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CLIFFORD-HERMITE-MONOGENIC OPERATORS

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Abstract. In this paper we consider operators acting on a subspace $\mathscr{M}$ of the space $L_2(\mathbb{R}^m;\mathbb{C}_m)$ of square integrable functions and, in particular, Clifford differential operators with polynomial coefficients. The subspace $\mathscr{M}$ is defined as the orthogonal sum of spaces $\mathscr{M}_{s,k}$ of specific Clifford basis functions of $L_2(\mathbb{R}^m;\mathbb{C}_m)$.

Every Clifford endomorphism of $\mathscr{M}$ can be decomposed into the so-called Clifford-Hermite-monogenic operators. These Clifford-Hermite-monogenic operators are characterized in terms of commutation relations and they transform a space $\mathscr{M}_{s,k}$ into a similar space $\mathscr{M}_{s',k'}$. Hence, once the Clifford-Hermite-monogenic decomposition of an operator is obtained, its action on the space $\mathscr{M}$ is known. Furthermore, the monogenic decomposition of some important Clifford differential operators with polynomial coefficients is studied in detail.

Keywords: differential operators, Clifford analysis

MSC 2000: 47B99, 30G35

1. Introduction

Let $\mathbb{C}_m$ be the complex Clifford algebra generated by the orthonormal basis $(e_1,\ldots,e_m)$ of $\mathbb{R}^m$ and determined by the relations $e_je_k+e_ke_j=-2\delta_{j,k}$ $(1\leqslant j,k\leqslant m)$. Then in $\mathbb{R}^m$ we consider the vector variable $\underline{x}=\sum\limits_{j=1}^{m}e_jx_j$ and the so-called Dirac operator $\partial_{\underline{x}}=\sum\limits_{j=1}^{m}e_j\partial_{x_j}$. Left nullsolutions of $\partial_{\underline{x}}$ are called left monogenic functions; they are at the heart of Clifford analysis.

In this paper we study the so-called Clifford-Hermite-monogenic operators. For introducing this subject we first explain the already existing notion of the monogenic operator in the polynomial framework.

Let $\mathscr{P}^s$ be the space of scalar-valued polynomials in $\mathbb{R}^m$. Then a Clifford algebra-valued polynomial, Clifford polynomial for short, is an element of $\mathscr{P}=\mathscr{P}^s\otimes\mathbb{C}_m$.

The inner product on the space $\mathscr{P}$ is defined as

$$\langle P(\underline{x}), Q(\underline{x})\rangle = \{\overline{P}(\partial_{\underline{x}})Q(\underline{x})\}_{\underline{x}=\underline{0}}$$

with $P(\partial_{\underline{x}})$ the differential operator obtained by replacing each x_i by ∂_{x_i} in $P(\underline{x})$ and the bar denoting the so-called conjugation in the Clifford algebra.

The subspaces $\mathscr{P}_k$ of homogeneous Clifford polynomials of degree k, $k \in \mathbb{N}$, are the polynomial eigenspaces of the Euler operator

$$E = \left\langle \underline{x}, \partial_{\underline{x}} \right\rangle = \sum_{j=1}^{m} x_j \partial_{x_j},$$

i.e.

$$ER_k = kR_k, \quad R_k \in \mathscr{P}_k.$$

Obviously, every Clifford polynomial can be decomposed into homogeneous ones.

Let $\mathrm{End}(\mathscr{P}^s)$ be the algebra consisting of endomorphisms of $\mathscr{P}^s$, then the elements of $\mathrm{End}(\mathscr{P}^s) \otimes \mathbb{C}_m$ are called Clifford polynomial operators. A Clifford polynomial operator A transforms a Clifford polynomial $P \in \mathscr{P}$ into another Clifford polynomial $AP \in \mathscr{P}$.

Like a Clifford polynomial, every Clifford polynomial operator can be decomposed into homogeneous parts:

$$A = \sum_{l \in \mathbb{Z}} A_l \tag{1}$$

with A_l a homogeneous polynomial operator of degree l, i.e. $A_l \mathscr{P}_k \subset \mathscr{P}_{k+l}$, $k \in \mathbb{N}$. Moreover, the homogeneous operators A_l are determined by the commutation relation $[E, A_l] = EA_l - A_l E = lA_l$.

A left monogenic homogeneous Clifford polynomial P_k of degree k is called a left inner spherical monogenic of order k. The set of all left inner spherical monogenics of order k is denoted by $M_l^+(k)$.

The left inner spherical monogenics are polynomial eigenfunctions of the so-called spherical Dirac operator

$$\Gamma = -\sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i}),$$

i.e.

$$\Gamma P_k = -kP_k, \quad P_k \in M_l^+(k).$$

Every homogeneous Clifford polynomial R_k of degree k admits a canonical decomposition of the form

$$R_k(\underline{x}) = \sum_{s=0}^{k} \underline{x}^s P_{k-s}(\underline{x}), \quad P_{k-s} \in M_l^+(k-s).$$

This so-called monogenic decomposition also yields a monogenic decomposition of the space $\mathscr{P}$ of Clifford polynomials:

$$\mathscr{P} = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \oplus_\perp M_{s,k},$$

where

$$M_{s,k} = \{\underline{x}^s P_k(\underline{x});\ P_k \in M_l^+(k)\}.$$

Similarly to the polynomial setting, the decomposition (1) of a Clifford polynomial operator into homogeneous ones can be further refined to a decomposition into the so-called monogenic operators $A^{\pm}_{\lambda,\kappa}$:

$$A = \sum_{\lambda,\kappa} (A^+_{\lambda,\kappa} + A^-_{\lambda,\kappa}). \tag{2}$$

These monogenic operators $A^{\pm}_{\lambda,\kappa}$ transform each space $M_{s,k}$ into a similar space $M_{s',k'}$. Hence, once the monogenic decomposition (2) of a Clifford polynomial operator is obtained, its action on the space $\mathscr{P}$ of Clifford polynomials is known. As the spaces $M_{s,k}$ are the simultaneous eigenspaces of the operators E and Γ, the monogenic operators are characterized in terms of commutation relations involving E and Γ. In [5] the monogenic decomposition of differential operators acting on Clifford polynomials was studied in detail.

