Antonio Boccuto; Beloslav Riečan

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 289–310

Persistent URL: http://dml.cz/dmlcz/128259

Czechoslovak Mathematical Journal, 58 (133) (2008), 289–310

THE SYMMETRIC CHOQUET INTEGRAL WITH RESPECT TO RIESZ-SPACE-VALUED CAPACITIES

Antonio Boccuto, Perugia, Beloslav Riečan, Banská Bystrica

(Received February 2, 2005)

Abstract. A definition of "Šipoš integral" is given, similarly to [3], [5], [10], for real-valued functions and with respect to Dedekind complete Riesz-space-valued "capacities". A comparison of Choquet and Šipoš-type integrals is given, and some fundamental properties and some convergence theorems for the Šipoš integral are proved.

Keywords: Riesz spaces, capacities, integration, symmetric Choquet integral, monotone and dominated convergence theorems

MSC 2000: 28A70

1. Introduction

In [3] we introduced a "monotone-type" (that is, Choquet-type) integral for real-valued functions, with respect to finitely additive positive set functions, with values in a Dedekind complete Riesz space. A "Lebesgue-type" integral for such kind of functions was investigated in [7]. In [4] we gave some comparison results for these types of integrals.

In [10], a Choquet-type integral for real-valued functions with respect to Riesz-space-valued "capacities", that is, monotone set functions not necessarily finitely additive, is investigated. The study of these integrals is motivated by several branches of mathematics (for example, stochastic processes, see [16]) and has also some applications to probability theory and economics, for example for the study of the fundamental properties of the so-called "utility functions" (see for instance [14], [19], [21], [22]) and the study of "qualitative probabilities", that is, set functions which associate to an event not necessarily a real number (indeed, in reality it is often

This paper was supported by the cooperation project between Slovak Academy of Sciences (S.A.V.) and Italian National Council of Researches (C.N.R.)

not so "natural" to represent the probability of a person about an event simply by means of an element of $[0,1]$, see for example [12]). For related topics, see also the bibliography [5] and [15].

In this paper we introduce a Šipoš-type, that is, "symmetric Choquet"-type integral, for real-valued functions with respect to Riesz-space-valued capacities, we investigate the fundamental properties and prove some convergence theorems (for real-valued capacities see [23] and [20], pp. 152–176).

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$ and $\tilde{\mathbb{R}}$ be the sets of all natural, real, positive, negative and extended real numbers, respectively.

A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

Throughout this paper we always suppose that R is a Dedekind complete Riesz space. In some suitable cases we add to R two extra element which we call $+\infty$ and $-\infty$, extending the ordering and operations. They have the same role as the usual $+\infty$ and $-\infty$ with the real numbers (see also [2], [13]). By the symbol $\overline{R}$ we denote the set $R \cup \{+\infty\} \cup \{-\infty\}$.

Definition 2.1. Given an element $r \in R$, we define $r^+ \equiv r \vee 0$, $r^- \equiv (-r) \vee 0$, $|r| \equiv r \vee (-r)$.

Definition 2.2. A sequence $(p_n)_n$ is called an (o)-*sequence* if $p_n \downarrow 0$, that is, if it is decreasing and $\inf\limits_n p_n = 0$. We say that a sequence $(r_n)_n$ is (o)-*convergent* (*order convergent*) to r if there exists an (o)-sequence $(p_n)_n \in R$ such that $|r_n - r| \leqslant p_n$ $\forall\, n \in \mathbb{N}$, and in this case we write $(o)\lim\limits_n r_n = r$.

Definition 2.3. A directed net $(r_\alpha)_{\alpha\in\Xi}$ is called an (o)-*net* if $r_\alpha \downarrow 0$, that is, if it is decreasing and $\inf\limits_{\alpha\in\Xi} r_\alpha = 0$. We say that the directed net $(r_\alpha)_{\alpha\in\Xi}$ is (o)-*convergent* to r if

$$(o)\limsup_\alpha r_\alpha \equiv \inf_\alpha[\sup_{\beta\geqslant\alpha} r_\beta] = (o)\liminf_\alpha r_\alpha \equiv \sup_\alpha[\inf_{\beta\geqslant\alpha} r_\beta] = r,$$

and in this case we write $(o)\lim\limits_{\alpha\in\Xi} r_\alpha = r$.

3. The symmetric Choquet integral for capacities

We begin with recalling the Choquet integral, introduced in [10], and we introduce and investigate the Šipoš (that is, the symmetric Choquet) integral for (extended) real-valued functions with respect to Riesz-space-valued capacities.

Definition 3.1. Let X be any nonempty set, and let $\mathscr{A} \subset \mathscr{P}(X)$ be a σ-algebra (we suppose this for the sake of simplicity, though several results remain true if we consider more general structures). We say that a set function $P\colon \mathscr{A} \to R$ is a *capacity* if $P(\emptyset) = 0$ and $P(A) \leqslant P(B)$ whenever $A, B \in \mathscr{A}$, $A \subset B$; P is said to be *submodular* if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) + P(A \cap B) \leqslant P(A) + P(B);$$

supermodular, if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) + P(A \cap B) \geqslant P(A) + P(B);$$

subadditive, if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) \leqslant P(A) + P(B);$$

superadditive, if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) \geqslant P(A) + P(B).$$

An R-valued capacity P is said to be *continuous from below* if for every increasing sequence $(E_n)_n$ of elements of $\mathscr{A}$ we have

$$P\Big(\bigcup_{n=1}^{\infty} E_n\Big) = (o)\lim_n P(E_n) = \sup_n P(E_n);$$

continuous from above, if for every decreasing sequence $(E_n)_n$ of elements of $\mathscr{A}$ we have

$$P\Big(\bigcap_{n=1}^{\infty} E_n\Big) = (o)\lim_n P(E_n) = \inf_n P(E_n);$$

continuous, if it is continuous both from below and from above.

A function $P\colon \mathscr{A} \to R$ is called a *mean* (or a *finitely additive set function*) if $P(A) \geqslant 0\ \forall\, A \in \mathscr{A}$ and $P(A \cup B) = P(A) + P(B)$ whenever $A \cap B = \emptyset$. It is easy to check that every mean is a capacity, but the converse is in general not true. We say that a set function P is a *measure* or that P is *σ-additive* if it is a continuous mean. Similarly to [3], given a function $f\colon X \to \tilde{\mathbb{R}}$ and a capacity P, for all $t \in \mathbb{R}$ set $\Sigma_t^f (\Sigma_t) \equiv \{x \in X\colon f(x) \geqslant t\}$; and, for every $t \in \mathbb{R}$, let $u_f(t) = u(t) \equiv P(\Sigma_t)$. We say that a function $f\colon X \to \tilde{\mathbb{R}}$ is *measurable* if $\Sigma_t^f \in \mathscr{A}$, $\forall\, t \in \mathbb{R}$.

