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COMMUTATIVE IDEMPOTENT RESIDUATED LATTICES

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Abstract. We investigate the variety of residuated lattices with a commutative and idempotent monoid reduct.

Keywords: residuated lattice, semilattice, finitely based variety, minimal variety

MSC 2000: 06F05

A *residuated lattice* is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, e, /, \backslash)$ such that $(A, \vee, \wedge)$ is a lattice, $(A, \cdot, e)$ is a monoid and for every $a, b, c \in A$

$$ab \leqslant c \Leftrightarrow a \leqslant c/b \Leftrightarrow b \leqslant a \setminus c.$$

The last condition is equivalent to the fact that $(A, \vee, \wedge, \cdot, e)$ is a lattice-ordered monoid and for every $a, b \in A$ there is a greatest c such that $cb \leqslant a$ (denoted a/b) and a greatest d such that $bd \leqslant a$ (denoted $b \setminus a$). It is easy to see that the class $\mathcal{RL}$ of all residuated lattices is a variety. We are concerned about the variety $\mathcal{CI}d\mathcal{RL}$ of *commutative idempotent* (CI) *residuated lattices*, i.e. the subvariety of $\mathcal{RL}$ given by the equations

$$xy \approx yx \quad \text{and} \quad xx \approx x.$$

In other words, residuated lattices whose semigroup reduct is a semilattice. For example, every Heyting algebra is a CI residuated lattice, where $ab = a \wedge b$ and $a/b = b \setminus a = b \to a$ for every a, b (see e.g. [3, p. 30]).

Foundation of the theory of residuated lattices goes as far back as 1930's, when Dilworth and Ward [5] studied lattices of ring ideals. A recent introduction can be

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found in [4] and [10] and commutative residuated lattices were particularly studied in [9]. We will use the notation and terminology of these papers. We also assume a basic familiarity with universal algebra, standard references are [3] and [12].

In CI residuated lattices, we drop the operation $\backslash$, since owing to the commutativity $x/y \approx y \setminus x$. The lattice order will be denoted by $\leqslant$. We put $a \preceq b$ iff $ab = a$; hence $\preceq$ is the semilattice order, where $\cdot$ is regarded as the meet; e is its top element. When refering to an order, we mean the lattice order $\leqslant$, unless explicitly stated otherwise. We put $A^+ = \{a \in A\colon\ a \geqslant e\}$ and $A^- = \{a \in A\colon\ a \leqslant e\}$ and we call $\mathbf{A}^+$ the positive cone and $\mathbf{A}^-$ the negative cone of $\mathbf{A}$ (regarded as lattice-ordered monoids; indeed, they may not be closed under residuation).

The bottom element (in the lattice order) is denoted by 0 and the top element is denoted by 1, if they exist; it is easy to see that, in any residuated lattice, if 0 exists, then 1 exists, $0a = a0 = 0$ and $a/0 = 1/a = 1$ (see also [4]); in particular, 0 is also the bottom element of the semilattice order in any CI residuated lattice.

1. Motivation

Our interest in this particular variety comes from the following observation.

1.1. Observation. *Let $\mathcal{V}$ be a non-trivial subvariety of residuated lattices based (relatively to $\mathcal{RL}$) by equations in the language of monoids. Then $\mathcal{V}$ contains $\mathcal{CI}d\mathcal{RL}$ as a subvariety. (In other words, any monoid equation with a non-trivial residuated lattice model is implied by commutativity and idempotency.)*

Proof. Let $u \approx v$ be an equation in the language of monoids valid in $\mathcal{V}$. In order to prove that every CI residuated lattice is in $\mathcal{V}$, it is enough to show that $u \approx v$ holds in every semilattice. Indeed, this happens iff the terms u and v contain the same variables. Hence, suppose that a variable x occurs in the term u and does not occur in the term v. Put all the other variables equal to e and obtain an equation $x^n \approx e$ for some n, valid in $\mathcal{V}$. However, this implies that $\mathcal{V}$ is trivial, because any non-trivial lattice-ordered monoid contains an element a comparable to e and we get a contradiction either by $e < a \leqslant a^2 \leqslant \ldots \leqslant a^n = e$ if $a > e$, or similarly if $a < e$. $\square$

Our motivation was the following result of Bahls, Cole, Galatos, Jipsen and Tsinakis [1].