In this paper we consider operators acting on the space

$$\mathscr{M} = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \oplus_\perp \mathscr{M}_{s,k}.$$

Each function $f \in \mathscr{M}$ can be decomposed into a finite orthogonal sum

$$f = {\sum_s}' {\sum_k}' f_{s,k}$$

where $f_{s,k} \in \mathscr{M}_{s,k} = \mathrm{span}\{\varphi_{s,k,j}(\underline{x});\ j = 1, 2, \ldots, \dim(M_l^+(k))\}$ with

$$\varphi_{s,k,j}(\underline{x}) = \exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big),$$

$s, k \in \mathbb{N}$, $j = 1, 2, \ldots, \dim(M_l^+(k))$. The functions $\varphi_{s,k,j}(\underline{x})$ are orthogonal with respect to the standard inner product on the space $L_2(\mathbb{R}^m; \mathbb{C}_m)$ of square integrable functions. Furthermore, the set

$$\{P_k^{(j)}(\underline{x});\ j = 1, 2, \ldots, \dim(M_l^+(k))\}$$

constitutes an orthonormal basis of the space $M_l^+(k)$ and the $H_{s,m,k}(\underline{x})$ are the so-called generalized Clifford-Hermite polynomials, which are a generalization to Clifford analysis of the classical Hermite polynomials on the real line.

As the set

$$\{\varphi_{s,k,j}(\underline{x});\ s, k \in \mathbb{N}, j = 1, 2, \ldots, \dim(M_l^+(k))\}$$

is an orthogonal basis for the space $L_2(\mathbb{R}^m; \mathbb{C}_m)$, this space is precisely the closure of $\mathscr{M}$: $\overline{\mathscr{M}} = L_2(\mathbb{R}^m; \mathbb{C}_m)$.

Like in the polynomial framework, every Clifford endomorphism of $\mathscr{M}$ can be decomposed into the so-called Clifford-Hermite-monogenic (abbreviated CH-monogenic) operators. These CH-monogenic operators transform a space $\mathscr{M}_{s,k}$ into another such space $\mathscr{M}_{s',k'}$. As the spaces $\mathscr{M}_{s,k}$ are simultaneous eigenspaces of the operators

$$O_1 = \frac{1}{2}(\partial_{\underline{x}} - \underline{x})(\partial_{\underline{x}} + \underline{x}) \quad \text{and} \quad O_2 = \frac{1}{2}(\partial_{\underline{x}} + \underline{x})(\partial_{\underline{x}} - \underline{x}),$$

our CH-monogenic operators are characterized in terms of commutation relations involving O_1 and O_2.

The outline of the paper is as follows. For the reader who is not familiar with Clifford analysis, we recall some of its basics in Section 2. Also in this section we clarify the notion of Clifford differential operator with polynomial coefficients. Next, we introduce the space $\mathscr{M}$ and show that the spaces $\mathscr{M}_{s,k}$ are simultaneous eigenspaces of the commuting operators O_1 and O_2 (Section 3). In Section 4 we define our CH-monogenic and CH-anti-monogenic operators of degree (λ, κ) by means of commutation relations involving O_1 and O_2. We prove that such operators indeed transform $\mathscr{M}_{s,k}$ into an $\mathscr{M}_{s',k'}$ where s' and k' are determined in terms of s, k, the dimension m and the degrees of monogenicity λ and κ of the CH-monogenic operator under consideration. In the final section we establish the CH-monogenic decomposition of some important Clifford differential operators with polynomial coefficients.

2. Clifford analysis

Clifford analysis (see e.g. [1] and [3]) offers a function theory which is a higher dimensional analogue of the theory of holomorphic functions of one complex variable.

Consider functions defined in $\mathbb{R}^m$ $(m > 1)$ and taking values in the Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for $\mathbb{R}_m$ is given by $(e_A\colon\ A \subset \{1, \ldots, m\})$ where $e_\emptyset = 1$ is the identity element. The non-commutative multiplication in the Clifford algebra $\mathbb{R}_m$ is governed by the rules

$$e_j e_k + e_k e_j = -2\delta_{j,k}, \quad j, k = 1, \ldots, m.$$

Conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j, \quad j = 1, \ldots, m$$

with the additional rule $\overline{i} = -i$ in the case of $\mathbb{C}_m$.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j.$$

The product of two vectors splits up into a scalar part and a so-called bivector part:

$$\underline{x}\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y},$$

where

$$\underline{x} \cdot \underline{y} = -\left\langle \underline{x}, \underline{y} \right\rangle = -\sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j).$$

In particular, we have

$$\underline{x}^2 = -\left\langle \underline{x}, \underline{x} \right\rangle = -|\underline{x}|^2.$$

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$ is called left monogenic in an open region of $\mathbb{R}^m$, if in that region

$$\partial_{\underline{x}} F = 0.$$

Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$:

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j},$$

an elliptic vector operator of the first order, splitting the Laplace operator in $\mathbb{R}^m$:

$$\Delta_m = -\partial_{\underline{x}}^2.$$

The notion of right monogenicity is defined in a similar way by letting the Dirac operator act from the right.

In the sequel the monogenic homogeneous polynomials will play an important rôle.

A left or right, monogenic homogeneous polynomial P_k of degree k $(k \geqslant 0)$ is called a left, respectively right, inner spherical monogenic of order k. The set of all left, respectively right, inner spherical monogenics of order k is denoted by $M_l^+(k)$, respectively $M_r^+(k)$. The dimension of $M_l^+(k)$ is given by

$$\dim(M_l^+(k)) = \frac{(m+k-2)!}{(m-2)!\,k!}.$$

In this paper we consider the so-called Clifford differential operators with polynomial coefficients. These operators have the form

$$P(\underline{x}, \partial_{\underline{x}}) = \sum_{\underline{\alpha}} p_{\underline{\alpha}}(\underline{x}) \partial_{\underline{x}}^{\underline{\alpha}},$$

where we use the notation

$$\partial_{\underline{x}}^{\underline{\alpha}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_m}^{\alpha_m}$$

and where $p_{\underline{\alpha}}(\underline{x}) \in \mathscr{P}$ is a Clifford polynomial.

Let $\mathscr{D}(\mathbb{C}_m)$ denote the algebra of Clifford differential operators with polynomial coefficients. Then clearly $\mathscr{D}(\mathbb{C}_m) \subset \mathrm{End}(\mathscr{P}^s) \otimes \mathbb{C}_m$. This algebra $\mathscr{D}(\mathbb{C}_m)$ is generated by the basic operators $\{e_j, x_j, \partial_{x_j};\ j = 1, 2, \dots, m\}$.

