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HENSTOCK-KURZWEIL AND MCSHANE PRODUCT INTEGRATION; DESCRIPTIVE DEFINITIONS

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Abstract. The Henstock-Kurzweil and McShane product integrals generalize the notion of the Riemann product integral. We study properties of the corresponding indefinite integrals (i.e. product integrals considered as functions of the upper bound of integration). It is shown that the indefinite McShane product integral of a matrix-valued function A is absolutely continuous. As a consequence we obtain that the McShane product integral of A over $[a,b]$ exists and is invertible if and only if A is Bochner integrable on $[a,b]$.

Keywords: Henstock-Kurzweil product integral, McShane product integral, Bochner product integral

MSC 2000: 28B05

Let an interval $[a,b]\subset\mathbb{R}$, $-\infty<a<b<+\infty$ be given. A pair (τ,J) of a point $\tau\in[a,b]$ and a compact interval $J\subset[a,b]$ is called a *tagged interval*, where τ is the *tag* of J.

A finite collection $\{(\tau_j,J_j)\colon\ j=1,\dots,k\}$ of tagged intervals is called an *M-system* if

$$\operatorname{Int}(J_i)\cap\operatorname{Int}(J_j)=\emptyset\qquad\text{for } i\neq j$$

(where $\operatorname{Int}(J)$ denotes the interior of the interval J). An *M-partition* is an M-system which moreover satisfies

$$\bigcup_{j=1}^{k} J_j=[a,b].$$

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An M-system (M-partition) $\{(\tau_j, J_j)\colon\ j = 1, \ldots, k\}$ for which

$$\tau_j \in J_j, \quad j = 1, \ldots, k$$

is called a *K-system (K-partition)* on $[a, b]$.

In the sequel we assume that every system of tagged intervals $\{(\tau_i, J_i)\}_{i=1}^{k}$ is ordered in such a way that

$$\sup J_i \leqslant \inf J_{i+1}, \quad i = 1, \ldots, k-1.$$

In other words, the notation $\{(\tau_i, [\xi_i, \eta_i])\}_{i=1}^{k}$ implies

$$a \leqslant \xi_1 \leqslant \eta_1 \leqslant \ldots \leqslant \xi_k \leqslant \eta_k \leqslant b.$$

Given a positive function $\delta\colon\ [a, b] \to (0, +\infty)$ called a *gauge* on $[a, b]$, a tagged interval (τ, J) is said to be *δ-fine* if

$$J \subset (\tau - \delta(\tau), \tau + \delta(\tau)).$$

Using this concept we can speak about *δ-fine systems* and *δ-fine partitions* $\{(\tau_j, J_j);\ j = 1, \ldots, k\}$ of the interval $[a, b]$ whenever (τ_j, J_j) is δ-fine for every $j = 1, \ldots, k$.

It is a well-known fact that given a gauge $\delta\colon\ [a, b] \to (0, +\infty)$ there exists a δ-fine K-partition of $[a, b]$. This result is called *Cousin's lemma*.

Assume that Y is a real Banach space with the norm $\|\cdot\|_Y$. Let us consider a function $f\colon\ [a, b] \to Y$ and assume that μ is the Lebesgue measure on the real line.

Definition 1. Assume that $f\colon\ [a, b] \to Y$ is given. The function f is called *McShane integrable* if there is an element $M_f \in Y$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - M_f \right\|_Y < \varepsilon$$

for every δ-fine M-partition $\{(t_i, J_i);\ i = 1, \ldots, k\}$ of $[a, b]$. The vector M_f is called *the McShane integral* of f over $[a, b]$.

Definition 2. Assume that $f\colon\ [a, b] \to Y$ is given. The function f is called *Henstock-Kurzweil integrable* if there is an element $K_f \in Y$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - K_f \right\|_Y < \varepsilon$$

for every δ-fine K-partition $\{(t_i, J_i);\ i = 1, \ldots, k\}$ of $[a, b]$. The vector K_f is called *the Henstock-Kurzweil integral* of f over $[a, b]$.

Assume now that X is a real Banach space. Denote by $L(X)$ the Banach space of bounded linear operators on X with the usual operator norm given by

$$\|A\| = \|A\|_{L(X)} = \sup_{\|x\|=1} \|Ax\|_X$$

for $A \in L(X)$. By I the identity operator in $L(X)$ will be denoted.

Let $\mathfrak{J}$ be the set of all compact subintervals in $[a,b]$. Assume that a point-interval function $V\colon\ [a,b] \times \mathfrak{J} \to L(X)$ is given. We denote

$$P(V,D) = \prod_{i=k}^{1} V(t_i,J_i) = V(t_k,J_k)V(t_{k-1},J_{k-1})\ \ldots\ V(t_1,J_1),$$

where $D = \{(t_i,J_i)\}_{i=1}^k$ is an arbitrary M-partition of $[a,b]$.

Definition 3. A function $V\colon\ [a,b] \times \mathfrak{J} \to L(X)$ is called *McShane product integrable over* $[a,b]$ if there exists $Q \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge $\delta\colon\ [a,b] \to (0,+\infty)$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine M-partition $D = \{(t_i,J_i);\ i = 1,\ldots,k\}$ of $[a,b]$.

The operator Q is called *the McShane product integral* of V over $[a,b]$ and we use the notation $Q = (\mathrm{M})\prod\limits_a^b V(t,\mathrm{d}t)$.

Definition 4. A function $V\colon\ [a,b] \times \mathfrak{J} \to L(X)$ is called *Henstock-Kurzweil product integrable over* $[a,b]$ if there exists $Q \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge $\delta\colon\ [a,b] \to (0,+\infty)$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine K-partition $D = \{(t_i,J_i);\ i = 1,\ldots,k\}$ of $[a,b]$.

The operator Q is called *the Henstock-Kurzweil product integral* of V over $[a,b]$ and we use the notation $Q = (\mathrm{HK})\prod\limits_a^b V(t,\mathrm{d}t)$.

Remark 5. A similar concept of product integration was introduced by J. Jarník and J. Kurzweil in [2] (see also [5]) for the case of $n \times n$-matrix valued point-interval functions V with K-partitions. The corresponding product integral was called the *Perron product integral* in [2]. This terminology originates in the well known fact that a real function $g\colon [a,b] \to \mathbb{R}$ is Perron integrable to the value $\int_a^b g(t)\,\mathrm{d}t \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\left|\sum_{i=1}^{k} g(t_i)\mu(J_i) - \int_a^b g(t)\,\mathrm{d}t\right| < \varepsilon$$

for every δ-fine K-partition $D = \{(t_i, J_i);\ i = 1,\dots,k\}$ of $[a,b]$.

The Henstock-Kurzweil and McShane product integrals generalize the notion of the Riemann product integral. A function $V\colon [a,b] \times \mathfrak{J} \to L(X)$ is called *Riemann product integrable* if there exists $Q \in L(X)$ such that for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every K-partition $D = \{(\tau_i, [\alpha_i, \alpha_{i+1}])\}_{i=1}^{k}$ of the interval $[a,b]$ which satisfies $\alpha_{i+1} - \alpha_i < \delta$ for $i = 1,\dots,k$. The study of the Riemann product integral was initiated in the work of V. Volterra; a modern treatment of the theory which is due to P. R. Masani can be found in [3].

Since evidently every δ-fine K-partition is also a δ-fine M-partition we obtain the following statement.

Proposition 6. *If $V\colon [a,b] \times \mathfrak{J} \to L(X)$ is McShane product integrable then it is also Henstock-Kurzweil product integrable and*

$$(\mathrm{HK})\prod_a^b V(t,\mathrm{d}t) = (\mathrm{M})\prod_a^b V(t,\mathrm{d}t).$$

Let us mention that a similar statement holds also for the integrals based on integral sums presented in Definitions 1 and 2.

We now introduce a condition concerning the point-interval function $V\colon [a,b] \times \mathfrak{J} \to L(X)$.

Condition (C). *For every $t \in [a,b]$ and $\zeta > 0$ there exists $\sigma = \sigma(t) > 0$ such that*

$$\|V(t,J) - I\| < \zeta$$

for any interval $J \subset [a,b]$ such that $J \subset (t-\sigma, t+\sigma)$.

Typical cases of V satisfying condition **(C)** are

$$V_1(t,J) = I + A(t)\mu(J)$$

and

$$V_2(t,J) = \mathrm{e}^{A(t)\mu(J)}$$

where $A\colon [a,b] \to L(X)$ and μ is the Lebesgue measure on the real line. The corresponding product integrals are usually denoted $\prod\limits_a^b (I + A(t)\,\mathrm{d}t)$ and $\prod\limits_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$. These integrals are particularly interesting since they can be used to solve the differential equation $x'(t) = Ax(t)$, where $x\colon [a,b] \to X$.