1.2. Theorem. *Let $\mathcal{V}$ be a non-trivial subvariety of residuated lattices based (relatively to $\mathcal{RL}$) by equations in the language of lattices. Then $\mathcal{V}$ does not satisfy any non-trivial monoid equation (more precisely, for every equation ε in the language $\cdot, e$, if $\mathcal{V} \vDash \varepsilon$, then all monoids satisfy ε).*

Proof. Let $\mathbf{L}$ be a bounded lattice. We construct a residuated lattice $\mathbf{L}'$, whose monoid reduct is the free monoid over the alphabet L and whose lattice reduct satisfies the same lattice equations as $\mathbf{L}$ (it generates the same variety as $\mathbf{L}$). We identify words of length n over L with n-tuples of elements of L and define a lattice structure on the free monoid to be the ordinal sum of $\mathbf{L}^0$ (consisting of the empty word), $\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^3, \ldots$ (with the empty word on top). One can check that the resulting structure becomes a residuated lattice. Now, if a monoid identity holds in $\mathcal{V}$, it holds in $\mathbf{L}'$ for every $\mathbf{L}$ satisfying the relative base of $\mathcal{V}$. Hence it holds in free monoids and thus in every monoid. See [1] for details. □

Is there a similar theorem, with the role of lattice and monoid reducts interchanged?

1.3. Theorem. *The variety $\mathcal{CI}d\mathcal{RL}$ does not satisfy any non-trivial lattice equation (more precisely, for every equation ε in the language $\vee$, $\wedge$, if $\mathcal{CI}d\mathcal{RL} \models \varepsilon$, then all lattices satisfy ε).*

Proof. Let $\mathbf{L}$ be a bounded lattice. We construct a CI residuated lattice $\mathbf{L}'$, whose lattice reduct satisfies the same lattice equations as $\mathbf{L}$ (it generates the same variety as $\mathbf{L}$). Let us denote by 1 the top element of $\mathbf{L}$ and by e the bottom element of $\mathbf{L}$. Let L' be the disjoint union of L and $\{0\}$. The lattice structure on L' is defined so that 0 is added to $\mathbf{L}$ as a new bottom element. We define the multiplication by $00 = 0a = a0 = 0$ for every $a \in L$ and $ab = a \vee b$ for every $a, b \in L$. It is easy to check that this is a lattice-ordered CI monoid and it admits residuation as follows: $a/0 = 1$, $0/a = 0$, $a/b = a$ for $b \leqslant a$ and $a/b = 0$ for $b \nleqslant a$, $a, b \in L$. Now, if a lattice identity holds in $\mathcal{CI}d\mathcal{RL}$, it holds in $\mathbf{L}'$ for every bounded lattice $\mathbf{L}$ and thus it holds in all lattices. □

1.4. Corollary. *Let $\mathcal{V}$ be a non-trivial subvariety of residuated lattices based (relatively to $\mathcal{RL}$) by equations in the language of monoids. Then $\mathcal{V}$ does not satisfy any non-trivial lattice equation.*

Proof. According to Observation 1.1, the variety $\mathcal{CI}d\mathcal{RL}$ is a subvariety of $\mathcal{V}$ and thus Theorem 1.3 applies. □

2. BASIC PROPERTIES

2.1. Lemma. *Let $\mathbf{A}$ be a lattice-ordered idempotent monoid and $a, b \in A$.*

(1) $a \wedge b \leqslant ab \leqslant a \vee b$.

(2) *If* $a, b \geqslant e$, *then* $ab = a \vee b$.

(3) *If* $a, b \leqslant e$, *then* $ab = a \wedge b$.
(4) *If* $a \leqslant e \leqslant ab$, *then* $ab = b$.
(5) *If* $ab \leqslant e \leqslant a$, *then* $ab = b$.

Proof. (1) $a \wedge b \leqslant a, b \leqslant a \vee b$, hence $a \wedge b = (a \wedge b)(a \wedge b) \leqslant ab \leqslant (a \vee b)(a \vee b) = a \vee b$.

(2) If $a \geqslant e$, then $ab \geqslant eb = b$ and similarly also $ab \geqslant a$. Thus $ab \geqslant a \vee b$. The other inequality was proven in (1). Similarly for (3).

(4) $b = eb \leqslant abb = ab \leqslant eb = b$. Similarly for (5). □

The following two statements about congruence lattices of CI residuated lattices are immediate consequences of results in [4] and [9]. The second sentence of Proposition 2.2 appears also in [8] (in a more general setting).