Important examples of such Clifford differential operators are:

- the Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$,
- the vector multiplication operator $f \to \underline{x} f$, with $\underline{x} = \sum_{j=1}^{m} x_j e_j$,
- the Euler operator $E = \langle \underline{x}, \partial_{\underline{x}} \rangle = \sum_{j=1}^{m} x_j \partial_{x_j} = r\partial_r$; $r = |\underline{x}|$,
- the spherical Dirac operator $\Gamma = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i})$.

3. The space $\mathscr{M}$ and the operators O_1 and O_2

The set

$$\left\{\varphi_{s,k,j}(\underline{x}) = \exp(-\tfrac{1}{2}|\underline{x}|^2)H_{s,m,k}\big(\sqrt{2}\underline{x}\big)P_k^{(j)}\big(\sqrt{2}\underline{x}\big)\right\}$$

with $s, k \in \mathbb{N}$, $j = 1, 2, \ldots, \dim(M_l^+(k))$, constitutes an orthogonal basis for $L_2(\mathbb{R}^m; \mathbb{C}_m)$ (see [2]). Here

$$\{P_k^{(j)}(\underline{x});\ j = 1, 2, \ldots, \dim(M_l^+(k))\}$$

is an orthonormal basis of $M_l^+(k)$ and $H_{s,m,k}(\underline{x})$ are the generalized Clifford-Hermite polynomials introduced by Sommen in [4]. The first generalized Clifford-Hermite polynomials are given by:

$$\begin{aligned}
H_{0,m,k}(\underline{x}) &= 1,\\
H_{1,m,k}(\underline{x}) &= \underline{x},\\
H_{2,m,k}(\underline{x}) &= \underline{x}^2 + 2k + m = -r^2 + 2k + m,\\
H_{3,m,k}(\underline{x}) &= \underline{x}^3 + (2k + m + 2)\underline{x} = \underline{x}(-r^2 + 2k + m + 2),\\
H_{4,m,k}(\underline{x}) &= \underline{x}^4 + 2(2k + m + 2)\underline{x}^2 + (2k + m)(2k + m + 2)\\
&= r^4 - 2(2k + m + 2)r^2 + (2k + m)(2k + m + 2),\\
H_{5,m,k}(\underline{x}) &= \underline{x}^5 + 2(2k + m + 4)\underline{x}^3 + (2k + m + 4)(2k + m + 2)\underline{x}\\
&= \underline{x}(r^4 - 2(2k + m + 4)r^2 + (2k + m + 4)(2k + m + 2)).
\end{aligned}$$

Note that $H_{s,m,k}(\underline{x})$ is a polynomial of degree s in the variable $\underline{x}$ with real coefficients depending on k. Furthermore, $H_{2s,m,k}(\underline{x})$ only contains even powers of $\underline{x}$, while $H_{2s+1,m,k}(\underline{x})$ only contains odd ones.

The basis functions $\varphi_{s,k,j}(\underline{x})$ satisfy the orthogonality relation

$$\int_{\mathbb{R}^m} \bar{\varphi}_{s,k_1,j_1}(\underline{x})\varphi_{t,k_2,j_2}(\underline{x})\,\mathrm{d}V(\underline{x}) = \frac{\gamma_{s,k_1}}{2^{m/2}}\delta_{s,t}\delta_{k_1,k_2}\delta_{j_1,j_2}$$

with $\mathrm{d}V(\underline{x})$ the Lebesgue measure on $\mathbb{R}^m$ and γ_{s,k_1} a real constant depending on the parity of s. In other words, they are orthogonal with respect to the inner product on $L_2(\mathbb{R}^m; \mathbb{C}_m)$ defined as

$$\langle f, g\rangle = \int_{\mathbb{R}^m} \overline{f}(\underline{x})g(\underline{x})\,\mathrm{d}V(\underline{x}), \quad f, g \in L_2(\mathbb{R}^m; \mathbb{C}_m).$$

Note that this inner product is Clifford algebra-valued.

In what follows we study the action of operators on the space $\mathscr{M} \subset L_2(\mathbb{R}^m; \mathbb{C}_m)$ given by the algebraic orthogonal sum

$$\mathscr{M} = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \oplus_{\perp} \mathscr{M}_{s,k}$$

with

$$\mathscr{M}_{s,k} := \operatorname{span}\{\varphi_{s,k,j}(\underline{x});\ j = 1, 2, \ldots, \dim(M_l^+(k))\}.$$

Each function $f \in \mathscr{M}$ is to be written as a finite sum:

$$f = \sum_s{}' \sum_k{}' f_{s,k}, \quad f_{s,k} \in \mathscr{M}_{s,k}$$

and naturally

$$\overline{\mathscr{M}} = L_2(\mathbb{R}^m; \mathbb{C}_m).$$

As the set

$$\left\{H_{s,m,k}\left(\sqrt{2}\underline{x}\right) P_k^{(j)}\left(\sqrt{2}\underline{x}\right);\ s, k \in \mathbb{N},\ j = 1, 2, \ldots, \dim(M_l^+(k))\right\}$$

constitutes a basis for the space of Clifford polynomials and as $\mathscr{D}(\mathbb{C}_m) \subset \operatorname{End}(\mathscr{P}^s) \otimes \mathbb{C}_m$, every Clifford differential operator with polynomial coefficients transforms an element of $\mathscr{M}$ into another element of $\mathscr{M}$.

Now let us consider the operators

$$O_1 = \frac{1}{2}(\partial_{\underline{x}} - \underline{x})(\partial_{\underline{x}} + \underline{x}) \quad \text{and} \quad O_2 = \frac{1}{2}(\partial_{\underline{x}} + \underline{x})(\partial_{\underline{x}} - \underline{x}),$$

and more precisely their action on the spaces $\mathscr{M}_{s,k}$.

We start with some preliminary calculus.

