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ON EIGENVECTORS OF MIXED GRAPHS WITH EXACTLY ONE NONSINGULAR CYCLE

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Abstract. Let G be a mixed graph. The eigenvalues and eigenvectors of G are respectively defined to be those of its Laplacian matrix. If G is a simple graph, [M. Fiedler: A property of eigenvectors of nonnegative symmetric matrices and its applications to graph theory, Czechoslovak Math. J. 25 (1975), 619–633] gave a remarkable result on the structure of the eigenvectors of G corresponding to its second smallest eigenvalue (also called the algebraic connectivity of G). For G being a general mixed graph with exactly one nonsingular cycle, using Fiedler's result, we obtain a similar result on the structure of the eigenvectors of G corresponding to its smallest eigenvalue.

Keywords: mixed graphs, Laplacian eigenvectors

MSC 2000: 05C50, 15A18

1. Introduction

Let $G = (V, E)$ be a *mixed graph* with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, \ldots, e_m\}$, which is obtained from an undirected graph by orienting some of its edges. Then some edges of G have a special head and tail, while others do not. The notion of a mixed graph generalizes both the classical approach of orienting all edges [4] and the unoriented approach [9]. However, it should be pointed out that mixed graphs are considered here as the underlying undirected graphs as concerns defining degrees, paths, cycles, connectivity, etc., and have no multi-edges or loops in this paper.

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Let $e \in E(G)$. The sign of e is denoted by $\operatorname{sgn} e$ and defined as $\operatorname{sgn} e = 1$ if e is unoriented and $\operatorname{sgn} e = -1$ otherwise. Set $a_{ij} = \operatorname{sgn}\{v_i, v_j\}$ if $\{v_i, v_j\} \in E(G)$ and $a_{ij} = 0$ otherwise. Then $A(G) = [a_{ij}]$ is called the *adjacency matrix* of G. The *incidence matrix* of G is the $n \times m$ matrix $M = M(G) = [m_{ij}]$ whose entries are given by $m_{ij} = 1$ if e_j is an unoriented edge incident with v_i or e_j is an oriented edge with head v_i, $m_{ij} = -1$ if e_j is an oriented edge with tail v_i, and $m_{ij} = 0$ otherwise. The *Laplacian matrix* of G is defined as $L = L(G) = MM^T$, where M^T denotes the transpose of M. Denote by $d_G(v) = d(v)$ the degree of the vertex v in the graph G. It is easy to see that $L(G) = D(G) + A(G)$, where $D(G) = \operatorname{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$ is a diagonal matrix.

One can find that $L(G)$ is symmetric and positive semidefinite so that its eigenvalues can be arranged as follows: $0 \leqslant \lambda_1(G) \leqslant \lambda_2(G) \leqslant \ldots \leqslant \lambda_n(G)$. We simply say the eigenvalues and eigenvectors of $L(G)$ as those of G, respectively. The *spectrum* of G is defined by the multi-set $\{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$. G is called *singular* (or *nonsingular*) if $L(G)$ is singular (or nonsingular).

Clearly, if G is *all-oriented* (i.e. all edges of G are oriented), then $L(G)$ is the standard Laplacian matrix which is consistent with the Laplacian matrix of a simple graph (see [15]); and there are a lot of results involved with the relations between its spectrum and numerous graph invariants, such as connectivity, diameter, matching number, isoperimetric number, and expanding properties of a graph (see, e.g., [7], [10], [15], [16]). For many properties of mixed graphs, one can refer to [2], [3], [6], [14], [17].

Denote by $\vec{G}$ an all-oriented graph obtained from G by assigning to each unoriented edge of G an arbitrary orientation (one of the two possible directions). G is called *quasi-bipartite* if it does not contain a nonsingular cycle, or equivalently, if G contains no cycles with an odd number of unoriented edges (see [2, Lemma 1]). Note that the *signature matrix* is a diagonal matrix with ± 1 along its diagonal.