2.2. Proposition. *The congruence lattice of* $\mathbf{A}$ *is isomorphic to the lattice of filters on* $\mathbf{A}^-$. *In particular, if* A *is finite, then* $\mathbf{Con}(\mathbf{A}) \simeq (\mathbf{A}^-)^\partial$.

Proof. Blount and Tsinakis described in [4] a correspondence between congruences of a residuated lattice $\mathbf{A}$ and convex normal submonoids of $\mathbf{A}^-$. We prove that convex normal submonoids in CI residuated lattices are precisely filters.

Let $M \subseteq A^-$. Since $a \wedge b = ab$ for all $a, b \leqslant e$, M is closed under meet iff it is closed under multiplication. If $e \in M$ (it indeed is, whenever $\mathbf{M}$ is a submonoid or a filter), then M is convex iff it is an upper set. Hence, it remains to show that every filter is normal. Since $(ba)/b = (ab)/b \geqslant a$ for all a, b, every conjugation mapping $\gamma(x) = ((bx)/b) \wedge e$ maps a negative element onto a greater one. Consequently, congruences of a CI residuated lattice correspond to filters. □

2.3. Corollary. *A CI residuated lattice* $\mathbf{A}$ *is simple iff* $|A^-| = 2$. *It is subdirectly irreducible iff* e *is completely join-irreducible.*

It is well-known that residuated lattices are congruence distributive and congruence permutable. In particular, the negative cone of a non-trivial CI residuated lattice is always distributive (in fact, it is a Heyting algebra) and contains at least two elements.

3. FINITELY AND NON-FINITELY BASED SUBVARIETIES

3.1. Proposition. *CI residuated lattices have definable principle congruences.*

Proof. Principal congruences correspond to principal filters, which are, of course, first-order definable. It can be checked easily that a congruence corresponding to a definable convex normal submonoid is also definable (generally for residuated lattices). □

In fact, N. Galatos proved a stronger result in [8]: principal congruences in commutative n-potent residuated lattices are *equationally* definable. This result is indeed more complicated.

3.2. Corollary. *A subvariety $\mathcal{V}$ of $\mathcal{CIdRL}$ is finitely based iff the class of subdirectly irreducible algebras in $\mathcal{V}$ is first-order definable.*

Proof. This is an immediate consequence of a theorem of K. Baker and J. Wang [2]. □

A non-finitely based variety of lattices was found by R. McKenzie in [11]. He constructed an infinite independent family $\varepsilon_1, \varepsilon_2, \dots$ of lattice equations and finite lattices $\mathbf{B}_1, \mathbf{B}_2, \dots$ such that $\mathbf{B}_n \not\models \varepsilon_n$ and $\mathbf{B}_n \models \varepsilon_m$ for every $m \neq n$. We modify his construction to get an example of a non-finitely based subvariety of CI residuated lattices.

3.3. Proposition. *Let $\mathcal{V}$ be a variety with a lattice reduct and assume that for every finite lattice $\mathbf{L}$ there is an algebra $\mathbf{A_L} \in \mathcal{V}$ such that $\mathbf{L}$ and $(A_{\mathbf{L}}, \vee, \wedge)$ satisfy the same lattice equations. Then the subvariety of $\mathcal{V}$ based (relatively to $\mathcal{V}$) by $\varepsilon_1, \varepsilon_2, \dots$ is not finitely based.*

Proof. Let us denote the subvariety by $\mathcal{W}$. If there were a finite base Σ of $\mathcal{W}$, by the compactness theorem, only finitely many ε_i's would be necessary to prove that Σ holds in $\mathcal{W}$. Thus there is n such that $\mathcal{CIdRL}, \varepsilon_1, \dots, \varepsilon_n \models \Sigma$. Hence, since Σ is a base of $\mathcal{W}$, a CI residuated lattice is in $\mathcal{W}$ iff it satisfies $\varepsilon_1, \dots, \varepsilon_n$. But it means that $\mathbf{A}_{\mathbf{B}_{m+1}} \in \mathcal{W}$, because $\mathbf{B}_{m+1}$ satisfies all the equations $\varepsilon_1, \dots, \varepsilon_m$. On the other hand, $\mathcal{W} \models \varepsilon_{m+1}$ and $\mathbf{A}_{\mathbf{B}_{m+1}} \not\models \varepsilon_{m+1}$. This is a contradiction. □

Proposition 3.3 applies to the variety $\mathcal{CIdRL}$; we can take, for example, $\mathbf{A_L} = \mathbf{L}'$ from the proof of Theorem 1.3. It applies also to the variety of cancellative residuated lattices, if we take $\mathbf{A_L} = \mathbf{L}'$ from the proof of Theorem 1.2.