First, we have that

$$\begin{aligned}(3) \qquad &(\partial_{\underline{x}} - \underline{x})\left(\exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s,m,k}\left(\sqrt{2}\underline{x}\right) P_k^{(j)}\left(\sqrt{2}\underline{x}\right)\right) \\ &\quad = \exp(-\tfrac{1}{2}|\underline{x}|^2)(\partial_{\underline{x}} - 2\underline{x})\left(H_{s,m,k}\left(\sqrt{2}\underline{x}\right) P_k^{(j)}\left(\sqrt{2}\underline{x}\right)\right).\end{aligned}$$

The recurrence relation (see [4])

$$(\underline{x} - \partial_{\underline{x}})(H_{s,m,k}(\underline{x}) P_k^{(j)}(\underline{x})) = H_{s+1,m,k}(\underline{x}) P_k^{(j)}(\underline{x})$$

implies that

$$(2\underline{x} - \partial_{\underline{x}})\left(H_{s,m,k}\left(\sqrt{2}\underline{x}\right) P_k^{(j)}\left(\sqrt{2}\underline{x}\right)\right) = \sqrt{2}\, H_{s+1,m,k}\left(\sqrt{2}\underline{x}\right) P_k^{(j)}\left(\sqrt{2}\underline{x}\right).$$

Consequently, (3) becomes

$$
\begin{aligned}
(\partial_{\underline{x}} - \underline{x})\big(\exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big)\big) \\
= -\sqrt{2}\exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s+1,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big).
\end{aligned}
$$

Hence we can qualify the operator $\partial_{\underline{x}} - \underline{x}$ as a creation operator, since it increases the degree of the generalized Clifford-Hermite polynomial.

Next, we have that

$$
\begin{aligned}
(\partial_{\underline{x}} + \underline{x})\big(\exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big)\big) \\
= \exp(-\tfrac{1}{2}|\underline{x}|^2) \partial_{\underline{x}}\big(H_{s,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big)\big).
\end{aligned}
\tag{4}
$$

In [2] we proved that

$$
\partial_{\underline{x}}(H_{s,m,k}(\underline{x}) P_k^{(j)}(\underline{x})) = -C_{s,m,k} H_{s-1,m,k}(\underline{x}) P_k^{(j)}(\underline{x})
$$

with

$$
C_{s,m,k} = \begin{cases} s & \text{for } s \text{ even,} \\ s - 1 + m + 2k & \text{for } s \text{ odd,} \end{cases}
$$

from which we readily obtain that

$$
\partial_{\underline{x}}\big(H_{s,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big)\big) = -\sqrt{2}\, C_{s,m,k} H_{s-1,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big).
$$

Hence we have that

$$
\begin{aligned}
(\partial_{\underline{x}} + \underline{x})\big(\exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big)\big) \\
= -\sqrt{2}\, C_{s,m,k} \exp(-\tfrac{1}{2}|\underline{x}|^2) H_{s-1,m,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big).
\end{aligned}
$$

Consequently, the operator $\partial_{\underline{x}} + \underline{x}$ is an annihilation operator, since it decreases the degree of the generalized Clifford-Hermite polynomial.

From the above results it follows at once that the spaces $\mathscr{M}_{s,k}$ are simultaneous eigenspaces of the operators O_1 and O_2.

Proposition 1. *One has*

$$
O_1(\varphi_{s,k,j}(\underline{x})) = C_{s,m,k} \varphi_{s,k,j}(\underline{x})
$$

and

$$
O_2(\varphi_{s,k,j}(\underline{x})) = C_{s+1,m,k} \varphi_{s,k,j}(\underline{x}),
$$

with

$$C_{s,m,k} = \begin{cases} s & \textit{for } s \textit{ even,} \\ s-1+m+2k & \textit{for } s \textit{ odd.} \end{cases}$$

As the spherical Dirac operator Γ can be written as

$$\Gamma = -\frac{1}{2}(\underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} - m),$$

the operators O_1 and O_2 also take the form

$$(5.)\qquad O_1 = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) + \left(\Gamma - \frac{m}{2}\right) \quad \text{and} \quad O_2 = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) - \left(\Gamma - \frac{m}{2}\right)$$

Using these expressions, we now easily obtain that O_1 and O_2 commute.

Proposition 2. *One has*

$$[O_1, O_2] = 0.$$

Proof. As the spherical Dirac operator Γ commutes with the Laplace operator Δ_m and with the multiplication operator r, i.e.

$$[\Delta_m, \Gamma] = 0 \quad \text{and} \quad [r, \Gamma] = 0,$$

we have

$$\left[\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2), \Gamma\right] = \left[\frac{1}{2}(-\Delta_m + r^2), \Gamma\right] = 0$$

which, in view of (5), immediately implies that O_1 and O_2 commute. □

4. Clifford-Hermite-monogenic operators

As already mentioned in the introduction (Section 1), the Clifford-Hermite-monogenic operators (CH-monogenic operators for short) which we introduce now will transform elements of a space $\mathscr{M}_{s,k}$ into elements of a space $\mathscr{M}_{s',k'}$ for some s' and k'. As the spaces $\mathscr{M}_{s,k}$ are simultaneous eigenspaces of the operators O_1 and O_2, the CH-monogenicity property is expressed in terms of commutation relations involving these operators.

Definition.

(i) A Clifford endomorphism A of $\mathscr{M}$ is called CH-monogenic of degree (λ, κ), notation $A \in \chi^+_{\lambda,\kappa}$, if

$$[O_1, A] = \lambda A \quad \text{and} \quad [O_2, A] = \kappa A.$$

(ii) A Clifford endomorphism B of $\mathscr{M}$ is called CH-anti-monogenic of degree (λ, κ), notation $B \in \chi^-_{\lambda,\kappa}$, if

$$O_1 B = B O_2 + \lambda B \quad \text{and} \quad O_2 B = B O_1 + \kappa B.$$

Remarks.

1. As mentioned in Section 3, Clifford differential operators with polynomial coefficients belong to the set of endomorphisms of $\mathscr{M}$.
2. The operators O_1 and O_2 themselves are CH-monogenic of degree $(0, 0)$.
3. A Clifford endomorphism A of $\mathscr{M}$ is CH-monogenic of degree (λ, κ) if and only if
$$\Big[\frac{1}{2}(-\Delta_m + r^2), A\Big] = \frac{\lambda + \kappa}{2} A \quad \text{and} \quad \Big[\Gamma - \frac{m}{2}, A\Big] = \frac{\lambda - \kappa}{2} A.$$
4. A Clifford endomorphism B of $\mathscr{M}$ is CH-anti-monogenic of degree (λ, κ) if and only if
$$\frac{1}{2}(-\Delta_m + r^2) B = B \frac{1}{2}(-\Delta_m + r^2) + \frac{\lambda + \kappa}{2} B$$
and
$$\Big(\Gamma - \frac{m}{2}\Big) B = -B\Big(\Gamma - \frac{m}{2}\Big) + \frac{\lambda - \kappa}{2} B.$$

CH-monogenic and CH-anti-monogenic operators are closed under composition as shown in the next proposition.