The following result was proved in [4] for the McShane product integral and in [2] for the Henstock-Kurzweil product integral (in the case $X = \mathbb{R}^n$).

Theorem 7. *Consider a function $V\colon [a,b] \times \mathfrak{J} \to L(X)$ such that the McShane (Henstock-Kurzweil) integral $\prod\limits_a^b V(t,\mathrm{d}t)$ exists and is invertible. If the function V satisfies condition* **(C)**, *then for every $s \in [a,b]$ the McShane (Henstock-Kurzweil) product integrals*

$$\prod_a^s V(t,\mathrm{d}t),\ \prod_s^b V(t,\mathrm{d}t)$$

exist, the equality

$$\prod_s^b V(t,\mathrm{d}t) \prod_a^s V(t,\mathrm{d}t) = \prod_a^b V(t,\mathrm{d}t)$$

holds and there exists a constant $K > 0$ such that

$$\Big\| \prod_a^s V(t,\mathrm{d}t) \Big\| \leqslant K, \quad \Big\| \Big(\prod_a^s V(t,\mathrm{d}t) \Big)^{-1} \Big\| \leqslant K$$

for $s \in [a,b]$.

Example 8. We now demonstrate the existence of a function $A\colon [a,b] \to L(\mathbb{R}^n)$ such that the McShane product integral $(\mathrm{M})\prod\limits_a^b \mathrm{e}^{A(x)\,\mathrm{d}x}$ is not invertible. Define $f(x) = -1/x$ for $x \in (0,1]$ and $f(0) = 0$. We will show that

$$(\mathrm{M}) \prod_0^1 \mathrm{e}^{f(x)\,\mathrm{d}x} = 0.$$

To simplify the notation we have identified the real function $x \mapsto f(x)$ with a 1×1 matrix valued function $x \mapsto \{f(x)\}$. Choose an arbitrary $N \in \mathbb{N}$ and define

$$\delta(x) = \frac{1}{16} \cdot \frac{1}{2^N}, \quad x \in [0,1].$$

This is a constant function and we can write δ instead of $\delta(x)$. Let

$$D = \{(\tau_j, [\alpha_{j-1}, \alpha_j]);\ j = 1, \ldots, m\}$$

be a δ-fine M-partition of $[0, 1]$, i.e.

$$\tau_j - \delta < \alpha_{j-1} \leqslant \alpha_j < \tau_j + \delta$$

for $j = 1, \ldots, m$. Since $\alpha_j - \alpha_{j-1} < 2 \cdot \delta = 1/(8 \cdot 2^N)$, to every $i \in \{1, \ldots, N\}$ we can find indices $j_1(i)$ and $j_2(i)$ such that

$$\begin{aligned} \alpha_{j_1(i)} &\in \Bigl(\frac{1}{2^i} + \frac{1}{8} \cdot \frac{1}{2^i}, \frac{1}{2^i} + \frac{2}{8} \cdot \frac{1}{2^i}\Bigr], \\ \alpha_{j_2(i)} &\in \Bigl[\frac{1}{2^{i-1}} - \frac{2}{8} \cdot \frac{1}{2^i}, \frac{1}{2^{i-1}} - \frac{1}{8} \cdot \frac{1}{2^i}\Bigr). \end{aligned}$$

Consequently,

$$\begin{gathered} 1 < j_1(N) < j_2(N) < j_1(N-1) < j_2(N-1) < \ldots < j_1(1) < j_2(1) < m, \\ \alpha_{j_2(i)} - \alpha_{j_1(i)} \geqslant \frac{1}{2^{i-1}} - \frac{2}{8} \cdot \frac{1}{2^i} - \frac{1}{2^i} - \frac{2}{8} \cdot \frac{1}{2^i} = \frac{1}{2^{i+1}} \end{gathered}$$

and for every $j \in \mathbb{N}$ such that $j_1(i) + 1 \leqslant j \leqslant j_2(i)$ we have

$$\begin{aligned} \tau_j &> \alpha_{j-1} - \delta \geqslant \alpha_{j_1(i)} - \delta > \frac{1}{2^i} + \frac{1}{8} \cdot \frac{1}{2^i} - \frac{1}{16} \cdot \frac{1}{2^N} > \frac{1}{2^i}, \\ \tau_j &< \alpha_j + \delta \leqslant \alpha_{j_2(i)} + \delta < \frac{1}{2^{i-1}} - \frac{1}{8} \cdot \frac{1}{2^i} + \frac{1}{16} \cdot \frac{1}{2^N} < \frac{1}{2^{i-1}}, \end{aligned}$$

i.e.

$$2^{i-1} < \frac{1}{\tau_j} < 2^i.$$

Finally,

$$\begin{aligned} -\sum_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1}) &\geqslant \sum_{i=1}^{N} \sum_{j=j_1(i)+1}^{j_2(i)} \frac{1}{\tau_j}(\alpha_j - \alpha_{j-1}) \\ &> \sum_{i=1}^{N} 2^{i-1}(\alpha_{j_2(i)} - \alpha_{j_1(i)}) \geqslant \sum_{i=1}^{N} 2^{i-1} \frac{1}{2^{i+1}} = \frac{N}{4} \end{aligned}$$

and

$$0 < \prod_{j=1}^{m} \mathrm{e}^{f(\tau_j)(\alpha_j - \alpha_{j-1})} = \exp\Bigl(\sum_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1})\Bigr) < \mathrm{e}^{-\frac{N}{4}}.$$

If we choose $N \in \mathbb{N}$ such that $N \geqslant -4 \log \varepsilon$, we have

$$0 < \prod_{j=1}^{m} \mathrm{e}^{f(\tau_j)(\alpha_j - \alpha_{j-1})} < \varepsilon$$

for every δ-fine M-partition of $[0,1]$, which means that

$$(\mathrm{M}) \prod_{0}^{1} \mathrm{e}^{f(x)\,\mathrm{d}x} = 0.$$

Observe that because δ is a constant function, the Riemann product integral exists as well and

$$(\mathrm{R}) \prod_{0}^{1} \mathrm{e}^{f(x)\,\mathrm{d}x} = 0.$$

Example 9. Define again $f(x) = -1/x$ for $x \in (0,1]$ and $f(0) = 0$. We will prove that

$$(\mathrm{M}) \prod_{0}^{1} (1 + f(x)\,\mathrm{d}x) = 0.$$

Given $\varepsilon > 0$ we have to show there is a gauge $\delta\colon\ [0,1] \to (0,\infty)$ such that

$$\left| \prod_{j=1}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) \right| < \varepsilon$$

for every δ-fine M-partition $D = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^{m}$ of the interval $[0,1]$.

The first condition that we impose on δ is that $\delta(x) < x/2$ for $x \in (0,1]$, which will guarantee that

$$1 + f(\tau_j)(\alpha_j - \alpha_{j-1}) > 0$$

for $j = 1, \ldots, m$. This is indeed true in the case τ_j=0. Otherwise the inequality

$$\tau_j - \delta(\tau_j) < \alpha_{j-1} \leqslant \alpha_j < \tau_j + \delta(\tau_j)$$

implies

$$1 + f(\tau_j)(\alpha_j - \alpha_{j-1}) = 1 - \frac{1}{\tau_j}(\alpha_j - \alpha_{j-1}) > 1 - \frac{2 \cdot \delta(\tau_j)}{\tau_j} > 0.$$

The well-known inequality

$$x_1 \ldots x_m \leqslant \Big(\frac{x_1 + \ldots + x_m}{m} \Big)^m$$

(which holds for non-negative numbers $x_1, \dots, x_m$) yields the estimate

$$0 < \prod_{j=1}^{m}(1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) \leqslant \left(\frac{\sum\limits_{j=1}^{m}(1 + f(\tau_j)(\alpha_j - \alpha_{j-1}))}{m} \right)^m$$
$$= \left(1 + \frac{\sum\limits_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1})}{m} \right)^m .$$

If we now require

$$\delta(x) < \frac{1}{16} \cdot \frac{1}{2^N}, \quad x \in [0,1],$$

where N is an arbitrary fixed natural number, we have (see Example 8)

$$\sum_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1}) < -N/4.$$

Since

$$\lim_{k \to \infty} \left(1 - \frac{N/4}{k}\right)^k = \mathrm{e}^{-N/4},$$

there exists $k_0(N) \in \mathbb{N}$ such that

$$\left| \left(1 - \frac{N/4}{k}\right)^k - \mathrm{e}^{-N/4} \right| < 1/N$$

for every $k \geqslant k_0(N)$. If $\delta(x) < 1/(2 \cdot k_0(N))$, then every δ-fine M-partition satisfies $\alpha_j - \alpha_{j-1} < 1/k_0(N)$ and therefore consists of $m \geqslant k_0(N)$ subintervals of $[0,1]$. From these facts we conclude that

$$0 < \prod_{j=1}^{m}(1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) < \left(1 - \frac{N/4}{m}\right)^m < \mathrm{e}^{-N/4} + 1/N.$$