We now recall a Riemann-type integral for functions defined in an interval of the real line and taking values in a Dedekind complete Riesz space (see also [3], [4]).

Definition 3.2. Given an interval $[a, b] \subset \mathbb{R}$, we call any finite set $\{x_0, x_1, \ldots, x_n\} \subset [a, b]$ where $x_0 = a$, $x_n = b$ and $x_i < x_{i+1}\ \forall\, i = 0, \ldots, n-1$ a *division of* $[a, b]$. We call the quantity $\delta(D) \equiv \max\limits_{i=0}^{n-1}(x_{i+1} - x_i)$ the *mesh* of D. We write $D_1 \geqslant D_2$ if $\delta(D_1) \leqslant \delta(D_2)$. A function $u\colon\ [a, b] \to R$ is said to be *Riemann integrable* if there exists an element $I \in R$ and an (o)-sequence $(p_j)_j$ such that

$$\sup_{\delta(D)\leqslant\frac{1}{j}} \left|\sum_{i=0}^{n-1} u(z_i)(x_{i+1} - x_i) - I\right| \leqslant p_j \quad \forall\, z_i \in [x_i, x_{i+1}]\ (i = 0, \ldots, n-1),$$

and we write $\int_a^b u(t)\,\mathrm{d}t \equiv I$.

The quantity $\sum\limits_{i=0}^{n-1} u(z_i)(x_{i+1} - x_i)$ is called the *Riemann sum* of u associated with the division $\{x_0, x_1, \ldots, x_n\}$ with respect to the points $z_i \in [x_i, x_{i+1}]$, $i = 0, \ldots, n-1$.

We note that, similarly to the classical case, it is easy to check that every monotone function $u\colon\ [a, b] \to R$ is Riemann integrable.

We now introduce the Choquet integral for nonnegative functions with respect to Riesz space-valued capacities (see also [5], [10]).

Definition 3.3. A measurable nonnegative function $f \in \tilde{\mathbb{R}}^X$ is said to be *Choquet integrable* if the quantity

$$\int_0^{+\infty} u(t)\,\mathrm{d}t \equiv \sup_{a>0} \int_0^a u(t)\,\mathrm{d}t = (o) \lim_{a\to+\infty} \int_0^a u(t)\,\mathrm{d}t$$

exists in R where $u(t) = P(\Sigma_t)$, $t \geqslant 0$. If f is Choquet integrable, we denote its integral by the symbol $(C)\int_X f\,\mathrm{d}P$.

We now introduce the Šipoš integral, that is, the symmetric Choquet integral, for extended real-valued functions with respect to Riesz space-valued capacities. We begin with

Definitions 3.4. A measurable function $f\colon\ X \to \mathbb{R}$ is said to be *simple* if its range is finite.

Let $\mathscr{F}$ be the family of all finite subsets of $\mathbb{R}$ which contain zero. Given $F \in \mathscr{F}$ and $a \in \mathbb{R}$, set

$$F + a \equiv \{d \in \mathbb{R}\colon\ d = b + a,\ \text{with } b \in F\}$$

and

$$aF \equiv \{d \in \mathbb{R}\colon\ d = ab,\ \text{with } b \in F\}.$$

Let now $F \in \mathscr{F}$, $F = \{b_k, b_{k-1}, \ldots, b_1, b_0, a_0, a_1, \ldots, a_n\}$, where $b_k < b_{k-1} < \ldots < b_1 < b_0 = 0 = a_0 < a_1 < \ldots < a_n$, and let f be a measurable function. As in [20], p. 153, set

$$f_F = \sum_{i=1}^{n} (a_i - a_{i-1}) \chi_{A_i} + \sum_{j=1}^{k} (b_j - b_{j-1}) \chi_{B_j},$$

where

$$\begin{aligned} A_i &= \{x \in X \colon\ f(x) \geqslant a_i\}, \quad i = 0, 1, \ldots, n; \\ B_j &= \{x \in X \colon\ f(x) \leqslant b_j\}, \quad j = 0, 1, \ldots, k. \end{aligned} \tag{1}$$

If P is an R-valued capacity, we define the *integral sum* (with respect to P) associated with f and F as follows:

$$S_F(f) = \sum_{i=1}^{n} (a_i - a_{i-1}) P(A_i) + \sum_{j=1}^{k} (b_j - b_{j-1}) P(B_j)$$

(where the A_i's and the B_j's are as in (1)) if the right-hand side contains no expression of the type $+\infty - \infty$; moreover, we put *by convention* $S_{\{0\}}(f) = 0$. We note that the set $\mathscr{F}$ is directed. We say that $f \colon X \to \tilde{\mathbb{R}}$ (not necessarily positive) is *Šipoš integrable* (*(S)-integrable*) if the limit

$$(o) \lim_{F \in \mathscr{F}} S_F(f) \tag{2}$$

exists in R and in this case we denote it by the symbol $(S)\int_X f \, \mathrm{d}P$. If the limit in (2) is $+\infty$ or $-\infty$, we write

$$(S)\int_X f \, \mathrm{d}P = +\infty$$

or

$$(S)\int_X f \, \mathrm{d}P = -\infty,$$

respectively, though f is, of course, not (S)-integrable. Furthermore, given a set $A \subset X$, $A \in \mathscr{A}$, we say that f is (S)-*integrable on A* if $f\chi_A$ is (S)-integrable, and in this case we put, *by definition*,

$$(S)\int_A f \, \mathrm{d}P \equiv (S)\int_X f \chi_A \, \mathrm{d}P. \tag{3}$$

Proposition 3.5. *Let $f\colon X \to \tilde{\mathbb{R}}$ be a measurable function. The following assertions hold:*

a) *If $f \geqslant 0$ and $\mathscr{F} \ni F_1 \subset F_2 \in \mathscr{F}$, then $S_{F_1}(f) \leqslant S_{F_2}(f)$.*

b) *If $f \geqslant 0$, then $(S)\int_X f \,\mathrm{d}P$ exists in $R \cup \{+\infty\}$ and $(S)\int_X f \,\mathrm{d}P \geqslant 0$. Moreover, in this case we have*

$$(S)\int_X f \,\mathrm{d}P = \sup_{F \in \mathscr{F}} S_F(f).$$

c) *$(S)\int_X \cdot \,\mathrm{d}P$ is a monotone functional.*

d) *If $(S)\int_X f \,\mathrm{d}P$ exists in R, then for every $c \in \mathbb{R}$ we have*

$$(S)\int_X (cf) \,\mathrm{d}P = c \cdot (S)\int_X f \,\mathrm{d}P.$$

Proof. The proof is similar to the one of Lemma 7.3 (ii), p. 156, and Theorem 7.10 (i)–(iii), p. 155, of [20]. □

We now prove that, for measurable non negative extended real-valued functions, the Šipoš and Choquet integrals do coincide.