4. More examples

A complete lattice $\mathbf{L}$ is called *infinitely join distributive*, if $\bigvee_{x \in X} (x \wedge y) = \Big(\bigvee_{x \in X} x\Big) \wedge y$ holds for any $X \subseteq L$ and $y \in L$.

Example. Let $\mathbf{D}$ be a complete infinitely join distributive lattice. Then the algebra $(D, \vee, \wedge, \wedge, 1, /)$ is a CI residuated lattice, where $a/b = \bigvee\{c\colon\ c \wedge b \leqslant a\}$. (Indeed, since a/b is the greatest c such that $c \wedge b \leqslant a$, we must have $\bigvee\{c\colon\ c \wedge b \leqslant a\}$. And the big join is less than a, if $\mathbf{D}$ is infinitely join distributive.)

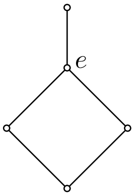
Example. Let $\mathbf{L}$ be a bounded lattice and $\mathbf{D}$ a complete infinitely join distributive lattice; suppose $L \cap D = \emptyset$. We construct a CI residuated lattice $\mathbf{L} \sqcup \mathbf{D}$ on the set $L \cup D$. Let $\mathbf{L}$, $\mathbf{D}$ be sublattices of $\mathbf{L} \sqcup \mathbf{D}$ with all elements of L greater then any element of D. Denote e the bottom element of $\mathbf{L}$ and t the top element of $\mathbf{D}$, while $0, 1$ refer to the top and bottom of $\mathbf{L} \sqcup \mathbf{D}$. Put $ab = a \vee b$ for $a, b \in L$, $ab = a \wedge b$ for $a, b \in D$ and $ab = ba = b$ for $a \in L$, $b \in D$. It is easy to check that this is a lattice-ordered CI monoid and that it admits residuation as follows:

- $a/b = a$ for $e \leqslant b \leqslant a$.
- $a/b = 1$ for $b \leqslant a$, $b \leqslant e$.
- $a/b = a$ for $a \leqslant e \leqslant b$.
- $a/b = t$ for $b \nleqslant a$, $a, b \geqslant e$.
- $a/b = \bigvee\{c \in D \colon\ c \wedge b \leqslant a\}$ for $b \nleqslant a$, $a, b \leqslant e$.

Consequently, for every bounded lattice $\mathbf{L}$ and complete infinitely join distributive lattice $\mathbf{D}$, there is a CI residuated lattice $\mathbf{A}$ with $(A^+, \vee, \wedge) = \mathbf{L}$, $(A^-, \vee, \wedge) = \mathbf{D} + \{e\}$ and all elements comparable to e. Note that the lattice $\mathbf{L} \sqcup \mathbf{D}$ is subdirectly irreducible.

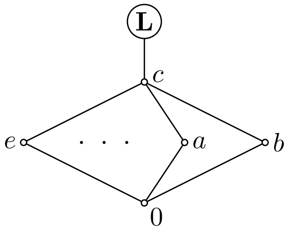
In particular, there exists a simple CI residuated lattice $\mathbf{L}'$ with $(L'^+, \vee, \wedge) = \mathbf{L}$ (take $\mathbf{D}$ trivial). By Lemma 2.1 (2), any simple CI residuated lattice with no elements incomparable to the unit is some $\mathbf{L}'$. Also, by Jónsson's lemma, $\mathbf{L}'$'s are the only subdirectly irreducible algebras in the variety they generate, hence they generate a proper subvariety of $\mathcal{CI}d\mathcal{RL}$. This variety is finitely based, according to Corollary 3.2. In fact, one can use the Galatos algorithm [7] and find a basis: it is based (relatively to $\mathcal{CI}d\mathcal{RL}$) by the single equation $((e/x) \wedge e) \vee ((y/x) \wedge e) \approx e$.

It is easy to check that there is (up to isomorphism) one 2-element CIRL, two 3-element CIRLs and four 4-element CIRLs. Using a computer, on can compute that there are twenty 5-element CIRLs; every 5-element lattice is a reduct of a CIRL; and in any 5-element lattice, one can choose $e \neq 0, 1$ arbitrarily, except for the following case:



We proved that every bounded lattice is a subreduct of a CI residuated lattice. However, there is a 6-element lattice, which is not a reduct of a CI residuated lattice.