It is now easy to complete the proof: Given $\varepsilon > 0$, the number N can be chosen to be greater than $\max(2/\varepsilon, -4\log(\varepsilon/2))$. The gauge $\delta\colon [0,1] \to (0,\infty)$ is an arbitrary function such that

$$\delta(x) < \min\left(\frac{x}{2}, \frac{1}{16} \cdot \frac{1}{2^N}, \frac{1}{2 \cdot k_0(N)}\right)$$

for $x \in (0,1]$ and

$$\delta(0) < \min\left(\frac{1}{16} \cdot \frac{1}{2^N}, \frac{1}{2 \cdot k_0(N)}\right).$$

Then

$$0 < \prod_{j=1}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) < \varepsilon$$

for every δ-fine M-partition of $[0,1]$, which means that

$$(\mathrm{M}) \prod_{0}^{1} (1 + f(x)\,\mathrm{d}x) = 0.$$

It is perhaps interesting to note that the Riemann product integral

$$(\mathrm{R}) \prod_{0}^{1} (1 + f(x)\,\mathrm{d}x)$$

does not exist. This follows from Masani's result (see [3]) which says that $(\mathrm{R}) \prod_{a}^{b}(I + A(x)\,\mathrm{d}x)$ can exist only for bounded functions; a direct verification is also easy: If the Riemann integral exists, it must be equal to the McShane integral which is zero. Therefore for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \prod_{j=1}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) \right| < \varepsilon$$

for every partition

$$0 = \alpha_0 \leqslant \tau_1 \leqslant \alpha_1 \leqslant \ldots \leqslant \tau_m \leqslant \alpha_m = 1$$

such that $\alpha_j - \alpha_{j-1} < \delta$, $j = 1, \ldots, m$. Take such a partition which moreover satisfies $\alpha_1 > 0$,

$$1 + f(\tau_j)(\alpha_j - \alpha_{j-1}) \neq 0,\ j = 1, \ldots, m$$

(this can achieved by choosing $\tau_j \neq \alpha_j - \alpha_{j-1}$) and

$$0 < \tau_1 < \frac{\alpha_1}{\left| \prod\limits_{j=2}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) \right|^{-1} + 1}.$$

Then

$$|1 + f(\tau_1)(\alpha_1 - \alpha_0)| = \left| 1 - \frac{\alpha_1}{\tau_1} \right| = \frac{\alpha_1}{\tau_1} - 1 > \left| \prod_{j=2}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) \right|^{-1}$$

and therefore

$$\left|\prod_{j=1}^{m}(1+f(\tau_j)(\alpha_j-\alpha_{j-1}))\right|>1,$$

which is a contradiction.

This example (together with Example 8) shows that the Riemann product integrals $\prod_a^b(I+A(x)\,\mathrm{d}x)$ and $\prod_a^b \mathrm{e}^{A(x)\,\mathrm{d}x}$ do not always coincide.

2. The indefinite product integral

Assume that $V\colon [a,b]\times\mathfrak{J}\to L(X)$ satisfies condition **(C)** and that the integral (M) $\prod_a^b V(t,\mathrm{d}t)$ exists and is invertible. Let

$$U_M(s)=(\mathrm{M})\prod_a^s V(t,\mathrm{d}t),\ s\in(a,b],\ U_M(a)=I \tag{1}$$

denote the indefinite McShane product integral of V defined for $s\in[a,b]$. By Theorem 7 this definition makes sense. We define in a similar way the indefinite Henstock-Kurzweil product integral

$$U_{HK}(s)=(HK)\prod_a^s V(t,\mathrm{d}t),\ s\in(a,b],\ U_{HK}(a)=I \tag{2}$$

provided (HK) $\prod_a^b V(t,\mathrm{d}t)$ exists and is invertible. Let us note that

$$(\mathrm{M})\prod_\alpha^\beta V(t,\mathrm{d}t)=U_M(\beta)U_M^{-1}(\alpha),$$
$$(\mathrm{HK})\prod_\alpha^\beta V(t,\mathrm{d}t)=U_{HK}(\beta)U_{HK}^{-1}(\alpha).$$

Also by Proposition 6 we have $U_M(s)=U_{HK}(s)$ if both functions are defined.

The following lemma appeared in [2]; for reader's convenience we repeat both its statement and proof.

Lemma 10. *Let $V\colon [a,b]\times\mathfrak{J}\to L(X)$ be McShane (Henstock-Kurzweil) product integrable with $\prod\limits_a^b V(t,\mathrm{d}t)=Q$, where Q is an invertible operator. Assume that V satisfies condition* **(C)**. *For $\varepsilon>0$ find a gauge $\delta\colon [a,b]\to(0,+\infty)$ such that*

$$\|P(V,D)-Q\|<\varepsilon$$

for every δ-fine M-partition (K-partition) D of $[a,b]$. Let $\{(\tau_j,[\xi_j,\eta_j])\}_{j=1}^r$ be a δ-fine M-system (K-system) on $[a,b]$. If we define

$$U^{-1}(\eta_j)V(\tau_j,[\xi_j,\eta_j])U(\xi_j)=I+Z_j$$

for $j=1,\dots,r$, U is the corresponding indefinite product integral, then

$$\|(I+Z_r)(I+Z_{r-1})\dots(I+Z_1)-I\|\leqslant\|Q^{-1}\|\varepsilon. \tag{3}$$

P r o o f. Denote $\eta_0=a$ and $\xi_{r+1}=b$. Since the product integral exists over all intervals of the form $[\eta_j,\xi_{j+1}]$, $j=0,\dots,r$, for any $\omega>0$ there exist gauges δ_j on $[\eta_j,\xi_{j+1}]$ such that $\delta_j(t)<\delta(t)$ and

$$\Bigl\|P(V,D_j)-\prod_{\eta_j}^{\xi_{j+1}}V(t,\mathrm{d}t)\Bigr\|=\|P(V,D_j)-U(\xi_{j+1})U^{-1}(\eta_j)\|<\omega \tag{4}$$

for every δ_j-fine M-partition (K-partition) D_j of $[\eta_j,\xi_{j+1}]$. Composing the partitions, we obtain that

$$D=D_0\circ(\tau_1,[\xi_1,\eta_1])\circ\dots D_{r-1}\circ(\tau_r,[\xi_r,\eta_r])\circ D_r$$

is a δ-fine M-partition (K-partition) of the interval $[a,b]$ and therefore

$$\begin{aligned}&\|Q-P(V,D)\|\\&\qquad=\|Q-P(V,D_r)V(\tau_r,[\xi_r,\eta_r])\dots P(V,D_1)V(\tau_1,[\xi_1,\eta_1])P(V,D_0)\|<\varepsilon.\end{aligned}$$

This yields

$$\begin{aligned}&\|I-Q^{-1}P(V,D_r)V(\tau_r,[\xi_r,\eta_r])\dots P(V,D_1)V(\tau_1,[\xi_1,eta_1])P(V,D_0)\|\\&=\|Q^{-1}(Q-P(V,D_r)V(\tau_r,[\xi_r,\eta_r])\dots P(V,D_1)V(\tau_1,[\xi_1,\eta_1])P(V,D_0))\|<\|Q^{-1}\|\varepsilon,\end{aligned}$$

which can be also written in the form

$$\begin{aligned}&\|I-U(b)^{-1}P(V,D_r)U(\eta_r)U^{-1}(\eta_r)V(\tau_r,[\xi_r,\eta_r])U(\xi_r)U^{-1}(\xi_r)\dots\\&\quad P(V,D_1)U(\eta_1)U^{-1}(\eta_1)V(\tau_1,[\xi_1,\eta_1])U(\xi_1)U^{-1}(\xi_1)P(V,D_0)\|<\|Q^{-1}\|\varepsilon.\end{aligned} \tag{5}$$

Now we take

$$U^{-1}(\xi_{j+1})P(V,D_j)U(\eta_j) - I = W_j$$

for $j = 0,1,\ldots,r$. Then using (4) and Theorem 7 we obtain

$$\begin{aligned}
\|W_j\| &= \|U^{-1}(\xi_{j+1})P(V,D_j)U(\eta_j) - I\| \\
&\leqslant \|U^{-1}(\xi_{j+1})\|\|P(V,D_j) - U(\xi_{j+1})U^{-1}(\eta_j)\|\|U(\eta_j)\| \\
&\leqslant \|U^{-1}(\xi_{j+1})\|\|U(\eta_j)\|\omega \leqslant K^2\omega
\end{aligned} \tag{6}$$

for $j = 0,1,\ldots,r$. Looking at the definitions of Z_j and W_j we rewrite the inequality (5) as

$$\|I - (I+W_r)(I+Z_r)\ldots(I+W_1)(I+Z_1)(I+W_0)\| \leqslant \|Q^{-1}\|\varepsilon.$$