Theorem 3.6. *Let $f\colon X \to \tilde{\mathbb{R}}$ be a nonnegative measurable function. Then f is Šipoš integrable if and only if it is Choquet integrable, and in this case*

$$(S)\int_X f \,\mathrm{d}P = (C)\int_X f \,\mathrm{d}P.$$

Proof. First of all we prove that every nonnegative bounded measurable function f is Šipoš integrable.

Let $K \in \mathbb{N}$ be such that $f(x) \leqslant K$ for all $x \in X$. Then the function $u(t) \equiv P(\{x \in X\colon f(x) \geqslant t\})$ vanishes on $[K, +\infty[$, and therefore u is Riemann integrable in $[0, K]$ and we get

$$R \ni \int_0^K u(t) \,\mathrm{d}t = \int_0^{+\infty} u(t) \,\mathrm{d}t = (C)\int_X f \,\mathrm{d}P. \tag{4}$$

Thus f is Choquet integrable. Moreover, thanks also to Proposition 3.5 a), it is easy to check that $0 \leqslant S_F(f) \leqslant KP(X)$ for every $F \in \mathscr{F}$. From this and Proposition 3.5 b) it follows that f is Šipoš integrable too and

$$(S)\int_X f \,\mathrm{d}P \leqslant KP(X).$$

Now, given $F \in \mathscr{F}$, let $\Sigma(u, F, K)$ be the Riemann sum of u associated with that division of $[0, K]$ whose elements, which we denote by α_j, $j = 1, \ldots, s$, are the points of F belonging to $[0, K]$ and the points 0 and K, with respect to the (right end-)points α_j's themselves, $j = 1, \ldots, s$. We have

$$\begin{aligned}
0 &\leqslant \left|(S)\int_X f \,\mathrm{d}P - (C)\int_X f \,\mathrm{d}P\right| \\
&= \left|(S)\int_X f \,\mathrm{d}P - \int_0^K u(t) \,\mathrm{d}t\right| \\
&= (o) \limsup_{F \in \mathscr{F}} \left|(S)\int_X f \,\mathrm{d}P - S_F(f) + S_F(f) - \int_0^K u(t) \,\mathrm{d}t\right| \\
&\leqslant (o) \limsup_{F \in \mathscr{F}} \left|(S)\int_X f \,\mathrm{d}P - S_F(f)\right| \\
&\qquad + (o) \limsup_{F \in \mathscr{F}} \left|\Sigma(u, F, K) - \int_0^K u(t) \,\mathrm{d}t\right| \\
&= (o) \limsup_{F \in \mathscr{F}} \left|\Sigma(u, F, K) - \int_0^K u(t) \,\mathrm{d}t\right| = 0.
\end{aligned}$$

From this the assertion follows, at least when f is bounded and measurable.

If f is not bounded, then we have (finite or $+\infty$)

$$\begin{aligned}
(S)\int_X f \,\mathrm{d}P &= \sup_{F \in \mathscr{F}} S_F(f) \\
&= \sup_{K \in \mathbb{N}} [\sup_{F \in \mathscr{F}} S_F(f \wedge K)] \\
&= \sup_{K \in \mathbb{N}} (S)\int_X (f \wedge K) \,\mathrm{d}P = \sup_{K \in \mathbb{N}} (C)\int_X (f \wedge K) \,\mathrm{d}P \\
&= \sup_{K \in \mathbb{N}} \int_0^K P(\{x \in X\colon\ f(x) \geqslant t\}) \,\mathrm{d}t \\
&= \int_0^{+\infty} P(\{x \in X\colon\ f(x) \geqslant t\}) \,\mathrm{d}t = (C)\int_X f \,\mathrm{d}P.
\end{aligned} \tag{5}$$

This concludes the proof. □

Proceeding in a way analogous to (5), it is possible to prove

Proposition 3.7. *A nonnegative measurable extended real-valued function f is (S)-integrable if and only if one of the following three elements*

$$(o)\lim_{K\to+\infty}(S)\int_X (f\wedge K)\,\mathrm{d}P,$$
$$\sup_{K\in\mathbb{R},K\geqslant 0}(S)\int_X (f\wedge K)\,\mathrm{d}P,$$
$$\sup_{K\in\mathbb{N}}(S)\int_X (f\wedge K)\,\mathrm{d}P$$

exists in R and then these quantities coincide with $(S)\int_X f\,\mathrm{d}P$, not only if they belong to R, but also if they are equal to $+\infty$.

We now prove

Theorem 3.8. *If f is a measurable extended real-valued function (not necessarily nonnegative) and $a\in\mathbb{R}$, $a\geqslant 0$, then*

$$(S)\int_X f\,\mathrm{d}P=(S)\int_X (f\wedge a)\,\mathrm{d}P+(S)\int_X (f-f\wedge a)\,\mathrm{d}P \tag{6}$$

(finite or $+\infty$), if one of the right-hand side expressions belongs to R.

P r o o f. We prove the theorem in the case $(S)\int_X (f\wedge a)\,\mathrm{d}P\in R$: the proof in the other case is analogous.