Proposition 3.
(i) *If* $A \in \chi^+_{\lambda,\kappa}$ *and* $B \in \chi^+_{\lambda',\kappa'}$ *then* $AB \in \chi^+_{\lambda+\lambda',\kappa+\kappa'}$ *and* $BA \in \chi^+_{\lambda+\lambda',\kappa+\kappa'}$.
(ii) *If* $A \in \chi^-_{\lambda,\kappa}$ *and* $B \in \chi^-_{\lambda',\kappa'}$ *then* $AB \in \chi^+_{\lambda+\kappa',\lambda'+\kappa}$ *and* $BA \in \chi^+_{\lambda'+\kappa,\lambda+\kappa'}$.
(iii) *If* $A \in \chi^+_{\lambda,\kappa}$ *and* $B \in \chi^-_{\lambda',\kappa'}$ *then* $AB \in \chi^-_{\lambda+\lambda',\kappa+\kappa'}$ *and* $BA \in \chi^-_{\lambda'+\kappa,\lambda+\kappa'}$.

Proof. The proof of (ii) goes as follows:

$$O_1 AB = (AO_2 + \lambda A)B = A(BO_1 + \kappa' B) + \lambda AB = ABO_1 + (\lambda + \kappa')AB$$

and

$$O_2 AB = (AO_1 + \kappa A)B = A(BO_2 + \lambda' B) + \kappa AB = ABO_2 + (\lambda' + \kappa)AB.$$

Similarly we find for the operator BA:

$$O_1 BA = BAO_1 + (\lambda' + \kappa)BA$$

and

$$O_2 BA = BAO_2 + (\lambda + \kappa')BA.$$

The proofs of (i) and (iii) are similar. $\square$

We now prove that the CH-monogenic operators indeed transform $\mathscr{M}_{s,k}$ into an $\mathscr{M}_{s',k'}$:

Proposition 4. *Let $A \in \chi^{+}_{\lambda,\kappa}$ and $f \in \mathscr{M}_{s,k}$. Then $Af \in \mathscr{M}_{s',k'}$ for some s' and k' depending on s, k, λ, κ and m.*

Proof. We start by observing that Af is a simultaneous eigenfunction of O_1 and O_2:

$$\begin{aligned}O_1(Af) &= (AO_1 + \lambda A)f = (C_{s,m,k} + \lambda)Af\\ O_2(Af) &= (AO_2 + \kappa A)f = (C_{s+1,m,k} + \kappa)Af.\end{aligned}$$

As Af belongs to the space $\mathscr{M}$, it can be written as

$$Af = \sum_{j=0}^{\infty}{}' \sum_{t=0}^{\infty}{}' \sum_{i=1}^{\dim(M_l^+(t))} \varphi_{j,t,i}(\underline{x}) a_{j,t,i}; \quad a_{j,t,i} \in \mathbb{C}_m.$$

Hence we also have

$$O_1(Af) = \sum_{j=0}^{\infty}{}' \sum_{t=0}^{\infty}{}' \sum_{i=1}^{\dim(M_l^+(t))} C_{j,m,t}\varphi_{j,t,i}(\underline{x}) a_{j,t,i}.$$

Comparing the above expression with

$$O_1(Af) = \sum_{j=0}^{\infty}{}' \sum_{t=0}^{\infty}{}' \sum_{i=1}^{\dim(M_l^+(t))} (C_{s,m,k} + \lambda)\varphi_{j,t,i}(\underline{x}) a_{j,t,i},$$

we obtain that either $a_{j,t,i} = 0$ or $C_{j,m,t} = C_{s,m,k} + \lambda$.

Similarly, comparing

$$O_2(Af) = \sum_{j=0}^{\infty}{}' \sum_{t=0}^{\infty}{}' \sum_{i=1}^{\dim(M_l^+(t))} C_{j+1,m,t}\varphi_{j,t,i}(\underline{x}) a_{j,t,i}$$

with

$$O_2(Af) = \sum_{j=0}^{\infty}{}' \sum_{t=0}^{\infty}{}' \sum_{i=1}^{\dim(M_l^+(t))} (C_{s+1,m,k} + \kappa)\varphi_{j,t,i}(\underline{x}) a_{j,t,i}$$

yields that either $a_{j,t,i} = 0$ or $C_{j+1,m,t} = C_{s+1,m,k} + \kappa$. Consequently, we must prove that at most one pair of indices (j,t) satisfies the set of equations

$$(6) \qquad \begin{cases} C_{j,m,t} = C_{s,m,k} + \lambda, \\ C_{j+1,m,t} = C_{s+1,m,k} + \kappa. \end{cases}$$

Hereto we must distinguish several cases.

Case A.

If s is even, the set of equations (6) becomes

$$
\begin{cases} C_{j,m,t} = s + \lambda, \\ C_{j+1,m,t} = s + m + 2k + \kappa. \end{cases} \tag{7}
$$

Case A.1: m odd, λ odd

From the first equation we obtain that j must be odd. For j odd, the set of equations (7) becomes

$$
\begin{cases} j - 1 + m + 2t = s + \lambda, \\ j + 1 = s + m + 2k + \kappa, \end{cases}
$$

leading to

$$
j = s + m + 2k + \kappa - 1 \quad \text{and} \quad 2t = \lambda + 2 - 2m - 2k - \kappa.
$$

In this case, the second equation implies that κ must be odd. As t must be positive, we thus have

$$
A\colon\ \mathscr{M}_{s,k} \to \begin{cases} 0 & \text{for } 2k > \lambda + 2 - 2m - \kappa, \\ \mathscr{M}_{s+m+2k+\kappa-1,\frac{1}{2}(\lambda+2-2m-2k-\kappa)} & \text{for } 2k \leqslant \lambda + 2 - 2m - \kappa. \end{cases}
$$