Now we have

$$\begin{aligned}
&\|I - (I+Z_r)\ldots(I+Z_1)\| \\
&\quad\leqslant \|I - (I+W_r)(I+Z_r)\ldots(I+W_1)(I+Z_1)(I+W_0)\| \\
&\qquad + \|(I+W_r)(I+Z_r)\ldots(I+W_1)(I+Z_1)(I+W_0) - (I+Z_r)\ldots(I+Z_1)\| \\
&\quad\leqslant \|Q^{-1}\|\varepsilon
\end{aligned}$$

because (6) implies that

$$\|(I+W_r)(I+Z_r)\ldots(I+W_1)(I+Z_1)(I+W_0) - (I+Z_r)\ldots(I+Z_1)\|$$

is arbitrarily small if $\omega > 0$ is small enough. □

Theorem 11. *Consider a function $V\colon [a,b]\times\mathfrak{J} \to L(X)$ which satisfies condition* **(C)**. *Assume that the McShane (Henstock-Kurzweil) product integral $\prod\limits_a^b V(t,\mathrm{d}t) = Q$ exists and is invertible. Then the indefinite integral*

$$\begin{aligned}
U(s) &= \prod_a^s (I + A(t)\,\mathrm{d}t),\ \ s \in (a,b], \\
U(a) &= I
\end{aligned} \tag{7}$$

is continuous at every point $s \in [a,b]$.

P r o o f. We present the proof for the McShane product integral only; the proof for the Henstock-Kurzweil integral case is similar and was given in [2] for the case $X = \mathbb{R}^n$, i.e. for the case of $n \times n$ matrices.

Given $\varepsilon > 0$ let $\delta\colon [a,b] \to (0,+\infty)$ be the gauge such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine M-partition $D = \{(t_i, J_i)\colon\ i = 1,\dots,k\}$ of $[a,b]$. By condition **(C)**, for every $s \in [a,b]$ and $\varepsilon > 0$ there exists $\sigma(s) > 0$ such that

$$\|V(s,J) - I\| < \varepsilon$$

for any interval $J \subset [a,b] \cap (s-\sigma, s+\sigma)$. Assume that $s \in [a,b)$ is given and let $t \in (s,b]$ satisfy $s < t < s + \delta_0(s)$, where $0 < \delta_0(s) < \min(\delta(s), \sigma(s))$. Let $D_1 = \{(t_i, [\alpha_{i-1}, \alpha_i])\}_{i=1}^{l}$ be a δ-fine M-partition of $[a,s]$ and let us set

$$D_2 = D_1 \circ (s, [s,t]).$$

Then D_2 is evidently a δ-fine M-partition of $[a,t]$. We have

$$\begin{aligned}
&U^{-1}(s)P(V,D_1) - I = U^{-1}(\alpha_l)P(V,D_1) - I\\
&\quad = U^{-1}(\alpha_l)V(t_l,[\alpha_{l-1},\alpha_l])V(t_{l-1},[\alpha_{l-2},\alpha_{l-1}])\dots V(t_1,[\alpha_0,\alpha_1]) - I\\
&\quad = U^{-1}(\alpha_l)V(t_l,[\alpha_{l-1},\alpha_l])U(\alpha_{l-1})U^{-1}(\alpha_{l-1})V(t_{l-1},[\alpha_{l-2},\alpha_{l-1}])U(\alpha_{l-2})\\
&\qquad U^{-1}(\alpha_{l-2})\dots U(\alpha_1)U^{-1}(\alpha_1)V(t_1,[\alpha_0,\alpha_1])U(\alpha_0) - I
\end{aligned}$$

because $U(\alpha_0) = U(a) = I$. Denote

$$U^{-1}(\alpha_j)V(t_j,[\alpha_{j-1},\alpha_j])U(\alpha_{j-1}) - I = Z_j$$

for $j = 1,\dots,l$. Then Lemma 10 and especially (3) imply

$$\|U^{-1}(s)P(V,D_1) - I\| = \|(I+Z_l)(I+Z_{l-1})\dots(I+Z_1) - I\| \leqslant \|Q^{-1}\|\varepsilon$$

and by Theorem 7 we get

$$\|P(V,D_1) - U(s)\| \leqslant \|U(s)\|\|U^{-1}(s)P(V,D_1) - I\| \leqslant K\|Q^{-1}\|\varepsilon.$$

In a fully analogous way we obtain

$$\|P(V,D_2) - U(t)\| \leqslant K\|Q^{-1}\|\varepsilon.$$

Now by the form of condition **(C)** from the beginning of the proof we have

$$\begin{aligned}\|U(t)-U(s)\| &\leqslant \|P(V,D_2)-U(t)\| \\ &\qquad + \|P(V,D_1)-U(s)\| + \|P(V,D_2)-P(V,D_1)\| \\ &\leqslant 2K\|Q^{-1}\|\varepsilon + \|P(V,D_2)-P(V,D_1)\| \\ &= 2K\|Q^{-1}\|\varepsilon + \|V(s,[s,t])P(V,D_1)-P(V,D_1)\| \\ &\leqslant 2K\|Q^{-1}\|\varepsilon + \|V(s,[s,t])-I\|\cdot\|P(V,D_1)\| \\ &\leqslant 2K\|Q^{-1}\|\varepsilon + K\varepsilon = K(2\|Q^{-1}\|+1)\varepsilon\end{aligned}$$

and this proves the continuity of U from the right at the point s. The left continuity of U at $s \in (a,b]$ can be shown analogously. □

The following lemma has been taken over from [2].

Lemma 12. *Let $A_1, A_2, \ldots, A_k \in L(X)$ with $\sum_{i=1}^{k} \|A_i\| \leqslant 1$. Then*

$$\Big\|(I+A_k)(I+A_{k-1})\ldots(I+A_1) - I - \sum_{i=1}^{k} A_i\Big\| \leqslant \Big(\sum_{i=1}^{k}\|A_i\|\Big)^2.$$

P r o o f. Put $\lambda_i = \|A_i\|$ for $i = 1, \ldots, k$ and $\lambda = \sum_{i=1}^{k} \lambda_i \leqslant 1$. Then

$$\begin{aligned}&(1+\lambda_k)(1+\lambda_{k-1})\ldots(1+\lambda_1) \\ &\qquad = 1 + \sum_{i=1}^{k}\lambda_i + \sum_{j_2>j_1}\lambda_{j_2}\lambda_{j_1} + \sum_{j_3>j_2>j_1}\lambda_{j_3}\lambda_{j_2}\lambda_{j_1} + \ldots + \lambda_k\lambda_{k-1}\ldots\lambda_1 \\ &\qquad \leqslant \mathrm{e}^{\lambda_k}\mathrm{e}^{\lambda_{k-1}}\ldots\mathrm{e}^{\lambda_1} = \mathrm{e}^{\lambda}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j_2>j_1}\lambda_{j_2}\lambda_{j_1} &+ \sum_{j_3>j_2>j_1}\lambda_{j_3}\lambda_{j_2}\lambda_{j_1} + \ldots + \lambda_k\lambda_{k-1}\ldots\lambda_1 \\ &\leqslant \mathrm{e}^{\lambda} - 1 - \lambda \leqslant \lambda^2\sum_{k=2}^{\infty}\frac{1}{k!} = \lambda^2(\mathrm{e}-2) \leqslant \lambda^2.\end{aligned} \tag{8}$$

Now

$$\begin{aligned}B &= (I+A_k)(I+A_{k-1})\ldots(I+A_1) - I - \sum_{i=1}^{k} A_i \\ &= \sum_{j_2>j_1} A_{j_2}A_{j_1} + \sum_{j_3>j_2>j_1} A_{j_3}A_{j_2}A_{j_1} + \ldots + A_kA_{k-1}\ldots A_1.\end{aligned}$$

Therefore by (8) we obtain

$$\|B\| \leqslant \sum_{j_2>j_1} \|A_{j_2}\|\|A_{j_1}\| + \sum_{j_3>j_2>j_1} \|A_{j_3}\|\|A_{j_2}\|\|A_{j_1}\|$$
$$+ \ldots + \|A_k\|\|A_{k-1}\| \ldots \|A_1\| < \lambda^2 = \Bigl(\sum_{i=1}^{k} \|A_i\|\Bigr)^2.$$

□

3. Finite-dimensional case

At this point we switch to the case $X = \mathbb{R}^n$; the operators in $L(\mathbb{R}^n)$ are now represented by real $n \times n$ matrices.