By Proposition 3.5 b), the quantity $(S)\int_X (f-f\wedge a)\,\mathrm{d}P$ exists in $R\cup\{+\infty\}$. We consider first the case $(S)\int_X (f-f\wedge a)\,\mathrm{d}P\in R$. By definition of the Šipoš integral and the (o)-convergence of nets, there exist two (o)-nets $(p_F)_{F\in\mathscr{F}}$ and $(q_F)_{F\in\mathscr{F}}$ such that

$$\left|S_F(f\wedge a)-(S)\int_X (f\wedge a)\,\mathrm{d}P\right|\leqslant p_F\quad\forall F\in\mathscr{F} \tag{7}$$

and

$$\left|S_F(f-f\wedge a)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P\right|\leqslant q_F\quad\forall F\in\mathscr{F}. \tag{8}$$

Fix arbitrarily $F_1,F_2\in\mathscr{F}$, let $F_0\equiv F_1\cup(F_2+a)$ and pick $F\supset F_0$. Since $F-a\supset F_2$, we get

$$\begin{aligned}&\left|S_F(f)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P-(S)\int_X (f\wedge a)\,\mathrm{d}P\right|\\&\quad\leqslant\left|S_{F-a}(f-f\wedge a)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P\right|\\&\qquad+\left|S_F(f\wedge a)-(S)\int_X (f\wedge a)\,\mathrm{d}P\right|\leqslant q_{F_1}+p_{F_2}.\end{aligned}$$

From (9), thanks also to Proposition 3.5 a), it follows that

$$(10)\qquad (o)\limsup_{F\in\mathscr{F}}\left|S_F(f)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P-(S)\int_X (f\wedge a)\,\mathrm{d}P\right|=0.$$

From (10) it follows that $(S)\int_X f\,\mathrm{d}P$ exists in R and formula (6) holds true. In the case

$$(S)\int_X (f-f\wedge a)\,\mathrm{d}P=+\infty,$$

proceeding with the analogous notation as above, we get

$$S_F(f)=S_F(f\wedge a)+S_{F-a}(f-f\wedge a)\geqslant S_F(f\wedge a)+\int_X (f\wedge a)\,\mathrm{d}P-p_{\{0\}},$$

and taking the supremum in (11) we obtain that $(S)\int_X f\,\mathrm{d}P=+\infty$ if $(S)\int_X (f-f\wedge a)\,\mathrm{d}P=+\infty$, and hence (6) holds with the value $+\infty$. This concludes the proof. $\square$

Theorem 3.9. *Let $f\colon X\to\tilde{\mathbb{R}}$ be a measurable function. If $(S)\int_X f^+\,\mathrm{d}P$ or $(S)\int_X f^-\,\mathrm{d}P$ belongs to R, then $(S)\int_X f\,\mathrm{d}P$ belongs to R too and*

$$(12)\qquad (S)\int_X f\,\mathrm{d}P=(S)\int_X f^+\,\mathrm{d}P-(S)\int_X f^-\,\mathrm{d}P.$$

Moreover, if f is (S)-integrable, then (12) *holds true, and f^+, f^- are (S)-integrable too.*

Proof. The first part is an easy consequence of Theorem 3.8 (see also [20], p. 159).

We now turn to the second part. In order to prove it, it is sufficient to prove that $(S)\int_X f^+\,\mathrm{d}P$ and $(S)\int_X f^-\,\mathrm{d}P$ belong to R. We now report in detail only the proof of the first property. Since f is (S)-integrable, there exists an (o)-net $(p_F)_{F\in\mathscr{F}}$ such that

$$(13)\qquad \left|S_F(f)-(S)\int_X f\,\mathrm{d}P\right|\leqslant p_F\quad\forall\,F\in\mathscr{F}.$$

Fix now arbitrarily $F_0\in\mathscr{F}$ and choose $F\in\mathscr{F}$ with $F\supset F_0$ and $F\cap\mathbb{R}^-=F_0\cap\mathbb{R}^-$. Proceeding analogously to [20], pp. 159–160, we get

$$(14)\qquad \begin{aligned}0&\leqslant S_{F\cap\mathbb{R}^+}(f^+)=S_F(f^+)=S_{-F}(f^-)+S_F(f)\\&=S_{-F_0}(f^-)+S_F(f)\leqslant S_{-F_0}(f^-)+(S)\int_X f\,\mathrm{d}P+p_{\{0\}}.\end{aligned}$$

We have

$$(15) \qquad (S)\int_X f^+ \,\mathrm{d}P = \sup_{F\in\mathscr{F}} S_F(f^+) = \sup_{F\in\mathscr{F}} S_{F\cap\mathbb{R}^+}(f^+).$$

By virtue of Proposition 3.5 a), the supremum in (15) is equal to the supremum with respect to those elements F of $\mathscr{F}$ which contain F_0 and such that $F\cap\mathbb{R}^- = F_0\cap\mathbb{R}^-$. Since F_0 was *fixed, it follows* from (14) and (15) that $(S)\int_X f^+\,\mathrm{d}P \in R$. □

Proposition 3.10. *Let $X \supset A_1 \supset A_2 \supset \ldots \supset A_n \in \mathscr{A}$. Let c_i be positive real numbers and $f_i \equiv c_i\chi_{A_i}$, $i = 1, 2, \ldots, n$. Then*

$$(S)\int_X \left(\sum_{i=1}^n f_i\right) \mathrm{d}P = \sum_{i=1}^n c_i P(A_i).$$

P r o o f. The proof is similar to the one of [8], Proposition 2.4, p. 65, and takes into account the equivalence between the Choquet and Šipoš integrals for nonnegative measurable functions. □

The proof of the following proposition is straightforward.

Proposition 3.11. *If f is measurable and $|f|$ is (S)-integrable, then f is (S)-integrable too. Moreover, if f is measurable, g is (S)-integrable and $|f| \leqslant g$, then f is (S)-integrable too.*

From now on, given a nonnegative measurable function $f\colon X \to \tilde{\mathbb{R}}$, let S_f be the set of all simple functions g such that $0 \leqslant g(x) \leqslant f(x)\ \forall\, x \in X$.

Proposition 3.12. *If $f \geqslant 0$ is (S)-integrable, then*

$$(S)\int_X f\,\mathrm{d}P = \sup_{g\in S_f} (S)\int_X g\,\mathrm{d}P.$$

Conversely, if $f \geqslant 0$ is measurable and such that the quantity $\sup\limits_{g\in S_f} (S)\int_X g\,\mathrm{d}P$ exists in R, then f is (S)-integrable and

$$(S)\int_X f\,\mathrm{d}P = \sup_{g\in S_f} (S)\int_X g\,\mathrm{d}P.$$

Furthermore, if f is nonnegative and (S)-integrable, then there exists a sequence of simple functions $(g_n)_n$ such that

$$(S)\int_X f\,\mathrm{d}P = \sup_n (S)\int_X g_n\,\mathrm{d}P.$$

Proof. The proof of the first two parts is similar to the one of [8], Proposition 2.5, p. 65. The proof of the last part, in the case of a bounded f, is similar to the one of [3], Proposition 3.12, p. 798, and takes into account Proposition 3.10; the general case follows from the case of a bounded function and Proposition 3.7. □

Proposition 3.13. *If f is (S)-integrable, then*

$$(o) \lim_{t \to +\infty} P(\{x \in X \colon\ |f(x)| \geqslant t\}) = 0 = P(\{x \in X \colon\ |f(x)| = +\infty\}).$$

Proof. The proof is similar to the one of [3], Proposition 3.10, p. 797, applied to f^+ and f^-, which are integrable by virtue of Theorem 3.9. □

We now show absolute continuity of the Šipoš integral. In order to do this, first we state a preliminary lemma (for the case $R = \mathbb{R}$, see [20], Lemma 7.5. (i), p. 163).