Case A.2: m odd, λ even

Now the first equation of (7) implies that j must be even. For j even, the set of equations (7) becomes

$$
\begin{cases} j = s + \lambda, \\ j + m + 2t = s + m + 2k + \kappa, \end{cases}
$$

which implies

$$
j = s + \lambda \quad \text{and} \quad 2t = 2k + \kappa - \lambda.
$$

Now κ must be even and we have

$$
A\colon\ \mathscr{M}_{s,k} \to \begin{cases} 0 & \text{for } 2k < \lambda - \kappa, \\ \mathscr{M}_{s+\lambda,(2k+\kappa-\lambda)/2} & \text{for } 2k \geqslant \lambda - \kappa. \end{cases}
$$

Case A.3: m even, λ odd

In this case $s + \lambda$ is odd. As $C_{j,m,t}$ is always even, we have

$$
A\colon\ \mathscr{M}_{s,k} \to 0.
$$

Case A.4: m even, λ even

As to the first equation of (7), both j even and j odd are possible. Hence we have to make a distinction between κ even and κ odd.

In the case when κ is even, both j even and j odd are possible for the second equation.

For j even, the set of equations (7) becomes

$$\begin{cases} j = s + \lambda, \\ j + m + 2t = s + m + 2k + \kappa, \end{cases}$$

and hence

$$j = s + \lambda \quad \text{and} \quad 2t = 2k + \kappa - \lambda.$$

For j odd, we have

$$\begin{cases} j - 1 + m + 2t = s + \lambda, \\ j + 1 = s + m + 2k + \kappa, \end{cases}$$

thus

$$j = s + m + 2k + \kappa - 1 \quad \text{and} \quad 2t = \lambda - 2m + 2 - 2k - \kappa.$$

As t must be positive, we have that $2k \geqslant \lambda - \kappa$ for j even, while $2k \leqslant \lambda - \kappa - (2m-2) < \lambda - \kappa$ for j odd.

This implies

$$A\colon\ \mathscr{M}_{s,k} \to \begin{cases} \mathscr{M}_{s+\lambda,(2k+\kappa-\lambda)/2} & \text{for } 2k \geqslant \lambda - \kappa, \\ 0 & \text{for } \lambda - \kappa - (2m-2) < 2k < \lambda - \kappa, \\ \mathscr{M}_{s+m+2k+\kappa-1,\frac{1}{2}(\lambda-2m+2-2k-\kappa)} & \text{for } 2k \leqslant \lambda - \kappa - (2m-2). \end{cases}$$

In the case when κ is odd, the second equation of (7) implies that neither j even, nor j odd is possible. Hence, we have

$$A\colon\ \mathscr{M}_{s,k} \to 0.$$

Case B.

The case when s is odd is treated in a similar way. $\square$

In a completely analogous manner we obtain the following result for the CH-anti-monogenic operators:

Proposition 5. *Let $B \in \chi^-_{\lambda,\kappa}$ and $f \in \mathscr{M}_{s,k}$. Then $Bf \in \mathscr{M}_{s',k'}$ for some s' and k' depending on s, k, λ, κ and m.*

Now we are able to prove that every Clifford endomorphism of $\mathscr{M}$ admits a decomposition into CH-(anti-) monogenic operators, which we call the Clifford-Hermite-monogenic decomposition (CHM-decomposition for short).

Theorem. *Every Clifford endomorphism of $\mathscr{M}$ admits a CHM-decompostion.*

Proof. It is clear that an arbitrary Clifford endomorphism A of $\mathscr{M}$ can be written as follows:

$$A = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} A_{s,k},$$

where, by definition,

$$A_{s,k}|_{\mathscr{M}_{s,k}} = A|_{\mathscr{M}_{s,k}} \quad \text{and} \quad A_{s,k}|_{\mathscr{M}_{s',k'}} = 0 \quad \text{whenever } (s,k) \neq (s',k').$$

As A is an endomorphism of $\mathscr{M}$, we also have

$$A_{s,k}(\mathscr{M}_{s,k}) = A(\mathscr{M}_{s,k}) \in \mathscr{M}$$

and hence

$$A_{s,k}(\mathscr{M}_{s,k}) = \sum_{s'=0}^{\infty} \sum_{k'=0}^{\infty} \mathscr{M}_{s',k,'}.$$

This implies that every $A_{s,k}$ in its turn can be decomposed as

$$A_{s,k} = \sum_{s'=0}^{\infty} \sum_{k'=0}^{\infty} A_{s,k}^{s',k'},$$

with

$$A_{s,k}^{s',k'} : \mathscr{M}_{s,k} \to \mathscr{M}_{s',k'} \quad \text{and} \quad A_{s,k}^{s',k'}|_{\mathscr{M}_{s'',k''}} = 0 \quad \text{whenever } (s,k) \neq (s'',k'').$$

Consequently, for the endomorphism A we obtain

$$A = \sum_{s,k} \sum_{s',k'} A_{s,k}^{s',k'}.$$

It is easily seen that every operator $A_{s,k}^{s',k'}$ is both CH-monogenic and CH-anti-monogenic. For example, if s and s' are even, we have

$$A_{s,k}^{s',k'} \in \chi^+_{s'-s,s'-s+2(k'-k)} \quad \text{and} \quad A_{s,k}^{s',k'} \in \chi^-_{s'-s-m-2k,s'-s+m+2k'}.$$

Collecting the CH-monogenic and CH-anti-monogenic operators of the same degree yields a decomposition of A of the form

$$A = \sum_{\lambda,\kappa} (A^{+}_{\lambda,\kappa} + A^{-}_{\lambda,\kappa}) \quad \text{with} \quad A^{\pm}_{\lambda,\kappa} \in \chi^{\pm}_{\lambda,\kappa}.$$

$\square$

5. CHM-decomposition of Clifford differential operators with polynomial coefficients

In this section we study the CHM-decomposition of Clifford differential operators with polynomial coefficients. It is sufficient to search for the CHM-decomposition of the basic operators $\{e_j, x_j, \partial_{x_j}\,;\ j = 1, 2, \ldots, m\}$, since they are generating the algebra $\mathscr{D}(\mathbb{C}_m)$.