For a matrix $A = (a_{i,j})_{i,j=1}^n$ we define a special norm

$$\|A\|_\bigstar = \max_{1\leqslant i,j\leqslant n} |a_{i,j}|. \tag{9}$$

Let us mention that all norms on $L(\mathbb{R}^n)$ are equivalent. This means in particular that if $\|\cdot\|$ is an arbitrary norm defined on the linear space of matrices, then there is a constant $L \geqslant 1$ such that

$$\frac{1}{L}\|A\|_\bigstar \leqslant \|A\| \leqslant L\|A\|_\bigstar.$$

The following important statement was presented in [2].

Lemma 13. *Let $0 < \theta < 1/9$. Assume that $Z_1, Z_2, \ldots, Z_r \in L(\mathbb{R}^n)$ are such that for every p-tuple $\{j_1, \ldots, j_p\} \subset \{1, 2, \ldots, r\}$ with $j_1 < j_2 < \ldots < j_p$ the inequality*

$$\|(I + Z_{j_p})(I + Z_{j_{p-1}}) \ldots (I + Z_{j_1}) - I\|_\bigstar \leqslant \theta \tag{10}$$

holds. Then

$$\sum_{j=1}^{r} \|Z_j\|_\bigstar \leqslant 4n^2\theta. \tag{11}$$

P r o o f. By (10) we have

$$\|Z_j\|_\bigstar = \|(I + Z_j) - I\|_\bigstar \leqslant \theta,\ j = 1, \ldots, r. \tag{12}$$

Denote $Z_j = (z_{j;i,k})_{i,k=1}^n$ and for $l, m \in \{1, 2, \dots, n\}$ set

$$J(l,m) = \{j \in \{1, \dots, r\};\ \|Z_j\|_\bigstar = \max_{i,k} |z_{j;i,k}| = |z_{j;l,m}|\}.$$

In case (11) is not valid we can find a couple $l, m \in \{1, 2, \dots, n\}$ such that

$$\sum_{j \in J(l,m)} \|Z_j\|_\bigstar > 4\theta.$$

Put

$$J_+ = \{j \in J(l,m)\ ; z_{j;l,m} \geqslant 0\}, \quad J_- = J(l,m) \setminus J_+.$$

Then either

$$\sum_{j \in J_+} z_{j;l,m} > 2\theta$$

or

$$-\sum_{j \in J_-} z_{j;l,m} > 2\theta.$$

Assume e.g. that the first inequality occurs. By (12) we have $z_{j;l,m} = \|Z_j\|_\bigstar \leqslant \theta$ for $j \in J_+$ and therefore there is a subset $J_+^* \subset J_+$ such that

$$2\theta < \sum_{j \in J_+^*} z_{j;l,m} \leqslant 3\theta. \tag{13}$$

Hence

$$2\theta < \Big\| \sum_{j \in J_+^*} Z_j \Big\|_\bigstar = \sum_{j \in J_+^*} \|Z_j\|_\bigstar = \sum_{j \in J_+^*} z_{j;l,m} \leqslant 3\theta < \frac{1}{3}. \tag{14}$$

The matrices Z_j, $j \in J_+^*$ satisfy the assumptions of Lemma 12 and therefore

$$\Big\| \prod_{j \in J_+^*} (I + Z_j) - I - \sum_{j \in J_+^*} Z_j \Big\|_\bigstar \leqslant \Big(\sum_{j \in J_+^*} \|Z_j\|_\bigstar \Big)^2 \leqslant 9\theta^2.$$

By (10) we get

$$\Big\| \sum_{j \in J_+^*} Z_j \Big\|_\bigstar \leqslant \Big\| \prod_{j \in J_+^*} (I + Z_j) - I - \sum_{j \in J_+^*} Z_j \Big\|_\bigstar + \Big\| \prod_{j \in J_+^*} (I + Z_j) - I \Big\|_\bigstar \leqslant 9\theta^2 + \theta$$

and by (14) also

$$2\theta < \Big\| \sum_{j \in J_+^*} Z_j \Big\|_\bigstar \leqslant 9\theta^2 + \theta.$$

Therefore $\theta > 1/9$, which is a contradiction. □

At this moment it should be pointed out that an analog of the preceding Lemma 13 does not hold for infinite-dimensional Banach spaces. The counter-example from [5], p. 389 concerns the Banach space $X = c_0$.

For this reason we restrict our considerations to the case $X = \mathbb{R}^n$ in the sequel. Using Lemma 13 we prove the next result (see [2]).

Theorem 14. *Consider a function $V\colon [a,b]\times\mathfrak{J} \to L(\mathbb{R}^n)$ which satisfies condition* **(C)**. *Assume the McShane (Henstock-Kurzweil) product integral $\prod\limits_a^b V(t,\mathrm{d}t) = Q$ exists and is invertible.*

Given $0 < \varepsilon < (9\|Q^{-1}\|_\bigstar)^{-1}$, find a gauge $\delta\colon [a,b] \to (0,+\infty)$ such that

$$\|P(V,D) - Q\|_\bigstar < \varepsilon$$

for every δ-fine M-partition (K-partition) D of $[a,b]$. Let $\{(\tau_j,[\xi_j,\eta_j])\}_{j=1}^r$ be a δ-fine M-system (K-system) on $[a,b]$. Define

$$U^{-1}(\eta_j)V(\tau_j,[\xi_j,\eta_j])U(\xi_j) = I + Z_j,\ j = 1,\dots,r,$$

U is the corresponding indefinite product integral. Then

$$\sum_{j=1}^r \|Z_j\|_\bigstar \leqslant 4n^2\|Q^{-1}\|_\bigstar\varepsilon \tag{15}$$

and

$$\sum_{j=1}^r \Big\|V(\tau_j,[\xi_j,\eta_j]) - \prod_{\xi_j}^{\eta_j} V(t,\mathrm{d}t)\Big\|_\bigstar \leqslant 4K^2n^2\|Q^{-1}\|_\bigstar\varepsilon, \tag{16}$$

where K is the constant from Theorem 7.

P r o o f. By (3) from Theorem 10 we have the inequality

$$\|(I+Z_{j_p})(I+Z_{j_{p-1}})\dots(I+Z_{j_1}) - I\|_\bigstar \leqslant \|Q^{-1}\|_\bigstar\varepsilon < \frac{1}{9}$$

for every p-tuple $\{j_1,\dots,j_p\} \subset \{1,2,\dots,r\}$ with $j_1 < j_2 < \dots < j_p$. Hence by Lemma 13 we obtain

$$\sum_{j=1}^r \|Z_j\|_\bigstar \leqslant 4n^2\|Q^{-1}\|_\bigstar\varepsilon.$$

To show (16) we take into account that for $j = 1, \dots, r$ we have

$$V(\tau_j, [\xi_j, \eta_j]) - \prod_{\xi_j}^{\eta_j} V(t, \mathrm{d}t) = V(\tau_j, [\xi_j, \eta_j]) - U(\eta_j)U^{-1}(\xi_j) = U(\eta_j)Z_jU^{-1}(\xi_j)$$

Hence

$$\left\| V(\tau_j, [\xi_j, \eta_j]) - \prod_{\xi_j}^{\eta_j} V(t, \mathrm{d}t) \right\|_\bigstar \leqslant \|U(\eta_j)\|_\bigstar \|Z_j\|_\bigstar \|U^{-1}(\xi_j)\|_\bigstar.$$

Now (15) and Theorem 7 imply (16). □

The following theorem also appeared in [2].

Theorem 15. *Consider a function $V\colon [a,b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ which satisfies condition* **(C)**. *Assume that the product integral* (HK) $\prod_a^b V(t, \mathrm{d}t) = Q$ *exists and is invertible. Then there exists a set $E \subset [a,b]$, $\mu(E) = 0$ such that for every $\varepsilon > 0$, $t \in [a,b] \setminus E$, there is $\vartheta > 0$ such that*

$$\|V(t, [x,y]) - U_{HK}(y)U_{HK}^{-1}(x)\|_\bigstar \leqslant \varepsilon(y - x) \tag{17}$$

provided $t - \vartheta < x \leqslant t \leqslant y < t + \vartheta$, $x, y \in [a,b]$.