Lemma 3.14. *If f is a nonnegative (S)-integrable function, then*

$$(o) \lim_{A \to +\infty} (S)\int_X (f - f \wedge A)\, \mathrm{d}P = 0.$$

Proof. Fix arbitrarily $F \in \mathscr{F}$, $F = \{b_k, b_{k-1}, \dots, b_1, b_0 = 0 = a_0, a_1, \dots, a_n\}$, where the elements of F are ordered in the increasing order, and let

$$f_F = \sum_{i=1}^{n} (a_i - a_{i-1}) \chi_{A_i}.$$

For $A \in \mathbb{R}^+$ large enough we get

$$f_F \leqslant f \wedge A \leqslant f. \tag{16}$$

Now, given $F \in \mathscr{F}$, let A satisfy condition (16). From (16) and the monotonicity of the Šipoš integral we have

$$S_F(f) = (S)\int_X f_F\, \mathrm{d}P \leqslant (S)\int_X (f \wedge A)\, \mathrm{d}P \leqslant (S)\int_X f\, \mathrm{d}P. \tag{17}$$

Moreover, by virtue of Theorem 3.8 and (17), we get

$$(S)\int_X (f - f \wedge A)\, \mathrm{d}P = (S)\int_X f\, \mathrm{d}P - (S)\int_X (f \wedge A)\, \mathrm{d}P \leqslant (S)\int_X f\, \mathrm{d}P - S_F(f). \tag{18}$$

From (18) and the Šipoš integrability of f it follows that

$$\begin{aligned} 0 &\leqslant (o) \limsup_{A \in \mathbb{R}^+} \left[(S)\int_X (f - f \wedge A)\, \mathrm{d}P \right] \\ &\leqslant (o) \limsup_{F \in \mathscr{F}} \left[(S)\int_X f\, \mathrm{d}P - S_F(f) \right] = 0. \end{aligned} \tag{19}$$

Thus the assertion follows. □

The next theorem is a consequence of Lemma 3.14

Theorem 3.15. *If $f\colon X \to \tilde{\mathbb{R}}$ is (S)-integrable, then the integral $(S)\int_{\cdot} f \,\mathrm{d}P$ is absolutely continuous, that is*

$$(o)\lim_n \int_{A_n} f dP = 0$$

whenever $(A_n)_n$ is a sequence in $\mathscr{A}$ such that $(o)\lim\limits_n P(A_n) = 0$.

P r o o f. The proof is similar to the one of [3], Proposition 3.17, p. 800, thanks to Lemma 3.14. $\square$

4. Convergence theorems

In this section we prove some convergence theorems for the Šipoš integral with respect to Riesz space-valued capacities, not necessarily finitely additive.

Throughout this section, we always assume that X is any nonempty set, $\mathscr{A} \subset \mathscr{P}(X)$ is a σ-algebra, R is a Dedekind complete Riesz space, and $P\colon \mathscr{A} \to R$ is a *continuous* capacity.

We begin with the following theorem (for the real case, see [20], Theorem 7.13, pp. 162–163):

Theorem 4.1. *Let $c \in R$, $c \geqslant 0$ $(f_n\colon X \to \tilde{\mathbb{R}})_n$ be an increasing sequence of nonnegative (S)-integrable functions with $(S)\int_X f_n dP \leqslant c$ for every $n \in \mathbb{N}$, and let $f \equiv \sup\limits_n f_n$ be the pointwise supremum.*

Then f is (S)-integrable, $(S)\int_X f \,\mathrm{d}P \leqslant c$ and

$$(S)\int_X f \,\mathrm{d}P = \sup_n (S)\int_X f_n \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P.$$

P r o o f. Fix arbitrarily $\varepsilon > 0$ and $F \in \mathscr{F}$, $F = \{a_0, a_1, \ldots, a_n\}$, where $0 = a_0 < a_1 < \ldots < a_n$. We choose δ such that

$$0 < 2\delta < \min\{(a_j - a_{j-1})\colon\ j = 1, 2, \ldots, k\}$$

and

$$\frac{\delta}{a_1 - \delta} < \varepsilon$$

such a δ does exist. Proceeding analogously to the proof of Theorem 7.13 of [20], we get

$$\begin{aligned} S_F(f) &\leqslant (o)\lim_n (S)\int_X f_n \,\mathrm{d}P + \frac{\delta}{a_1-\delta}(o)\lim_n (S)\int_X f_n \,\mathrm{d}P \\ &\leqslant (o)\lim_n (S)\int_X f_n \,\mathrm{d}P + \varepsilon c, \end{aligned}$$

and hence

$$(S)\int_X f \,\mathrm{d}P = \sup_{F\in\mathscr{F}} S_F(f) \leqslant (o)\lim_n (S)\int_X f_n \,\mathrm{d}P + \varepsilon c.$$

Due to arbitrariness of $\varepsilon \in \mathbb{R}^+$, (20) yields

$$(S)\int_X f \,\mathrm{d}P \leqslant (o)\lim_n (S)\int_X f_n \,\mathrm{d}P.$$

The converse inequality follows easily from the monotonicity of the integral $(S)\int_X f \,\mathrm{d}P$. □

We have the following consequences of Theorem 4.1:

Corollary 4.2. *If $(\alpha_n)_n$ is any decreasing sequence of positive real numbers with $\inf_n \alpha_n = 0$, then*

$$(o)\lim_{n\to+\infty} (S)\int_X (f\wedge\alpha_n) \,\mathrm{d}P = 0.$$

P r o o f. The proof is similar to the one of Lemma 7.5 (ii) of [20], p. 163. □

Corollary 4.3 (Fatou's Lemma)**.** *Let $c \in R$, $c \geqslant 0$ $(f_n\colon X \to \tilde{\mathbb{R}})_n$ be any sequence of nonnegative (S)-integrable functions with $(S)\int_X f_n dP \leqslant c$ for every $n \in \mathbb{N}$, and $f \equiv \liminf_n f_n$.*