5.1. The operators $f \to e_j f$, $j = 1, 2, \ldots, m$

For the special case when $m = 2$ we have that

$$\left[\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2), e_j\right] = 0 \quad \text{and} \quad \Gamma e_j = -e_j \Gamma.$$

Consequently, we find

$$\begin{aligned} O_1 e_j &= \left(\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2) + \Gamma - \frac{m}{2}\right) e_j \\ &= e_j \left(\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2) - \Gamma + \frac{m}{2}\right) - m e_j \\ &= e_j O_2 - m e_j \end{aligned}$$

and similarly

$$O_2 e_j = e_j O_1 + m e_j.$$

Hence, for $m = 2$, $e_j \in \chi^{-}_{-2,2}$.

For the general case when $m > 2$, we first introduce the operators

$$\tau_j = -(\Gamma e_j + e_j(\Gamma - m + 2)) \quad \text{and} \quad \delta_j = [\Gamma, e_j] = \Gamma e_j - e_j \Gamma$$

for which we prove the following lemma.

Lemma 1. *One has* $\tau_j \in \chi^+_{0,0}$ *and* $\delta_j \in \chi^-_{-2,2}$.

P r o o f. Naturally we have that

$$\left[\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2), \tau_j\right] = 0.$$

Furthermore, as the Laplace-Beltrami operator $\Delta^*_{\underline{\omega}} = \Gamma(m-2-\Gamma)$ is a scalar operator, we find

$$\begin{aligned}
[\Gamma, \tau_j] &= -\Gamma^2 e_j - \Gamma e_j(\Gamma - m + 2) + \Gamma e_j \Gamma + e_j(\Gamma - m + 2)\Gamma \\
&= -\Gamma(\Gamma - m + 2)e_j + e_j(\Gamma - m + 2)\Gamma \\
&= [\Delta^*_{\underline{\omega}}, e_j] = 0.
\end{aligned}$$

This implies

$$O_1 \tau_j = \tau_j O_1 \quad \text{and} \quad O_2 \tau_j = \tau_j O_2.$$

The operator δ_j satisfies

$$\left[\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2), \delta_j\right] = 0$$

and

$$\begin{aligned}
\Gamma \delta_j &= \Gamma^2 e_j - \Gamma e_j \Gamma \\
&= (m-2)\Gamma e_j - \Delta^*_{\underline{\omega}} e_j - \Gamma e_j \Gamma \\
&= (m-2)\Gamma e_j - e_j \Gamma(m-2-\Gamma) - \Gamma e_j \Gamma \\
&= (m-2)(\Gamma e_j - e_j \Gamma) - (\Gamma e_j - e_j \Gamma)\Gamma \\
&= (m-2)\delta_j - \delta_j \Gamma.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
O_1 \delta_j &= \left(\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2) + \Gamma - \frac{m}{2}\right)\delta_j \\
&= \delta_j \left(\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2) - \Gamma + \frac{m}{2}\right) - 2\delta_j \\
&= \delta_j O_2 - 2\delta_j
\end{aligned}$$

and similarly

$$\begin{aligned}
O_2 \delta_j &= \left(\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2) - \Gamma + \frac{m}{2}\right)\delta_j \\
&= \delta_j \left(\frac{1}{2}(\partial^2_{\underline{x}} - \underline{x}^2) + \Gamma - \frac{m}{2}\right) + 2\delta_j \\
&= \delta_j O_1 + 2\delta_j.
\end{aligned}$$

□

Next, we have that $\tau_j + \delta_j = e_j(m-2-2\Gamma)$. The operator $(m-2-2\Gamma)$ is invertible in the set of endomorphisms of $\mathscr{M}$, since it has eigenvalues $2k+m-2$ (for s even) and $-(m+2k)$ (for s odd) which, for $m>2$, are never zero. Consequently, we can write

$$e_j = (\tau_j + \delta_j)(m-2-2\Gamma)^{-1} = E_j^0 + E_j^1,$$

where

$$E_j^0 = \tau_j(m-2-2\Gamma)^{-1} = -(\Gamma e_j + e_j(\Gamma - m + 2))(m-2-2\Gamma)^{-1}$$

and

$$E_j^1 = \delta_j(m-2-2\Gamma)^{-1} = (\Gamma e_j - e_j\Gamma)(m-2-2\Gamma)^{-1}.$$

Naturally, the operator $(m-2-2\Gamma)$ is CH-monogenic of degree $(0,0)$ and hence $(m-2-2\Gamma)^{-1} \in \chi_{0,0}^{+}$.

In view of Proposition 3 we obtain that $E_j^0 \in \chi_{0,0}^{+}$ and $E_j^1 \in \chi_{-2,2}^{-}$.

Summarizing, the CHM-decomposition of e_j takes the form

$$e_j = E_j^0 + E_j^1 \quad \text{with} \quad E_j^0 \in \chi_{0,0}^{+} \quad \text{and} \quad E_j^1 \in \chi_{-2,2}^{-}.$$

Finally, what the action on the spaces $\mathscr{M}_{s,k}$ is concerned, we have by means of Propositions 4 and 5 that

$$E_j^0 \colon \mathscr{M}_{s,k} \to \mathscr{M}_{s,k}$$

and

$$E_j^1 \colon \mathscr{M}_{s,k} \to \begin{cases} \mathscr{M}_{s+1,k-1} & \text{for } s \text{ even and } k \geqslant 1, \\ 0 & \text{for } s \text{ even and } k = 0, \\ \mathscr{M}_{s-1,k+1} & \text{for } s \text{ odd.} \end{cases}$$

5.2. The operators $f \to \underline{x} f$ and $f \to \partial_{\underline{x}} f$

The operators $\underline{x}$ and $\partial_{\underline{x}}$ satisfy

$$\Gamma \underline{x} = \underline{x}(m-1-\Gamma) \quad \text{and} \quad \Gamma \partial_{\underline{x}} = \partial_{\underline{x}}(m-1-\Gamma).$$

This result will be combined with the following lemma.