4.1. Proposition. *Let* $\mathbf{L}$ *be a lattice and* $\mathbf{M}_n$ *be the* $(n+2)$*-element lattice with* n *atoms,* $n \geqslant 3$*. Then the ordinal sum* $\mathbf{L}'$ *of* $\mathbf{L}$ *and* $\mathbf{M}_n$ *(with* $\mathbf{L}$ *on top) is not a reduct of a CI residuated lattice.*



Proof. Assume there is a CI residuated lattice $\mathbf{A}$ with the lattice reduct $\mathbf{L}'$. First of all, note that the unit element must be one of the atoms—otherwise, $\mathbf{A}^-$ is not a non-trivial distributive lattice. Let us denote by e, a, b three distinct atoms and assume that e is the unit element. Let $c = e \vee a \vee b$ be the top element of $\mathbf{M}_n$. It is well known (see [4]) and easy to prove that in any residuated lattice multiplication distributes over joins, in symbols

$$x(y \vee z) \approx (xy) \vee (xz).$$

Using this identity, we get for every atom $x \neq e$ in $\mathbf{L}'$ that $xc = x(e \vee x) = x \vee x = x$. Another use of this identity yields $a = ac = a(e \vee b) = a \vee (ab)$ and similarly $b = b \vee (ab)$, so $ab \leqslant a$ and $ab \leqslant b$ and thus $ab = 0$. Now, choose $d \in L$. We have $(da) \vee (db) = d(a \vee b) = dc = d$ (because multiplication coincides with the join on positive elements). Hence, at least one of da, db must be greater than c; assume it is da. Then $c(db) \leqslant (da)(db) = d(ab) = d0 = 0$. However, this is possible iff $db = 0$, because $cx \geqslant c$ for every x positive and we have proved above that $cx = x$ for every atom $x \neq e$. But $db \geqslant eb = b$, a contradiction. ☐

A different argument yields examples of infinite lattices which are not reducts of any CI residuated lattice. Let $\mathbf{L}$ be an arbitrary simple atomless lattice (e.g. the dual of the lattice of subspaces of an infinite-dimensional vector space) and let $\mathbf{A}$ be a CI residuated lattice with the lattice reduct $\mathbf{L}$. By adding operations to a simple algebra, one gets again a simple algebra. Hence $\mathbf{A}$ is simple, but $\mathbf{A}^-$ cannot have two elements, because there are no atoms in $\mathbf{A}$, which contradicts Corollary 2.3.

The following propositions describe all totally ordered CI residuated lattices (i.e. those where the lattice reduct is a chain).

4.2. Proposition. *Let* $\mathbf{A} = (A, \vee, \wedge, \cdot, e)$ *be a structure such that* $(A, \vee, \wedge)$ *is a chain and* $(A, \cdot, e)$ *is a semilattice with a unit. Then the following are equivalent.*

(1) $\mathbf{A}$ *is a lattice-ordered monoid.*
(2) $ab = a \vee b$ *for every* $a, b \in A^+$, $ab = a \wedge b$ *for every* $a, b \in A^-$ *and the semilattice reduct is a chain.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 2.1. If a, b are both positive or both negative, 2.1 (2) or 2.1 (3) applies. Otherwise, since $\leqslant$ is a chain, we may assume that $a \leqslant e \leqslant b$. In this case, either $e \leqslant ab$ and 2.1 (4) applies, or $ab \leqslant e$ and 2.1 (5) applies.

(2) $\Rightarrow$ (1). Note that on the positive cone, $a \leqslant b$ iff $b \preceq a$, and on the negative cone, $a \leqslant b$ iff $a \preceq b$. Let $a \leqslant b$. We need to prove that $ac \leqslant bc$ for every $c \in A$. Since $(A, \preceq)$ is a chain, $ac \in \{a, c\}$ and $bc \in \{b, c\}$. Hence the only bad situation is either (a) $ac = a$, $bc = c$ and $a > c$, or (b) $ac = c$, $bc = b$ and $c > b$. We prove that none of them is actually possible. In (a), we have $c < a < b$ and $a \prec c \prec b$. The element a can't be positive, because in this case b is also positive and $a < b$ implies $b \prec a$. On the other hand, a can't be negative, because then c is also negative and $c < a$ implies $c \prec a$. This is a contradiction. In (b), we have $a < b < c$ and $b \prec c \prec a$ and a similar argument works. □

4.3. Corollary. *Let* $\mathbf{A} = (A, \vee, \wedge, \cdot, e)$ *be a structure such that* $(A, \vee, \wedge)$ *is a chain and* $(A, \cdot, e)$ *is a semilattice with a unit. Then the following are equivalent.*

(1) $(A, \vee, \wedge, \cdot, e, /)$ *is a residuated lattice for some* $/$.
(2) $ab = a \vee b$ *for every* $a, b \in A^+$, $ab = a \wedge b$ *for every* $a, b \in A^-$, *the semilattice reduct is a chain and for every* a, b *there is the greatest* c *such that* $ac \leqslant b$.