Then

$$(S)\int_X f \,\mathrm{d}P \leqslant (o)\liminf_n (S)\int_X f_n \,\mathrm{d}P.$$

P r o o f. First of all, we note that f is (S)-integrable, thanks to Theorem 4.1. For each $n \in \mathbb{N}$, let $h_n = \inf_{i\geqslant n} f_i$. Then $0 \leqslant h_n \uparrow f$ and

$$(S)\int_X h_n \,\mathrm{d}P \leqslant (S)\int_X f \,\mathrm{d}P \quad \forall n \in \mathbb{N}.$$

Again by Theorem 4.1, we get

$$\begin{aligned}(S)\int_X f\,\mathrm{d}P &= (o)\lim_n (S)\int_X h_n\,\mathrm{d}P\\ &= (o)\liminf_n (S)\int_X h_n\,\mathrm{d}P \leqslant (o)\liminf_n (S)\int_X f_n\,\mathrm{d}P.\end{aligned}$$

This concludes the proof. □

We now recall the following fundamental representation theorem for Riesz spaces ([1], [18], [24]).

Theorem 4.4. *Given a Dedekind complete Riesz space R, there exists a compact Stonian topological space Ω, unique up to homeomorphisms, such that R can be embedded as a solid subspace of $\mathscr{C}_\infty(\Omega) = \{f \in \tilde{\mathbb{R}}^\Omega\colon\ f$ is continuous, and $\{\omega\colon\ |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda\in\Lambda}$ is any family such that $a_\lambda \in R\,\forall\lambda$ and $a = \inf_\lambda a_\lambda \in R$ (where the infimum is taken with respect to R), then $a = \inf_\lambda a_\lambda$ with respect to $\mathscr{C}_\infty(\Omega)$, and the set $\{\omega\in\Omega\colon\ (\inf_\lambda a_\lambda)(\omega) \neq \inf_\lambda a_\lambda(\omega)\}$ is meager in Ω.*

We now turn to another version of the monotone convergence theorem. In order to prove it, we first establish

Lemma 4.5. *Let $[a,b]\subset\mathbb{R}$, $u_n\colon\ [a,b]\to R$ let $(n\in\mathbb{N}\cup\{0\})$ be monotone decreasing functions, such that*

$$u_n(t) = \inf_{s<t} u_n(s)\quad \forall\, t\in(a,b], \forall n\in\mathbb{N}\cup\{0\}; \tag{21}$$

$$u_n(t) \geqslant u_{n+1}(t)\quad \forall\, t\in[a,b],\ \forall\, n\geqslant 1; \tag{22}$$

$$\inf_n u_n(t) = u_0(t)\quad \forall\, t\in[a,b]. \tag{23}$$

Then

$$\int_a^b u_0(t)\,\mathrm{d}t = \inf_{n\geqslant 1}\int_a^b u_n(t)\,\mathrm{d}t = (o)\lim_{n\to+\infty}\int_a^b u_n(t)\,\mathrm{d}t.$$

P r o o f. First of all, we observe that for every $n\in\mathbb{N}\cup\{0\}$ the integral $\int_a^b u_n(t)\,\mathrm{d}t$ is the limit, for $l\to+\infty$, of the Riemann sums of the type

$$\sum_{i=1}^{2^l}(a_i^{(l)} - a_{i-1}^{(l)})u_n(a_i^{(l)}), \tag{24}$$

where the $a_i^{(l)}$'s, $l\in\mathbb{N}$, $i=1,2,\dots,2^l$, are taken in such a way that $a_0^{(l)} = a$, $a_{2^l}^{(l)} = b$, and the division generated by the $a_i^{(l)}$'s divides the interval $[a,b]$ in 2^l equal parts.

Denote by $\mathscr{Y}$ the set of points of all these divisions and let $\mathscr{Q}$ be the union of $\mathscr{Y}$ and the rational numbers contained in $[a,b]$; we note that $\mathscr{Q}$ is a countable dense subset of $[a,b]$.

Let now Ω be as in Theorem 4.4. We note that there exists a meager set $N^* \subset \Omega$ such that, for all $\omega \notin N^*$, we have

$$\left[\inf_n \left[\int_a^b u_n(t)\,\mathrm{d}t\right](\omega)\right] = \left[\inf_n \left[\int_a^b u_n(t)\,\mathrm{d}t\right]\right](\omega), \tag{25}$$

and for each $\omega \notin N^*$ and $s \in \mathscr{Q}$ we get

$$\begin{gathered} u_n(s)(\omega) \geqslant u_{n+1}(s)(\omega) \quad \forall\, n \geqslant 1, \\ \lim_{n\to+\infty} u_n(s)(\omega) = \inf_{n\geqslant 1} u_n(s)(\omega) = u_0(s)(\omega), \end{gathered}$$

and all quantities involved are *real* numbers. Now, for all $s \in [a,b] \cap \mathscr{Q}$, $\forall \omega \notin N^*$ and $\forall\, n \in \mathbb{N} \cup \{0\}$, set

$$w_{n,\omega}(s) = u_n(s)(\omega).$$

For each $t \in [a,b]$, $\omega \notin N^*$ and $n \in \mathbb{N} \cup \{0\}$, put

$$w_{n,\omega}(t) = \inf_{s\leqslant t, s\in\mathscr{Q}} w_{n,\omega}(s). \tag{26}$$

By (26) and since the $w_{n,\omega}$'s are decreasing, their integrals can be evaluated analogously to in (24), and thus we get, $\forall\, n \in \mathbb{N} \cup \{0\}$ and $\forall\, \omega \notin N^*$,

$$\int_a^b w_{n,\omega}(t)\,\mathrm{d}t = \left[\int_a^b u_n(t)\,\mathrm{d}t\right](\omega). \tag{27}$$

We note that

$$w_{n,\omega}(s) \downarrow w_{0,\omega}(s) \quad \forall \omega \notin N^*, \quad \forall\, s \in [a,b] \cap \mathscr{Q},\ s \geqslant 0. \tag{28}$$

Furthermore, $\forall\, \omega \notin N^*$ and $t \geqslant 0$, $t \in [a,b]$, by "interchanging the infima involved" we get

$$\begin{aligned} \inf_n w_{n,\omega}(t) &= \inf_n \Big[\inf_{s,t\in[a,b], s\leqslant t, s\in\mathscr{Q}} w_{n,\omega}(s)\Big] = \inf_{s,t\in[a,b], s\leqslant t, s\in\mathscr{Q}} \big[\inf_n w_{n,\omega}(s)\big] \\ &= \inf_{s,t\in[a,b], s\leqslant t, s\in\mathscr{Q}} [w_{0,\omega}(s)] = w_{0,\omega}(t), \end{aligned} \tag{29}$$

and thus

$$w_{n,\omega}(t) \downarrow w_{0,\omega}(t) \quad \forall\, \omega \notin N^*, \quad \forall\, t \in [a,b]. \tag{30}$$