Proof. Assume that $T \subset [a,b]$ is the set of all $t \in [a,b]$ for which (17) holds; set $E = [a,b] \setminus T$. Given $r \in \mathbb{N}$ denote by E_r the set of $t \in [a,b]$ such that there exist sequences $x_l = x_l(t)$, $y_l = y_l(t)$, $l \in \mathbb{N}$ with

$$x_l \leqslant t \leqslant y_l, \; y_l - x_l \to 0 \quad \text{as} \quad l \to \infty$$

and

$$\|V(t, [x_l, y_l]) - U_{HK}(y_l)U_{HK}^{-1}(x_l)\|_\bigstar \geqslant \frac{1}{r}(y_l - x_l). \tag{18}$$

Then $E = \bigcup_{r=1}^{\infty} E_r$. Assume that $\mu_e(E) > 0$, where $\mu_e(E)$ is the outer measure of the set $E \subset [a,b]$. Then there is an $r \in \mathbb{N}$ such that $\mu_e(E_r) > 0$. Choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{9}\|Q^{-1}\|_\bigstar,$$

$$4K^2n^2\|Q^{-1}\|_\bigstar \varepsilon < \frac{1}{2r}\mu_e(E_r) \tag{19}$$

($K>0$ is the constant from Theorem 7). Find a gauge δ on $[a,b]$ such that

$$\|P(V,D)-Q\|_\star<\varepsilon$$

for every δ-fine K-partition D of $[a,b]$. For $t\in E$ find $l_0(t)\in\mathbb{N}$ such that

$$t-\delta(t)<x_l(t)\leqslant t\leqslant y_l(t)<t+\delta(t)$$

for all $l\geqslant l_0$. The system of intervals

$$\{[x_l(t),y_l(t)];\ t\in E, l\geqslant l_0(t)\}$$

is a Vitali cover of the set E and by the Vitali covering theorem it contains a finite subsystem of intervals $\{[\xi_j,\eta_j]\}_{j=1}^s$ for which

$$\begin{gathered}\tau_j-\delta(\tau_j)<\xi_j\leqslant\tau_j\leqslant\eta_j<\tau_j+\delta(\tau_j),\ \tau_j\in E,\ j=1,2,\dots,s,\\ \eta_j\leqslant\xi_{j+1},\ j=1,2,\dots,s-1\end{gathered}$$

and

$$\mu_e\Bigl(E\setminus\bigcup_{j=1}^s[\xi_j,\eta_j]\Bigr)<\frac12\mu_e(E_r).$$

Hence

$$\sum_{j=1}^s(\eta_j-\xi_j)\geqslant\mu_e\Bigl(E\cap\bigcup_{j=1}^s[\xi_j,\eta_j]\Bigr)>\frac12\mu_e(E_r).$$

This inequality together with (18) and (19) yields

$$\begin{aligned}&\sum_{j=1}^s\|V(\tau_j,[\xi_j,\eta_j])-U_{HK}(\eta_j)U_{HK}^{-1}(\xi_j)\|_\star\\ &\quad=\sum_{j=1}^s\Bigl\|V(\tau_j,[\xi_j,\eta_j])-\prod_{\xi_j}^{\eta_j}V(t,\mathrm{d}t)\Bigr\|_\star\\ &\quad\geqslant\frac1r\sum_{j=1}^s(\eta_j-\xi_j)>\frac1{2r}\mu_e(E_r)\geqslant4K^2n^2\|Q^{-1}\|_\star\varepsilon,\end{aligned}$$

a contradiction to (16) from Theorem 14. Therefore $\mu_e(E_r)=0$ for every $r\in\mathbb{N}$ and $\mu_e(E)=0$, which yields $\mu(E)=0$. □

Let us now turn our attention to the classical case when

$$V(t,J)=I+A(t)\mu(J),\tag{20}$$

where $A\colon[a,b]\to L(\mathbb{R}^n)$ and μ is the Lebesgue measure on the real line. As was mentioned in Section 1, the function V given by (20) satisfies condition **(C)**. First we prove the following corollary of Theorem 15.

Corollary 16. *Consider a function $A\colon [a,b] \to L(\mathbb{R}^n)$ such that the product integral* (HK) $\prod_a^b (I + A(t)\,\mathrm{d}t)$ *exists and is invertible. Then for*

$$U_{HK}(s) = (\mathrm{HK}) \prod_a^s (I + A(t)\,\mathrm{d}t),\ s \in (a,b], \qquad U_{HK}(a) = I \tag{21}$$

the derivative $\dot U_{HK}(t)$ exists for almost all $t \in [a,b]$ and

$$\dot U_{HK}(t) = A(t) U_{HK}(t) \tag{22}$$

for almost all $t \in [a,b]$.

P r o o f. Given $\varepsilon > 0$, by Theorem 15 there exists a set $E \subset [a,b]$, $\mu(E) = 0$ such that for every $\varepsilon > 0$, $t \in [a,b] \setminus E$ there is $\vartheta > 0$ such that

$$\|I + A(t)(y-x) - U_{HK}(y)U_{HK}^{-1}(x)\|_{\bigstar} \leqslant \varepsilon(y-x)$$

provided $t - \vartheta < x \leqslant t \leqslant y < t + \vartheta$ and $x, y \in [a,b]$. Take $t \in [a,b] \setminus E$. Then

$$\|I + A(t)(y-t) - U_{HK}(y)U_{HK}^{-1}(t)\|_{\bigstar} \leqslant \varepsilon(y-t)$$

for $y \in [t, t+\vartheta)$. Hence

$$\left\| \frac{U_{HK}(t)U_{HK}^{-1}(t) - U_{HK}(y)U_{HK}^{-1}(t)}{y-t} + A(t) \right\|_{\bigstar} \leqslant \varepsilon$$

and

$$\left\| \frac{U_{HK}(y) - U_{HK}(t)}{y-t} U_{HK}^{-1}(t) - A(t) \right\|_{\bigstar} \leqslant \varepsilon$$

for $y \in (t, t+\vartheta)$. This means that

$$\left\| \frac{U_{HK}(y) - U_{HK}(t)}{y-t} - A(t)U_{HK}(t) \right\|_{\bigstar} \leqslant \varepsilon \|U_{HK}(t)\|_{\bigstar},$$

i.e. $\dot U_{HK}^{+}(t)$ (the derivative from the right of U_{HK} at the point t) exists and

$$\dot U_{HK}^{+}(t) = A(t)U_{HK}(t).$$

A similar relation for the derivative from the left leads to the conclusion that for $t \notin E$ the derivative $\dot U_{HK}(t)$ exists and

$$\dot U_{HK}(t) = A(t)U_{HK}(t).$$

$\square$

Theorem 17. *Assume that the product integral* (HK) $\prod_a^b (I + A(t)\,\mathrm{d}t)$ *exists and is invertible. Then the indefinite integral* $U_{HK}\colon\ [a,b] \to L(\mathbb{R}^n)$ *satisfies the following condition:*

(SL) *Let* $\eta > 0$, $N \subset [a,b]$, $\mu(N) = 0$. *Then there exists* $\delta\colon\ N \to (0,+\infty)$ *such that if* $\{(\tau_j, [\xi_j, \eta_j])\}_{j=1}^r$ *is a* δ*-fine* K*-system and* $\tau_j \in N$ *for* $j = 1,2,\dots,r$, *then*

$$\sum_{j=1}^{r} \|U_{HK}(\eta_j) - U_{HK}(\xi_j)\|_\bigstar \leqslant \eta. \tag{23}$$

Theorem 17 was proved in [2]; let us mention that the above presented condition **(SL)** is the so called strong Lusin condition. By the results of Corollary 16 and by Theorem 17 we know that the indefinite product integral $U_{HK}\colon\ [a,b] \to L(\mathbb{R}^n)$ possesses a derivative almost everywhere in $[a,b]$ and satisfies the strong Lusin condition. Since every McShane product integrable function is also Henstock-Kurzweil product integrable, the theorem is also valid for the indefinite product integral $U_M\colon\ [a,b] \to L(\mathbb{R}^n)$. We now prove an even stronger statement.

Theorem 18. *Assume that the product integral* (M) $\prod_a^b (I + A(t)\,\mathrm{d}t) = Q$ *exists and is invertible. Then the indefinite McShane product integral* U_M *satisfies the following condition:*

(AC) *For every* $\varrho > 0$ *there is a* $\sigma > 0$ *such that if* $\{[\xi_j, \eta_j]\}_{j=1}^r$ *are non-overlapping intervals in* $[a,b]$, *then* $\sum_{j=1}^{r} (\eta_j - \xi_j) < \sigma$ *implies*

$$\sum_{j=1}^{r} \|U_M(\eta_j) - U_M(\xi_j)\|_\bigstar < \varrho.$$

P r o o f. Given $\varrho > 0$ take $\varepsilon > 0$ such that $\varepsilon < (9\|Q^{-1}\|_\bigstar)^{-1}$ and

$$4K^3 n^2 \|Q^{-1}\|_\bigstar \varepsilon < \frac{\varrho}{2},$$

where $K > 0$ is the constant from Theorem 7. For this $\varepsilon > 0$ there is a gauge $\delta\colon\ [a,b] \to (0,+\infty)$ such that for $V(t,J) = A(t)\mu(J)$ we have

$$\|P(V,D) - Q\|_\bigstar < \varepsilon$$

for every δ-fine M-partition $D = \{(t_i, [u_i, v_i])\}_{i=1}^q$ of $[a,b]$. Let us fix such an M-partition D and put

$$\sigma = \frac{\varrho}{2K(\max\limits_{i=1,\dots,q} \|A(t_i)\|_\bigstar + 1)}.$$

Assume that $[\xi_j,\eta_j] \subset [a,b]$, $j = 1,\dots,r$, are non-overlapping intervals with $\sum\limits_{j=1}^{r}(\eta_j-\xi_j) < \sigma$ and consider the sum $\sum\limits_{j=1}^{r}\|U_M(\eta_j)-U_M(\xi_j)\|_\bigstar$.