Lemma 1.1 ([17, Lemma 2.2]). *Let G be a connected mixed graph. Then G is singular if and only if G is quasi-bipartite.*

Theorem 1.2 ([2, Theorem 4]). *Let G be a mixed graph. Then G is quasi-bipartite if and only if there exists a signature matrix D such that $D^T L(G) D = L(\vec{G})$.*

Suppose G is connected. If G is singular (or $\lambda_1(G) = 0$), then the spectrum of G is exactly that of $\vec{G}$, and the least nonzero eigenvalue of G is equal to $\lambda_2(\vec{G}) \triangleq \alpha(\vec{G}) > 0$, which is called the *algebraic connectivity* of $\vec{G}$ by Fiedler [7]. The algebraic connectivity of a simple graph has received much attention, see, e.g., [5], [15], [16] and the references therein. Fiedler gave a remarkable result on the structure of the eigenvectors corresponding to the algebraic connectivity (also called *Fiedler vectors*) of a simple graph (see [8, Theorem 3.14] or Theorem 2.5 in following section). Motivated by Fiedler's result, Fiedler vectors also received much attention recently, see, e.g., [1], [12], [13]. If G is nonsingular, then $\lambda_1(G) > 0$ and few results can be found for $\lambda_1(G)$ and the corresponding eigenvectors as yet. In this case, G contains at least one nonsingular cycle.

In this paper, we mainly discuss the eigenvectors of a mixed graph G with exactly one nonsingular cycle; and by Fiedler's result on a simple graph, we obtain a similar result on the structure of the eigenvectors of G corresponding to its least eigenvalue.

2. Main result

Let $G = (V, E)$ be a mixed graph with $V = \{v_1, v_2, \dots, v_n\}$, and let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be a nonzero vector. It will be convenient to adopt the following terminology from [8]: x is said to give a valuation of the vertices of V, that is, with vertex v_i of V we associate the value x_i, i.e., $x(v_i) = x_i$. Then λ is an eigenvalue of G with the corresponding eigenvector $x = (x_1, x_2, \dots, x_n)$ if and only if

$$(2.1) \qquad [\lambda - d(v_i)]x(v_i) = \sum_{e=\{v_i,v_j\}\in E} (\operatorname{sgn} e)x(v_j), \quad i = 1, 2, \dots, n.$$

Let D be a signature matrix of order n. Then $D^T L(G)D$ is the Laplacian matrix of a graph with the same underlying graph as that of G. So each signature matrix of order n gives a re-sign of the edges of G (that is, some oriented edges of G may turn to be unoriented and vice versa), and preserves the spectrum and the singularity of each cycle of G. We now use the notation ${}^DG$ to denote the graph obtained from G by a re-sign under the signature D, and assume that the label of the vertices of ${}^DG$ is the same as that of G.

Recall that a connected graph is called 2-connected if it has no *points of articulation* (or *cutpoints*). Let G be a connected graph. A *block* of G is a maximal 2-connected subgraph; or equivalently, the blocks of G are the subgraphs induced by the edges in a single equivalent class, given via the following relation: any two edge are equivalent if and only if there is a cycle in the graph containing both the edges (see also [8]).

Denote by $\mathscr{G}_n^m$ the class of connected mixed graphs on n vertices containing exactly one nonsingular cycle with that cycle on m vertices. Since any cycle of a graph must be contained in one of its blocks, each graph $G \in \mathscr{G}_n^m$ has the following property: It contains exactly one nonsingular block with that block containing exactly one nonsingular cycle. We claim that the block containing exactly one nonsingular cycle is exactly that nonsingular cycle.

Lemma 2.1. *Let G be a 2-connected mixed graph on n vertices containing exactly one nonsingular cycle. Then G is exactly that nonsingular cycle.*

Proof. Let C be the nonsingular cycle of G on m vertices. If $m = n$, then no edges except those of C join two vertices of the cycle. Otherwise, C is split into two cycles C_1 and C_2 with a common edge. By definition, one of C_1 and C_2 is nonsingular and the other is singular. Then G has at least two nonsingular cycles.