From (25), (27) and (30), and applying the classical (dominated) convergence theorem for real-valued functions, we get, $\forall\,\omega \notin N^*$:

$$\begin{aligned}(31)\qquad \left[\int_a^b u_0(t)\,\mathrm{d}t\right](\omega) &= \int_a^b w_{0,\omega}(t)\,\mathrm{d}t = \inf_n \left[\int_a^b w_{n,\omega}(t)\,\mathrm{d}t\right] \\ &= \inf_n \left[\left[\int_a^b u_n(t)\,\mathrm{d}t\right](\omega)\right] = \left[\inf_n \left[\int_a^b u_n(t)\,\mathrm{d}t\right]\right](\omega).\end{aligned}$$

From this, since N^* is meager and the complement of every meager subset of Ω is dense in Ω, it follows that

$$\int_a^b u_0(t)\,\mathrm{d}t = \inf_n \int_a^b u_n(t)\,\mathrm{d}t.$$

Thus we get the assertion. □

We now are in position to prove

Theorem 4.6. *Let $(f_n\colon X \to \tilde{\mathbb{R}})_n$ be a decreasing sequence of nonnegative (S)-integrable functions and let $f = \inf_n f_n$ be the pointwise infimum. Then f is (S)-integrable and*

$$(S)\!\int_X f\,\mathrm{d}P = \inf_n (S)\!\int_X f_n\,\mathrm{d}P = (o)\lim_n (S)\!\int_X f_n\,\mathrm{d}P.$$

P r o o f. First of all, since $0 \leqslant f \leqslant f_1$, it follows from Proposition 3.11 that f is integrable. Moreover, we observe that, proceeding similarly as in the first half of p. 164 of [20] and taking into account Lemma 3.14 we can suppose, without loss of generality, that the functions f_n and f are equibounded by a positive number A.

For each $t \geqslant 0$ and $n \in \mathbb{N}$, $n \geqslant 1$, let $u_n(t) = P(\{x \in X\colon\ f_n(x) \geqslant t\})$, and $\forall\,t \geqslant 0$ let $u_0(t) = P(\{x \in X\colon\ f(x) \geqslant t\})$.

Proceeding analogously to the proof of Theorem 3.6 we get

$$(32)\qquad (S)\!\int_X f\,\mathrm{d}P = \int_0^A u_0(t)\,\mathrm{d}t$$

and

$$(33)\qquad (S)\!\int_X f_n\,\mathrm{d}P = \int_0^A u_n(t)\,\mathrm{d}t \quad \forall\,n \geqslant 1.$$

Since P is a continuous capacity and $f_n \downarrow f$, the functions u_n, $n \in \mathbb{N} \cup \{0\}$, satisfy conditions (21), (22) and (23). Applying Lemma 4.5 with $[a, b] = [0, A]$ and using (32) and (33), we conclude that

$$\begin{aligned}(S)\int_X f \,\mathrm{d}P &= \int_0^A u_0(t)\,\mathrm{d}t = \inf_{n \geqslant 1} \int_a^b u_n(t)\,\mathrm{d}t \\ &= (o) \lim_{n \to +\infty} \int_a^b u_n(t)\,\mathrm{d}t = \inf_n (S)\int_X f_n \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P,\end{aligned}$$

which is the assertion. □

We now prove

Theorem 4.7. *Let $c \in R$, let $(f_n)_n$ be a sequence of (S)-integrable functions and f a measurable function such that $f_n \downarrow f$ and*

$$\int_X f_n \,\mathrm{d}P \geqslant c \quad \forall\, n \in \mathbb{N}.$$

Then f is (S)-integrable and

$$(S)\int_X f \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P = \inf_n (S)\int_X f_n \,\mathrm{d}P.$$

Proof. Since $f_n \downarrow f$, we have $f_n^+ \downarrow f^+$ and $f_n^- \uparrow f^-$. Further,

$$\begin{aligned}0 \leqslant (S)\int_X f_n^- \,\mathrm{d}P &= (S)\int_X f_n^+ \,\mathrm{d}P - (S)\int_X f_n \,\mathrm{d}P \\ &\leqslant (S)\int_X f_1^+ \,\mathrm{d}P - (S)\int_X f_n \,\mathrm{d}P \leqslant (S)\int_X f_1^+ \,\mathrm{d}P - c,\end{aligned}$$

and thus we get that the integrals $(S)\int_X f_n^- \,\mathrm{d}P$, $n \in \mathbb{N}$, are bounded from above by an element of R. By Theorem 4.1, f is Šipoš-integrable and

$$(S)\int_X f^- \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n^- \,\mathrm{d}P = \sup_n (S)\int_X f_n^- \,\mathrm{d}P. \tag{34}$$

Moreover, by Theorem 4.6, we get integrability of f^+ and

$$(S)\int_X f^+ \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n^+ \,\mathrm{d}P = \inf_n (S)\int_X f_n^+ \,\mathrm{d}P. \tag{35}$$

Thus, from (34), (35) and Theorem 3.9 we obtain

$$\begin{aligned}(o)\lim_n (S)\int_X f_n \,\mathrm{d}P &= (o)\lim_n (S)\int_X f_n^+ \,\mathrm{d}P - (o)\lim_n (S)\int_X f_n^- \,\mathrm{d}P \\ &= (S)\int_X f^+ \,\mathrm{d}P - (S)\int_X f^- \,\mathrm{d}P = (S)\int_X f \,\mathrm{d}P,\end{aligned}$$

which is the assertion. □

The proof of the next theorem is similar to those of Theorem 4.7 and of Theorem 7.15, p. 166 of [20], if we take into account that $f_n \uparrow f$ implies $f_n^+ \uparrow f^+$ and $f_n^- \downarrow f^-$.