By subdividing the intervals $[\xi_j,\eta_j]$ if necessary, we can assume that every interval $[\xi_j,\eta_j]$ belongs to an interval $[u_i,v_i]$ of the fixed partition D. For each $i = 1,\dots,q$ let

$$M_i = \{j;\ 1 \leqslant j \leqslant r \ \text{ with } \ [\xi_j,\eta_j] \subset [u_i,v_i]\}$$

and let us take $\tau_j = t_i$ if $j \in M_i$. Now we have

$$\begin{aligned}&\|U_M(\xi_j)-U_M(\eta_j)\|_\bigstar = \|[I-U_M(\eta_j)U_M^{-1}(\xi_j)]U_M(\xi_j)\|_\bigstar\\ &\quad\leqslant \|U_M(\xi_j)\|_\bigstar(\|I+A(\tau_j)(\eta_j-\xi_j)-U_M(\eta_j)U_M^{-1}(\xi_j)\|_\bigstar+\|A(\tau_j)(\eta_j-\xi_j)\|_\bigstar)\end{aligned}$$

and by Theorem 7 we get

$$\begin{aligned}\sum_{j=1}^{r}\|U_M(\eta_j)-U_M(\xi_j)\|_\bigstar \leqslant K\sum_{j=1}^{r}\|I+A(\tau_j)(\eta_j-\xi_j)-U_M(\eta_j)U_M^{-1}(\xi_j)\|_\bigstar\\ +K\max_{i=1,\dots,q}\|A(t_i)\|_\bigstar\sum_{j=1}^{r}(\eta_j-\xi_j).\end{aligned}$$

It is easy to check that for the points τ_j and the intervals $[\xi_j,\eta_j]$, $j = 1,\dots,r$ the assumption of Theorem 14 is satisfied if ξ_j and η_j are ordered properly. Using (16) from Theorem 14 we obtain the inequality

$$\begin{aligned}\sum_{j=1}^{r}\|U_M(\eta_j)-U_M(\xi_j)\|_\bigstar &\leqslant 4K^3n^2\|Q^{-1}\|_\bigstar\varepsilon+K\max_{i=1,\dots,q}\|A(t_i)\|_\bigstar\sigma\\ &< \frac{\varrho}{2}+\frac{\varrho}{2}=\varrho\end{aligned}$$

and the statement is proved. $\square$

Condition **(AC)** in Theorem 18 says that the indefinite product integral $U_M\colon [a,b] \to L(\mathbb{R}^n)$ is absolutely continuous in $[a,b]$.

The special norm $\|\cdot\|_\bigstar$ of matrices was used in the previous parts for technical reasons only. Note that according to (9) the proofs can be modified in a straightforward way for any norm of matrices.

Definition 19. Functions $V_1, V_2\colon [a,b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ are called M-equivalent (K-equivalent) if for every $\varepsilon > 0$ there is a gauge $\delta\colon [a,b] \to (0,\infty)$ such that

$$\sum_{j=1}^{m} \|V_1(\tau_j,[\alpha_{j-1},\alpha_j]) - V_2(\tau_j,[\alpha_{j-1},\alpha_j])\| < \varepsilon$$

for every δ-fine M-partition (K-partition) $\{(\tau_j,[\alpha_{j-1},\alpha_j])\}_{j=1}^m$ of $[a,b]$.

Theorem 20. *Let $V_1, V_2\colon [a,b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ be M-equivalent (K-equivalent) functions. Assume that V_1 satisfies condition* **(C)** *and that the McShane (Henstock-Kurzweil) integral $\prod_a^b V_1(t,\mathrm{d}t)$ exists and is invertible. Then the McShane (Henstock-Kurzweil) integral $\prod_a^b V_2(t,\mathrm{d}t)$ exists as well and the two product integrals have the same value.*

P r o o f. The Henstock-Kurzweil version is proved in [2]; the proof for the McShane product integral can be carried out in the same way (replacing K-partitions with M-partitions). □

Corollary 21. *Consider a function $A\colon [a,b] \to L(\mathbb{R}^n)$. Then the following conditions are equivalent:*

1) (M) $\prod_a^b (I + A(t)\,\mathrm{d}t)$ *exists and is invertible.*
2) (M) $\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$ *exists and is invertible.*

If one of these conditions is fulfilled, then

$$(\mathrm{M}) \prod_a^b (I + A(t)\,\mathrm{d}t) = (\mathrm{M}) \prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}.$$

P r o o f. The functions

$$V_1(t,[x,y]) = I + A(t)(y-x),$$
$$V_2(t,[x,y]) = \mathrm{e}^{A(t)(y-x)}$$

satisfy condition **(C)**. According to Theorem 20 it is sufficient to show that V_1 and V_2 are equivalent. For $x < y$ we have

$$\|I + A(t)(y-x) - \mathrm{e}^{A(t)(y-x)}\| = \left\| \sum_{k=2}^{\infty} \frac{A(t)^k (y-x)^k}{k!} \right\| \leqslant \|A(t)\|^2 (y-x)^2 \mathrm{e}^{\|A(t)\|(y-x)}.$$

Let $\delta\colon\ [a,b] \to (0,\infty)$ be an arbitrary function such that

$$\delta(t) < \min\Bigl(\frac{1}{2\|A(t)\|}, \frac{\varepsilon}{2e(b-a)\|A(t)\|^2}\Bigr)$$

whenever $\|A(t)\| > 0$. Then for every δ-fine M-partition $\{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ we have

$$\alpha_j - \alpha_{j-1} < 2\delta(\tau_j)$$

and

$$\begin{aligned}&\sum_{j=1}^m \|I + A(\tau_j)(\alpha_j - \alpha_{j-1}) - \mathrm{e}^{A(\tau_j)(\alpha_j-\alpha_{j-1})}\|\\ &\qquad \leqslant \sum_{j=1}^m \|A(\tau_j)\|^2(\alpha_j - \alpha_{j-1})^2 \mathrm{e}^{\|A(\tau_j)\|(\alpha_j-\alpha_{j-1})} < \sum_{j=1}^m \frac{\varepsilon(\alpha_j - \alpha_{j-1})}{b-a} = \varepsilon.\end{aligned}$$

$\square$

5. Bochner product integral

Let X be a Banach space. Assume that $B\colon\ [a,b] \to L(X)$ is a step-function, i.e. there exist points

$$a = s_0 < s_1 < \ldots < s_{m-1} < s_m = b$$

and operators $B_1, \ldots, B_m \in L(X)$ such that $B(x) = B_k$ for $x \in (s_{k-1}, s_k)$, $k = 1, 2, \ldots, m$. We have

$$\prod_a^b \mathrm{e}^{B(t)\,\mathrm{d}t} = \prod_{k=m}^1 \prod_{s_{k-1}}^{s_k} \mathrm{e}^{B(t)\,\mathrm{d}t} = \mathrm{e}^{B_m(s_m-s_{m-1})}\mathrm{e}^{B_{m-1}(s_{m-1}-s_{m-2})} \ldots \mathrm{e}^{B_1(s_1-s_0)}$$

(where the product integrals exist for example in the sense of Riemann).

Definition 22. A function $f\colon\ [a,b] \to X$ is called Bochner integrable if there is a sequence of step functions $f_k\colon\ [a,b] \to X$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} (\mathrm{L}) \int_a^b \|f_k(x) - f(x)\|\,\mathrm{d}x = 0,$$

where (L) denotes the Lebesgue integral.

In the monograph [1] the following definition of product integral is given (in finite-dimensional case).

Definition 23. Assume that $A\colon [a,b] \to L(X)$ is Bochner integrable. The Bochner product integral $(\mathrm{B})\prod\limits_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$ is defined by

$$(\mathrm{B})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t} = \lim_{n\to\infty}\prod_a^b \mathrm{e}^{A_n(t)\,\mathrm{d}t} \tag{24}$$

where $\{A_n\}_{n=1}^{\infty}$ is any sequence of step-functions convergent to A in the L^1 sense, i.e.

$$\lim_{n\to\infty}(\mathrm{L})\int_a^b \|A_n(s)-A(s)\|\,\mathrm{d}s = 0.$$

It is known that a function $f\colon [a,b]\to\mathbb{R}^n$ is Bochner integrable if and only if it is Lebesgue integrable. For $X=\mathbb{R}^n$ we have $L(X)=\mathbb{R}^{n\times n}$ and a function $A\colon [a,b]\to\mathbb{R}^{n\times n}$ is Bochner product integrable if and only if its components are Lebesgue integrable functions.

