}{t^{k_i}} = 0, \quad \lim_{t\to\infty} \frac{x_i(t)}{t^{k_i-1}} = \infty.$$

P r o o f. Let $a_{ij}$, $b_{ij}$, $1 \leqslant i, j \leqslant N$ be positive constants. Then we choose $\delta_i$ such that $0 < 2\delta_i = \min\{a_{ij}, b_{ij};\ 1 \leqslant j \leqslant N\}$, $1 \leqslant i \leqslant N$. We put $2\delta_i/(1-\beta_i) = \bar{\delta}_i$.

Let $T \geqslant \max\{\gamma(t_0)\ 1\}$ be such that (13) holds and

$$(33) \qquad \int\limits_{T}^{\infty} t^{n-k_i-1} \sum_{j=1}^{N} (\quad {}_j)^{+}_{k_i}(t) f_{ij}\left(\bar{\delta}_j\left(g_{ij}(t)\right)^{k_j}\right) \mathrm{d}t < \delta_i (n-k_i-1)!,$$

$$(34) \qquad \int\limits_{T}^{\infty} t^{n-k_i} \sum_{j=1}^{N} (p_{ij})^{-}_{k_i}(t) f_{ij}\left(\bar{\delta}_j\left(g_{ij}(t)\right)^{k_j}\right) \mathrm{d}t < \frac{\delta_i}{2(n-k_i)!}, \ 1 \leqslant i \leqslant N.$$

Let $C[T_0, \infty)$ be the space defined in the proof of Theorem 1. We consider the closed convex subset $S$ of $C[T_0, \infty)$ defined by

$$(35) \qquad S = \Big\{ Z = (z_1, \dots, z_N) \in C[T_0, \infty);\ z_i(t) = \delta_i \frac{T^{k_i-1}}{(k_i-1)!} \text{ for } t \in [T_0, T],\ \frac{\delta_i t^{k_i-1}}{2(k_i-1)!} \leqslant z_i(t) \leqslant \frac{2\delta_i t^{k_i}}{(k_i-1)!} \text{ for } t \geqslant T,\ 1 \leqslant i \leqslant N \Big\}.$$

With every $(z_1, \dots, z_N) \in S$ we associate the functions $(x_1, \dots, x_N)$ defined by the formula (25). From (25) in view of (26), (35), (2) and (3) we obtain

$$(36) \qquad \begin{aligned} z_i(t) \leqslant x_i(t) &\leqslant \frac{2\delta_i}{(k_i-1)!}\Big[t^{k_i} + \beta_i\left(h_i(t)\right)^{k_i} + \dots \\ &\quad + \beta_i^{n_i(t)-1}\left(H_i(n_i(t)-1, t)\right)^{k_i} + \beta_i^{n_i(t)} T^{k_i}\Big] \\ &\leqslant \frac{\bar{\delta}_i t^{k_i}}{(k_i-1)!}, \ t \geqslant T, \ 1 \leqslant i \leqslant N. \end{aligned}$$

Define an operator $F = (F_1, \dots, F_N)\colon S \to C[T_0, \infty)$ by

$$F_i Z(t) = \begin{cases} \dfrac{\delta_i T^{k_i-1}}{(k_i-1)!}, \; t \in [T_0, T], \\ \dfrac{\delta_i t^{k_i-1}}{(k_i-1)!} + (-1)^{n_{k_i}} \displaystyle\int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_T^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\ \qquad \times \displaystyle\sum_{j=1}^N p_{ij}(u) f_{ij}\big(x_j\big(g_{ij}(u)\big)\big)\, du\, ds, \; t \geqslant T, \; 1 \leqslant i \leqslant N. \end{cases} \tag{37}$$

Clearly, $S$ is a closed convex subset of $C[T_0, \infty)$. We show that $F(S) \subset S$. Let $Z = (z_1, \dots, z_N) \in S$. From (37) in view of (9), (4) and (36) we get for $t \geqslant T$, $1 \leqslant i \leqslant N$:

$$\begin{aligned} F_i Z(t) &\leqslant \frac{\delta_i t^{k_i-1}}{(k_i-1)!} + \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_T^\infty \frac{(u)^{n-k_i-1}}{(n-k_i-1)!} \\ &\quad \times \sum_{j=1}^N (p_{ij})^+_{k_i}(u) f_{ij}\big(\bar{\delta}_j \big(g_{ij}(u)\big)^{k_j}\big)\, du\, ds \\ &\leqslant \frac{\delta_i t^{k_i-1}}{(k_i-1)!} + \frac{\delta_i (t-T)^{k_i}}{k_i!} \leqslant \frac{2\delta_i t^{k_i}}{(k_i-1)!}, \; t \geqslant T, \; 1 \leqslant i \leqslant N \end{aligned} \tag{38}$$

Using (4), (35) and (34) we easily derive that

$$\begin{aligned} &\int_T^t \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \sum_{j=1}^N (p_{ij})^-_{k_i}(u) f_{ij}\big(x_j\big(g_{ij}(u)\big)\big)\, du\, ds \leqslant \\ &\int_T^\infty \frac{(u-T)^{n-k_i}}{(n-k_i)!} \sum_{j=1}^N (p_{ij})^-_{k_i}(u) f_{ij}\big(\bar{\delta}_j \big(g_{ij}(u)\big)^{k_j}\big)\, du \leqslant \frac{\delta_i}{2}, \; 1 \leqslant i \leqslant N. \end{aligned} \tag{39}$$