If $m < n$, there exists a vertex u out of the cycle C joining some vertex v of C by an edge, as G is connected. Since G is also 2-connected, $G - v$ is connected, and contains a path P which joins u and some vertex w ($w \neq v$) of C and contains no vertices of C except w. Let P_1, P_2 be two different paths on the cycle C which join v and w. Then we obtain two cycles containing the vertices u, v and those of P: C_1 having also vertices of P_1, C_2 having also vertices of P_2. By definition, among C_1 and C_2 there must exist a nonsingular cycle. Hence G has at least two nonsingular cycles. The result follows by the above discussion. □

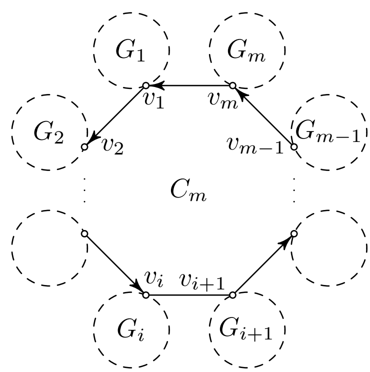

Figure 2.1. C_m has exactly one unoriented edge $\{v_i, v_{i+1}\}$; G_i is connected and all-oriented, and has exactly one common vertex v_i with C_m for $i = 1, 2, \ldots, m$.

Theorem 2.2. *Let $G \in \mathscr{G}_n^m$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, where $v_1, v_2, \ldots, v_m$ are the vertices of its nonsingular cycle. Then there exists a signature matrix D such that ${}^{D}G$ is obtained from a nonsingular cycle C_m on m vertices with exactly one unoriented edge by appending a connected all-oriented graph G_i on n_i ($n_i \geqslant 1$) vertices to the vertex v_i of C_m with v_i identifying some vertex of G_i for each $i = 1, 2, \ldots, m$, i.e., ${}^{D}G$ is the graph as in Fig. 2.1, where $n_1 + n_2 + \ldots + n_m = n$.*

Proof. By Lemma 2.1, the graph G has the structure as in Fig. 2.1. Let $e = \{v_i, v_{i+1}\}$ be an unoriented edge of the nonsingular cycle of G. Then $G - e$ is quasi-bipartite by definition. By Theorem 1.2, there exists a signature matrix D

such that ${}^{D}(G-e)$ is all-oriented. Since G is nonsingular, ${}^{D}G$ is the graph as in Fig. 2.1. $\square$

Lemma 2.3. *Let G be a mixed graph of order n and let e be an (oriented or unoriented) edge of G. Then*

$$\lambda_1(G-e) \leqslant \lambda_1(G) \leqslant \ldots \leqslant \lambda_{n-1}(G) \leqslant \lambda_n(G-e) \leqslant \lambda_n(G).$$

Proof. The matrices $N(G) = M(G)^T M(G)$ and $L(G) = M(G)M(G)^T$ have the same nonzero eigenvalues, so the same holds for $N(G-e)$ and $L(G-e)$. Since $N(G-e)$ is a principal submatrix of $N(G)$, the result follows by the Courant-Fischer Theorem (see [11, Theorem 4.3.15]). $\square$

Lemma 2.4. *Let $G \in \mathscr{G}_n^m$ be a graph on vertices $v_1, v_2, \ldots, v_n$ as in Fig. 2.1. Let H be a copy of G in which we only replace the label v_i by u_i for each $i = 1, 2, \ldots, n$. Let $e = \{v_i, v_{i+1}\}$ and $e' = \{u_i, u_{i+1}\}$. Let W be an all-oriented graph on $2n$ vertices obtained from the union $(G-e) \cup (H-e')$ by inserting two oriented edges $\{v_i, u_{i+1}\}, \{v_{i+1}, u_i\}$. Order the vertices of W as $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Then*

$$\lambda_1(G) = \alpha(W),$$

and if $x \in \mathbb{R}^n$ is an eigenvector of G corresponding to $\lambda_1(G)$, then $(x, -x) \in \mathbb{R}^{2n}$ is an eigenvector of W corresponding to $\alpha(W)$.