In particular, for A *finite, the conditions are equivalent to*

(3) $ab = a \vee b$ *for every* $a, b \geqslant e$, $ab = a \wedge b$ *for every* $a, b \leqslant e$ *and the semilattice reduct is a chain with* 0 *in bottom.*

Proof. (1) $\Leftrightarrow$ (2) follows obviously from the previous proposition. If (1), (2) are true, then (3) follows from the fact that 0 exists and $0a = a0 = 0$ for all a in any residuated lattice with 0. And if (3) holds, then there is always some c, namely $c = 0$, such that $ac \leqslant b$, and thus there is also the greatest such c. (Note that it is enough to assume that the dual of $(A, \vee, \wedge)$ is well-ordered with a top element, not necessarily finite.) □

5. Minimal varieties

Minimal subvarieties of residuated lattices were investigated by several authors, particularly by N. Galatos in [6]. He found also minimal subvarieties of $\mathcal{CIdRL}$—they are just two. We briefly reprove his result.

A residuated lattice is called *integral* if all its elements are negative. Let $\mathbf{C}_2$ be the two-element CI residuated lattice, $C_2 = \{0, 1\}$, $e = 1$. Let $\mathbf{C}_3$ be the three-element non-integral CI residuated lattice, $C_3 = \{0, e, 1\}$, $0 < e < 1$. (Note that, in fact, $\mathbf{C}_2$ is the only two-element residuated lattice and $\mathbf{C}_3$ is the only non-integral three-element residuated lattice.) Let $\mathcal{V}_2$, $\mathcal{V}_3$ be the varieties generated by $\mathbf{C}_2$, $\mathbf{C}_3$, respectively. It is clear from Jónsson's lemma that $\mathcal{V}_2$ and $\mathcal{V}_3$ are minimal varieties.

5.1. Theorem. *$\mathcal{V}_2$ and $\mathcal{V}_3$ are the only minimal subvarieties of $\mathcal{CIdRL}$.*

Proof. We show that every non-trivial subvariety $\mathcal{V}$ of $\mathcal{CIdRL}$ contains $\mathbf{C}_2$ or $\mathbf{C}_3$. According to the well known Magari theorem, $\mathcal{V}$ contains a (non-trivial) simple algebra $\mathbf{A}$. Indeed, $|A^-| = 2$, so $\mathbf{A}$ has the bottom and thus also the top element. We show that $B = \{0, e, 1\}$ is a subalgebra of $\mathbf{A}$—then it is isomorphic to one of $\mathbf{C}_2$, $\mathbf{C}_3$, depending on whether $e = 1$ or not. The set B is indeed closed under join, meet and multiplication. In any bounded residuated lattice the equations $x/0 \approx 1$, $x/e \approx x$ and $1/x \approx 1$ hold and $0/1 \leqslant e/1 < e$. Hence in a simple CI residuated lattice $0/1 = e/1 = 0$ and we are done. □

$\mathcal{V}_2$ is known as the variety of generalized Boolean algebras and it is based (relatively to $\mathcal{CIdRL}$) by $x \leqslant e$ and $y/(y/x) \approx x \vee y$. A finite base for the variety $\mathcal{V}_3$ can be found in [6] (or computed by the Galatos algorithm).

In fact, N. Galatos proved in [6] that $\mathbf{C}_2$ or $\mathbf{C}_3$ is a subalgebra of any idempotent residuated lattice $\mathbf{A}$ satisfying $e/x \approx x\backslash e$. If $\mathbf{A}$ is integral, then $\{a, e\}$ is a subalgebra isomorphic to $\mathbf{C}_2$ for every $a \neq e$ and if $\mathbf{A}$ is not integral, then $\{e/a, e, e/(e/a)\}$ is a subalgebra isomorphic to $\mathbf{C}_3$ for every $a > e$. Consequently, every subvariety of $\mathcal{CIdRL}$ is either integral, or contains $\mathbf{C}_3$ (in other words, $\mathbf{C}_3$ is a splitting algebra).

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