Theorem 4.8. *Let $c \in R$, $c \geqslant 0$, let $(f_n)_n$ be a sequence of (S)-integrable functions and f a measurable function such that $f_n \uparrow f$ and*

$$\int_X f_n \,\mathrm{d}P \leqslant c \quad \forall\, n \in \mathbb{N}.$$

Then f is (S)-integrable and

$$(S)\int_X f \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P = \sup_n (S)\int_X f_n \,\mathrm{d}P.$$

We now state a version of the Lebesgue convergence dominated theorem, which is a consequence of Theorems 4.7 and 4.8 and whose proof is similar to the one of Theorem 7.16 of [20]:

Theorem 4.9. *If $(f_n)_n$ is a sequence of measurable functions which converges pointwise to a measurable function f and if g is an (S)-integrable function with $|f_n| \leqslant g$ $\forall\, n \in \mathbb{N}$, then f is (S)-integrable and*

$$(S)\int_X f \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P.$$

5. The submodular theorems

In this section we prove some theorems for the Šipoš integral in the case when the involved capacities are submodular.

Theorem 5.1. *Let $P\colon \mathscr{A} \to R$ be a submodular capacity and let $f, g\colon X \to \tilde{\mathbb{R}}$ be two nonnegative measurable functions. Then*

$$(S)\int_X (f \wedge g) \,\mathrm{d}P + (S)\int_X (f \vee g) \,\mathrm{d}P \leqslant (S)\int_X f \,\mathrm{d}P + (S)\int_X g \,\mathrm{d}P$$

(finite or $+\infty$). Moreover, if f and g are integrable, then $f \wedge g$ and $f \vee g$ are integrable too.

P r o o f. If $(S)\int_X f\,\mathrm{d}P = +\infty$ or $(S)\int_X g\,\mathrm{d}P = +\infty$ the assertion is trivial. Let f and g be both (S)-integrable. (S)-integrability of $f \wedge g$ follows immediately from Proposition 3.11.

We now prove that $f \vee g$ is (S)-integrable. To this aim, pick arbitrarily $F \in \mathscr{F}$ with $F = \{a_0, a_1, \ldots, a_n\}$, where $a_0 = 0 < a_1 < \ldots < a_n$. Set

$$A_i = \{x\colon\ f(x) \geqslant a_i\}, \quad B_i = \{x\colon\ g(x) \geqslant a_i\}, \qquad i = 0, 1, \ldots, n.$$

Proceeding analogously to the proof of Theorem 7.17 of [20], thanks to the submodularity of P we get:

$$S_F(f \wedge g) + S_F(f \vee g) \leqslant S_F(f) + S_F(g) \leqslant \int_X f\,\mathrm{d}P + \int_X g\,\mathrm{d}P. \tag{36}$$

From (36), taking into account the Dedekind completeness of R, we have

$$\begin{aligned}
&(o) \lim_{F\in\mathscr{F}} [S_F(f \wedge g) + S_F(f \vee g)] \\
&\quad = (o) \lim_{F\in\mathscr{F}} [S_F(f \wedge g)] + (o) \lim_{F\in\mathscr{F}} [S_F(f \vee g)] \\
&\quad = \sup_{F\in\mathscr{F}} [S_F(f \wedge g)] + \sup_{F\in\mathscr{F}} [S_F(f \vee g)] \in R.
\end{aligned} \tag{37}$$

From the (S)-integrability of the function $f \wedge g$ and from (37) we get

$$(o) \lim_{F\in\mathscr{F}} [S_F(f \vee g)] = \sup_{F\in\mathscr{F}} [S_F(f \vee g)] \in R,$$

that is, the (S)-integrability of $f \vee g$. Taking the order limits for $F \in \mathscr{F}$, from (36) and (37) we obtain:

$$\begin{aligned}
&(S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P \geqslant (o) \lim_{F\in\mathscr{F}} [S_F(f \wedge g) + S_F(f \vee g)] \\
&\quad = (S)\int_X (f \wedge g)\,\mathrm{d}P + (S)\int_X (f \vee g)\,\mathrm{d}P,
\end{aligned}$$

which is the assertion. □

Proceeding analogously to Theorem 5.1, it is possible to prove

Proposition 5.2. *If f and g are nonnegative measurable functions and P is an R-valued subadditive capacity, then*

$$(S)\int_X (f \vee g)\,\mathrm{d}P \leqslant (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

(For the real case, see [20], Corollary 7.5, p. 168.)

We now state the submodular theorem (see also [5]).

Proposition 5.3. *Let $P\colon \mathscr{A} \to R$ be a submodular capacity and let $f, g \in \tilde{\mathbb{R}}^X$ be two nonnegative (S)-integrable functions. Then*

$$(S)\int_X (f+g)\,\mathrm{d}P \leqslant (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

Moreover, if f and g are (S)-integrable, then $f+g$ is (S)-integrable too.

Proof. If either f or g is not (S)-integrable, then the assertion is trivial. If both f and g are (S)-integrable, then, by virtue of the inequality $0 \leqslant f+g \leqslant 2(f \vee g)$ and Proposition 3.11, we get that $f+g$ is (S)-integrable. For the remaining part, see [5]. □

Proceeding analogously to Corollary 7.6 of [20], p. 173, it is possible to prove

Theorem 5.4. *Let f be a measurable function and P an R-valued submodular capacity. Then f is (S)-integrable if and only if $|f|$ is (S)-integrable.*

Remark 5.5. We observe that, in general, the hypothesis of submodularity of P cannot be dropped, not even in the case $R = \mathbb{R}$: indeed, if P is a real-valued not submodular capacity, there exist some (S)-integrable functions f (with respect to P) such that $|f|$ is not (S)-integrable (see [20], Example 3.16, p. 161).

Similarly to [20], Corollary 7.7, p. 174 and Corollary 7.8, p. 175, it is easy to prove the following two theorems:

Theorem 5.6. *If $P\colon \mathscr{A} \to R$ is a mean and f, g are (S)-integrable, then*

$$(S)\int_X (f+g)\,\mathrm{d}P = (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

Theorem 5.7. *If $P\colon \mathscr{A} \to R$ is a capacity and f, g are (S)-integrable and comonotonic, then*

$$(S)\int_X (f+g)\,\mathrm{d}P = (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

Acknowledgements. The paper was suported by grant VEGA 1/0539/08 and grant APVV LPP-0046-06.

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Authors' addresses: Antonio Boccuto, Dipartimento di Matematica e Informatica, via Vanvitelli, 1-06123 Perugia, Italy, e-mail: boccuto@dipmat.unipg.it, boccuto@yahoo.it; Beloslav Riečan, Katedra Matematiky, Fakulta Prírodných Vied, Univerzita Mateja Bela, Tajovského 40, SK-974 01 Banská Bystrica, Slovakia, e-mail: riecan@mat.savba.sk, riecan@fpv.umb.sk.