From (37) with regard to (9) and (39) we obtain for $k_i \geqslant 2$, $1 \leqslant i \leqslant N$:

$$
\begin{aligned}
F_i Z(t) &\geqslant \frac{\delta_i t^{k_i-1}}{(k_i-1)!} \\
&\quad - \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \sum_{j=1}^N (p_{ij})_{k_i}^-(u) f_{ij}\big(x_j(g_{ij}(u))\big)\,\mathrm{d}u\,\mathrm{d}s \\
&\geqslant \frac{\delta_i t^{k_i-1}}{(k_i-1)!} \\
&\quad - \int_T^t \frac{(t-\sigma)^{k_i-2}}{(k_i-2)!} \int_T^\sigma \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \sum_{j=1}^N (p_{ij})_{k_i}^-(u) f_{ij}\big(x_j(g_{ij}(u))\big)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\sigma \\
&\geqslant \frac{\delta_i t^{k_i-1}}{(k_i-1)!} - \frac{\sigma_i}{2} \int_T^t \frac{(t-\sigma)^{k_i-2}}{(k_i-2)!}\mathrm{d}\sigma \\
&\geqslant \frac{\delta_i t^{k_i-1}}{2(k_i-1)!}, \qquad t \geqslant T,\ 1 \leqslant i \leqslant N.
\end{aligned}
$$

If $k_i = 1$, then form (37) in virtue of (39) we have

$$
\begin{aligned}
F_i Z(t) \geqslant \delta_i - \int_T^t \int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!} \sum_{j=1}^N (p_{ij})_1^-(u) \\
\times f_{ij}\big(x_j(g_{ij}(u))\big)\,du\,ds \geqslant \frac{1}{2}\delta_i,\ t \geqslant T,\ 1 \leqslant i \leqslant N.
\end{aligned}
$$

We have proved that $F(S) \subset S$.

Proceeding similarly as in the proof of Theorem 1 we obtain that the operator $F$ is continuous and $F(S)$ has a compact closure. Therefore by the Schauder-Tychonov fixed point theorem there exists a $\bar{Z} = (\bar{z}_1, \dots, \bar{z}_N) \in S$ such that $(F_1\bar{Z}, \dots, F_N\bar{Z}) = (\bar{z}_1, \dots, \bar{z}_N)$ and the components of $(\bar{x}_1, \dots, \bar{x}_M)$ satisfy for $t \geqslant T$ the system

$$
\begin{aligned}
\bar{z}_i(t) = \frac{\delta_i t^{k_i-1}}{(k_i-1)!} + (-1)^{n-k_i} \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\
\times \sum_{j=1}^N p_{ij}(u) f_{ij}\big(\bar{x}_j(g_{ij}(u))\big)\,\mathrm{d}u\,\mathrm{d}s,\ 1 \leqslant i \leqslant N,
\end{aligned}
\tag{40}
$$

where $\bar{z}_i(t) = \bar{x}_i(t) - a_i(t)\bar{x}_i\big(h_i(t)\big)$.

Differentiating (40) $(k_i - 1)$-times and $k_i$-times, we get

$$\bar{z}_i^{(k_i-1)}(t) = \delta_i + (-1)^{n_{k_i}} \int_T^t \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^N p_{ij}(u) f_{ij}\big(\bar{x}_j(g_{ij}(u))\big)\, du\, ds,\ t \geqslant T,\ 1 \leqslant i \leqslant N, \tag{41}$$

$$\bar{z}_i^{(k_i)}(t) = (-1)^{n-k_i} \int_t^\infty \frac{(u-t)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^N p_{ij}(u) f_{ij}\big(\bar{x}_j(g_{ij}(u))\big)\, du\, ds,\ t \geqslant T,\ 1 \leqslant i \leqslant N, \tag{42}$$

respectively.

Then (42) implies that

$$\lim_{t\to\infty} \bar{z}_i^{(k_i)}(t) = 0. \tag{43}$$

From (41) on the basis of (9), (39), (3), (36) and (35) we conclude

$$\begin{aligned}
\bar{z}_i^{(k_i-1)}(t) &\geqslant \delta_i + \int_T^t \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \Big[(p_{ih})^+_{k_i}(u) f_{ih}\big(\bar{x}_h(h_{ih}(u))\big)\Big] \\
&\quad - \sum_{j=1}^N p_{ij}(u)^-_{k_i} f_{ij}\big(\bar{x}_j(g_{ij}(u))\big)\, du\, ds \\
&\geqslant \frac{\delta_i}{2} + \int_T^t \frac{(u-T)^{n-k_i}}{(n-k_i)!} (p_{ih})^+_{k_i}(u) f_{ih}\big(\bar{x}_h(h_{ih}(u))\big)\, du \\
&\geqslant \frac{\delta_i}{2} + \int_T^t \frac{(u-T)^{n-k_i}}{(n-k_i)!} (p_{ih})^+_{k_i}(u) f_{ih}\Big(\frac{\delta_h\big(g_{ih}(u)\big)^{k_h-1}}{2(k_h-1)!}\Big)\, du.
\end{aligned}$$

The last inequality together with (31) implies that

$$\lim_{t\to\infty} \bar{z}_i^{(k_i-1)}(t) = \infty. \tag{44}$$

By L'Hospital's rule, (43) and (44)

$$\lim_{t\to\infty} \frac{\bar{z}_i(t)}{t^{k_i}} = 0,\ \lim_{t\to\infty} \frac{\bar{z}(t)}{t^{k_i-1}} = \infty. \tag{45}$$

Then from (45) in view of (2), (3) and (36) we get

$$\lim_{t\to\infty} \frac{\bar{x}_i(t)}{t^{k_i}} = 0,\ \lim_{t\to\infty} \frac{\bar{x}_i(t)}{t^{k_i-1}} = \infty.$$

respectively.

The proof of Theorem 2 is complete. □

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