Proof. Let $x \in \mathbb{R}^n$ be an eigenvector of G corresponding to $\lambda_1(G)$. Then, by (2.1), $(x, -x) \in \mathbb{R}^{2n}$ is an eigenvector of W corresponding to the same eigenvalue $\lambda_1(G)$. Since W is all-oriented and connected,

$$\lambda_1(G) \geqslant \alpha(W)(= \lambda_2(W)) > 0.$$

We assert that $\lambda_1(G) = \alpha(W)$. Let $(y, z) \in \mathbb{R}^{2n}$ be a Fiedler vector of W, i.e., the eigenvector of W corresponding to $\alpha(W)$, where $y \in \mathbb{R}^n$, $z \in \mathbb{R}^n$. One can find that (z, y) is also a Fiedler vector of W by (2.1). If $y \neq z$, then $(y-z, z-y)$ is a Fiedler vector of W. Hence $y-z$ is an eigenvector of G corresponding to the eigenvalue $\alpha(W)$ by (2.1), and $\alpha(W) \geqslant \lambda_1(G)$. If $y = z$, then, also by (2.1), y is an eigenvector of $\vec{G}$ corresponding to the eigenvalue $\alpha(W)$. Note that the edge $\{v_i, v_{i+1}\}$ of $\vec{G}$ is oriented and is now denoted by $\vec{e}$, and $\vec{G} - \vec{e} = G - e$. By Lemma 2.3,

$$\alpha(W) \geqslant \alpha(\vec{G}) \geqslant \alpha(\vec{G} - \vec{e}) = \alpha(G-e) \geqslant \lambda_1(G).$$

So in both cases we have $\alpha(W) \geqslant \lambda_1(G)$. Hence $\lambda_1(G) = \alpha(W)$, and $(x, -x)$ is a Fiedler vector of W. $\square$

Lemma 2.4 establishes a relation between the least eigenvalue (and the corresponding eigenvectors) of a mixed graph on n vertices with exactly one nonsingular cycle and the algebraic connectivity (and respectively, the Fiedler vectors) of a simple graph on $2n$ vertices. We now introduce the remarkable result of Fiedler on the structure of Fiedler vectors of a simple (or all-oriented) graph.

Theorem 2.5 ([8, Theorem 3.14])**.** *Let G be a connected simple graph, and let y be a Fiedler vector of G which gives a valuation of the vertices of G. Then exactly one of the following cases occurs:*

Case A: There is a single block B_0 in G which contains both positively and negatively valuated vertices. Each other block has either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every path P which contains at most two points of articulation in each block, which starts in B_0 and contains just one vertex v in B_0, has the property that the values at points of articulation contained in P form either an increasing, or decreasing, or a zero sequence along this path according to whether $y(v) > 0$, $y(v) < 0$, or $y(v) = 0$; in the last case all vertices in P have value zero.

Case B: No block of G contains both positively and negatively valuated vertices. There exists a unique vertex w which has value zero and is adjacent to a vertex with a non-zero value. This vertex is a point of articulation. Each block contains (with the exception of w) either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every path P which contains at most two points of articulation in each block and which starts at w has the property that the values at its points of articulation either increase, in which case all values of vertices of P are (with the exception of w) positive, or decrease, in which case all values of vertices of P are (with the exception of w) negative, or all values of the vertices of P are zero. Every path containing both positively and negatively valuated vertices passes through w.

We now give an analog of Fiedler's result for a connected mixed graph with exactly one nonsingular cycle.