Definition 24. A function $f\colon [a,b]\to X$ has the property $\mathcal{S}^*\mathcal{M}$ if for every $\varepsilon>0$ there is a gauge $\delta\colon [a,b]\to(0,\infty)$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\|f(t_i)-f(s_j)\|\mu(J_i\cap L_j)<\varepsilon$$

for any δ-fine M-partitions $\{(t_i,J_i)\}_{i=1}^{k}$ and $\{(s_j,L_j)\}_{j=1}^{l}$ of $[a,b]$.

Theorem 25. *Let X be a finite-dimensional Banach space, $f\colon [a,b]\to X$. Then the following conditions are equivalent:*

1) *f is Bochner integrable.*
2) *f is McShane integrable.*
3) *f has the property $\mathcal{S}^*\mathcal{M}$.*

P r o o f. A function $f\colon [a,b]\to X$ is Bochner integrable if and only if it has the property $\mathcal{S}^*\mathcal{M}$ (see Theorem 5.1.4 in [6]). Moreover, in a finite-dimensional Banach space, a function is McShane integrable if and only if it has the property $\mathcal{S}^*\mathcal{M}$ (Proposition 5.2.1 in [6]). $\square$

The following theorem was proved in [1]:

Theorem 26. *Let $A\colon [a,b]\to L(\mathbb{R}^n)$ be Bochner integrable. Then the Bochner product integral* $(\mathrm{B})\prod\limits_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$ *is an invertible matrix.*

The following theorem was proved in [4]:

Theorem 27. *If $A\colon\ [a,b] \to L(X)$ has the property $\mathcal{S}^*\mathcal{M}$ then the product integrals* (B) $\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$ *and* (M) $\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$ *exist and*

$$(\mathrm{B})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t} = (\mathrm{M})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}.$$

Notice that the paper [4] uses a different terminology: Our McShane product integral is called the Bochner product integral there, while our Bochner product integral is referred to as the Lebesgue-type product integral and is denoted by (L) $\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$.

As a consequence of Theorem 25, Theorem 26, Theorem 27 and Corollary 21 we obtain the following statement.

Corollary 28. *Let $A\colon\ [a,b] \to L(\mathbb{R}^n)$ be Bochner integrable. Then*

$$(\mathrm{B})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t} = (\mathrm{M})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t} = (\mathrm{M})\prod_a^b (I + A(t)\,\mathrm{d}t),$$

where all the above product integrals are guaranteed to exist.

Theorem 29. *Let $A\colon\ [a,b] \to L(\mathbb{R}^n)$ be given. If there is an absolutely continuous function $U\colon\ [a,b] \to L(\mathbb{R}^n)$ such that $U^{-1}(s)$ exists for every $s \in [a,b]$ and $U'(s) = A(s)U(s)$ for almost all $s \in [a,b]$, then A is Bochner integrable.*

P r o o f. The function U^{-1} is measurable since the components u_{ij} $(i,j = 1,\ldots,n)$ of U are measurable and

$$U^{-1}(s) = \left\{\frac{(-1)^{i+j}\det U_{ji}(s)}{\det U(s)}\right\}_{i,j=1}^{n} \tag{25}$$

(where $U_{ji}(s)$ is the minor obtained from $U(s)$ by deleting the j-th row and i-th column). Since U is continuous and invertible on $[a,b]$ we have

$$m := \min_{x\in[a,b]} |\det U(x)| > 0.$$

It is also possible to find a constant $M > 0$ such that $|u_{ij}(s)| \leqslant M$ for every $s \in [a,b]$ and $i,j = 1,\ldots,n$. From (25) we get

$$\|U^{-1}(s)\|_{\bigstar} \leqslant \frac{|\det U_{ji}(s)|}{|\det U(s)|} \leqslant \frac{(n-1)!\,M^{n-1}}{m},$$

i.e. the function U^{-1} is bounded.

The components of U' are Lebesgue integrable (because U is absolutely continuous), U^{-1} is measurable and bounded. Therefore the components of $A(s) = U'(s)U^{-1}(s)$ are Lebesgue integrable and A is Bochner integrable. □

The following theorem may be regarded as a descriptive definition of the McShane product integral.

Theorem 30. *Consider a function $A\colon [a,b] \to L(\mathbb{R}^n)$. Then the McShane product integral* $(\mathrm{M}) \prod_a^b (I + A(t)\,\mathrm{d}t)$ *exists if and only if there is an absolutely continuous function $U\colon [a,b] \to L(\mathbb{R}^n)$ such that $U^{-1}(s)$ exists for every $s \in [a,b]$ and $U'(s) = A(s)U(s)$ for almost all $s \in [a,b]$; in this case*

$$(\mathrm{M}) \prod_a^b (I + A(t)\,\mathrm{d}t) = U(b)U^{-1}(a).$$

P r o o f. The first part of the theorem is easily proved by combining the results from Corollary 16, Theorem 18, Theorem 29 and Corollary 28: The McShane product integral $(\mathrm{M}) \prod_a^b (I + A(t)\,\mathrm{d}t)$ exists if and only if there is an absolutely continuous function $U\colon [a,b] \to L(\mathbb{R}^n)$ such that $U^{-1}(s)$ exists for every $s \in [a,b]$ and $U'(s) = A(s)U(s)$ for almost all $s \in [a,b]$.

Now take an arbitrary function U which satisfies the conditions stated above. Define

$$V(s) = (\mathrm{M}) \prod_a^s (I + A(t)\,\mathrm{d}t)$$

and let $W(s) = U^{-1}(s)V(s)$ for $s \in [a,b]$. The functions U and V are absolutely continuous. Using again the formula

$$U^{-1}(s) = \left\{ \frac{(-1)^{i+j} \det U_{ji}(s)}{\det U(s)} \right\}_{i,j=1}^{n}$$

we see that U^{-1} and consequently W are absolutely continuous functions. The equality $U'U^{-1} = V'V^{-1}$ almost everywhere implies

$$\begin{aligned} W' &= (U^{-1}V)' = \left(U^{-1}\right)' V + U^{-1}V' = -U^{-1}U'U^{-1}V + U^{-1}V' \\ &= -U^{-1}U'U^{-1}V + U^{-1}V'V^{-1}V = U^{-1}(V'V^{-1} - U'U^{-1})V = 0 \end{aligned}$$

almost everywhere on $[a,b]$, i.e. W is a constant function. The proof is completed by observing that

$$(\mathrm{M}) \prod_a^b (I + A(t)\,\mathrm{d}t) = V(b) = U(b)W(b) = U(b)W(a) = U(b)U^{-1}(a).$$

$\square$

Theorem 31. *Consider a function* $A\colon [a,b] \to L(\mathbb{R}^n)$ *such that the McShane product integral* $(\mathrm{M})\prod_a^b (I + A(t)\,\mathrm{d}t)$ *exists and is invertible. Then A is also Bochner integrable and*

$$(\mathrm{M})\prod_a^b (I + A(t)\,\mathrm{d}t) = (\mathrm{B})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}.$$

P r o o f. Define

$$U(s) = (\mathrm{M})\prod_a^s (I + A(t)\,\mathrm{d}t).$$

From Theorem 18 and Corollary 16 we know that U is absolutely continuous and $U'(s) = A(s)U(s)$ almost everywhere on $[a,b]$. According to Theorem 7 the matrix $U^{-1}(s)$ exists for every $s \in [a,b]$. To complete the proof apply Theorem 29 and Corollary 28. □

The following theorem describes the relation between the McShane product integral and the Bochner product integral.

Theorem 32. *For every* $A\colon [a,b] \to L(\mathbb{R}^n)$ *the following conditions are equivalent:*

1) *A is Bochner integrable.*
2) *The McShane product integral* $(\mathrm{M})\prod_a^b (I + A(t)\,\mathrm{d}t)$ *exists and is invertible.*
3) *The McShane product integral* $(\mathrm{M})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}$ *exists and is invertible.*

If one of these conditions is fulfilled, then

$$(\mathrm{B})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t} = (\mathrm{M})\prod_a^b (I + A(t)\,\mathrm{d}t) = (\mathrm{M})\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t}.$$

P r o o f. An easy consequence of Corollary 28, Theorem 21 and Theorem 31. □

As shown in Examples 8 and 9, the invertibility condition in the statement of Theorem 32 cannot be left out.

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