Theorem 2.6. *Let $G \in \mathscr{G}_n^m$ be a graph on vertices $v_1, v_2, \ldots, v_n$ as in Fig. 2.1. Let $x \in \mathbb{R}^n$ be an eigenvector of G corresponding to its least eigenvalue $\lambda_1(G)$, which gives a valuation of the vertices of G. Then there exists a nonsingular block B_0 (just the cycle C_m) of G which contains non-zero valuated vertices; each other block has either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only; every path P which contains at most two points of articulation in each block, which starts in B_0 and contains just one vertex v in B_0, has the property that the values at points of articulation contained in P form either* *an increasing, or decreasing, or zero sequence along this path according to whether* $x(v) > 0$, $x(v) < 0$, *or* $x(v) = 0$; *in the last case all vertices in* P *have value zero.*

Proof. Let W be the graph on $2n$ vertices which is obtained from G as in Lemma 2.4, and let C'_{2m} be the cycle of W on $2m$ vertices

$$v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_m, u_1, u_2, \ldots, u_i, v_{i+1}, v_{i+2}, \ldots, v_m.$$

By Lemma 2.4, $\xi = (x, -x) \in \mathbb{R}^{2n}$ is a Fiedler vector of W. Let ξ give a valuation of the vertices of W.

Assume that the values of the vertices on the cycle C_m of G are all zero. So are the values of those on the cycle C'_{2m} of W. If W belongs to case A of Theorem 2.5, then the single block of W with both positively and negatively valuated vertices is contained in some G_k or H_k, where H_k, as a copy of G_k, is a subgraph of H appending to the vertex u_k. Without loss of generality, let this single block be contained in G_k. Then by virtue of the structures of W and ξ, H_k also contains a block with both positively and negatively valuated vertices which is a contradiction. If W belongs to case B of Theorem 2.5, there is a unique vertex which has value zero and is adjacent to a non-zero valuated vertex (must be in some G_k or H_k), which also yields a contradiction by a similar argument.

By the above discussion, the cycle C_m of G contains non-zero valuated vertices. Then it follows from the structures of W and ξ that the corresponding cycle C'_{2m} of W contains both positively and negatively valuated vertices. Hence W belongs to case A of Theorem 2.5. The result follows by Theorem 2.5. □

Example. We give an illustration of Theorem 2.6 as follows. Consider the graph $G \in \mathscr{G}_{12}^4$ as in Fig. 2.2. By *Mathematica*, the least eigenvalue $\lambda_1(G) \approx 0.093$, and the corresponding eigenvector is listed on the vertices of the graph below with approximate value.

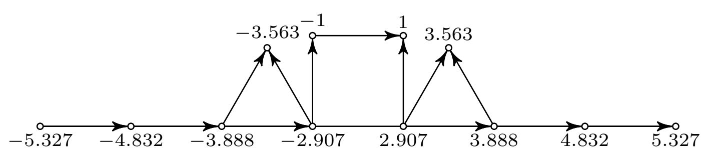

Fig. 2.2.

Theorem 2.7. *Let* G *be a mixed graph consisting of singular blocks* $B_1, B_2, \ldots, B_k$ *and nonsingular blocks* $B_{k+1}, B_{k+2}, \ldots, B_m$. *Then there exists a signature matrix* D *such that* ${}^D G$ *is the graph with* ${}^D B_1, {}^D B_2, \ldots, {}^D B_k$ *all-oriented.*

Proof. Let G' be the graph obtained from G by orienting all unoriented edges of the blocks $B_{k+1}, B_{k+2}, \ldots, B_m$. Then all blocks of G' are singular, and so is G'. By

Theorem 1.2, there exists a signature matrix D such that ${}^{D}G' = \vec{G}'$, an all-oriented graph. Noting that each nonsingular block of G contains no edges of singular blocks, we conclude that ${}^{D}G$ is a graph such that ${}^{D}B_1, {}^{D}B_2, \ldots, {}^{D}B_k$ are all-oriented. □

By Theorem 2.7, to discuss the eigenspaces of mixed graphs, it is enough to deal with graphs with all singular blocks all-oriented. A problem naturally arises whether Theorem 2.6 also holds for the mixed graph containing exactly one nonsingular block with two or more nonsingular cycles, or the mixed graph containing two or more nonsingular blocks? If not, what can we say for these graphs? These are still open problems.

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