Lemma 2. *One has*

$$\left[\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2), \underline{x}\right] = -\partial_{\underline{x}} \quad \text{and} \quad \left[\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2), \partial_{\underline{x}}\right] = -\underline{x}.$$

Proof. We obtain successively

$$\begin{aligned}
\left[\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2), \underline{x}\right] &= \frac{1}{2}(\partial_{\underline{x}}^2\underline{x} - \underline{x}\partial_{\underline{x}}^2) \\
&= -\frac{1}{2}\left(\sum_{k=1}^{m} \partial_{x_k}^2\right)\left(\sum_{j=1}^{m} x_j e_j\right) - \frac{1}{2}\underline{x}\partial_{\underline{x}}^2 \\
&= -\frac{1}{2}\sum_{k,j} \partial_{x_k}(\delta_{k,j}e_j + x_j e_j \partial_{x_k}) - \frac{1}{2}\underline{x}\partial_{\underline{x}}^2 \\
&= -\frac{1}{2}\sum_{k,j}(2\delta_{k,j}e_j\partial_{x_k} + x_j e_j \partial_{x_k}^2) - \frac{1}{2}\underline{x}\partial_{\underline{x}}^2 \\
&= -\partial_{\underline{x}} + \frac{1}{2}\underline{x}\partial_{\underline{x}}^2 - \frac{1}{2}\underline{x}\partial_{\underline{x}}^2 \\
&= -\partial_{\underline{x}}
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2), \partial_{\underline{x}}\right] &= -\frac{1}{2}(\underline{x}^2\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x}^2) \\
&= -\frac{1}{2}\underline{x}^2\partial_{\underline{x}} - \frac{1}{2}\left(\sum_{j=1}^{m} e_j\partial_{x_j}\right)\left(\sum_{k=1}^{m} x_k^2\right) \\
&= -\frac{1}{2}\underline{x}^2\partial_{\underline{x}} - \frac{1}{2}\sum_{j,k} e_j(2x_k\delta_{k,j} + x_k^2\partial_{x_j}) \\
&= -\frac{1}{2}\underline{x}^2\partial_{\underline{x}} - \underline{x} + \frac{1}{2}\underline{x}^2\partial_{\underline{x}} \\
&= -\underline{x}.
\end{aligned}$$

□

In view of the above, we now have

$$\begin{aligned}
O_1\underline{x} &= \left(\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) + \Gamma - \frac{m}{2}\right)\underline{x} \\
&= \underline{x}\left(\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) - \Gamma + \frac{m}{2}\right) - \underline{x} - \partial_{\underline{x}} \\
&= \underline{x}O_2 - \underline{x} - \partial_{\underline{x}}
\end{aligned}$$

and

$$\begin{aligned} O_2\underline{x} &= \Big(\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) - \Gamma + \frac{m}{2}\Big)\underline{x} \\ &= \underline{x}\Big(\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) + \Gamma - \frac{m}{2}\Big) + \underline{x} - \partial_{\underline{x}} \\ &= \underline{x}O_1 + \underline{x} - \partial_{\underline{x}}, \end{aligned}$$

while for the operator $\partial_{\underline{x}}$ we obtain

$$O_1\partial_{\underline{x}} = \partial_{\underline{x}}O_2 - \partial_{\underline{x}} - \underline{x} \quad \text{and} \quad O_2\partial_{\underline{x}} = \partial_{\underline{x}}O_1 + \partial_{\underline{x}} - \underline{x}.$$

Hence $\underline{x}$ and $\partial_{\underline{x}}$ are neither CH-monogenic nor CH-anti-monogenic. However, we do have the following result.

Lemma 3. *One has* $\underline{x} + \partial_{\underline{x}} \in \chi^-_{-2,0}$ *and* $\underline{x} - \partial_{\underline{x}} \in \chi^-_{0,2}$.

Proof. Straightforward. □

We now readily obtain the CHM-decomposition of $\underline{x}$ and $\partial_{\underline{x}}$:

$$\underline{x} = X^0 + X^1$$

with

$$X^0 = \frac{1}{2}(\underline{x} - \partial_{\underline{x}}) \in \chi^-_{0,2}, \quad X^1 = \frac{1}{2}(\underline{x} + \partial_{\underline{x}}) \in \chi^-_{-2,0},$$

and

$$\partial_{\underline{x}} = D^0 + D^1$$

with

$$D^0 = -\frac{1}{2}(\underline{x} - \partial_{\underline{x}}) \in \chi^-_{0,2}, \quad D^1 = \frac{1}{2}(\underline{x} + \partial_{\underline{x}}) \in \chi^-_{-2,0}.$$

5.3. The operators $f \to x_j f$, $j = 1, 2, \dots, m$

Once the CHM-decomposition of e_j, $j = 1, 2, \dots, m$ (Subsection 5.1) and $\underline{x}$ (Subsection 5.2) is obtained, the CHM-decomposition of x_j easily follows from

$$\begin{aligned} x_j &= -\frac{1}{2}(\underline{x}e_j + e_j\underline{x}) \\ &= -\frac{1}{2}((X^0 + X^1)(E_j^0 + E_j^1) + (E_j^0 + E_j^1)(X^0 + X^1)) \\ &= x_j^0 + x_j^1 + x_j^2 + x_j^3 + x_j^4 + x_j^5 \end{aligned}$$

with

$$
\begin{aligned}
x_j^0 &= -\frac{1}{2}X^0E_j^1 \in \chi^+_{2,0},\\
x_j^1 &= -\frac{1}{2}E_j^1X^0 \in \chi^+_{0,2},\\
x_j^2 &= -\frac{1}{2}E_j^1X^1 \in \chi^+_{-2,0},\\
x_j^3 &= -\frac{1}{2}X^1E_j^1 \in \chi^+_{0,-2},\\
x_j^4 &= -\frac{1}{2}(X^0E_j^0 + E_j^0X^0) \in \chi^-_{0,2},\\
x_j^5 &= -\frac{1}{2}(X^1E_j^0 + E_j^0X^1) \in \chi^-_{-2,0}.
\end{aligned}
$$

Here we have used the composition rules for CH-monogenic and CH-anti-monogenic operators derived in Proposition 3.

Finally, for the action on the spaces $\mathscr{M}_{s,k}$ we have

$$
\begin{aligned}
x_j^0\colon \mathscr{M}_{s,k} &\to \begin{cases} \mathscr{M}_{s+2,k-1} & \text{for } s \text{ even and } k \geqslant 1,\\ 0 & \text{for } s \text{ even and } k = 0,\\ \mathscr{M}_{s,k+1} & \text{for } s \text{ odd;} \end{cases}\\
x_j^1\colon \mathscr{M}_{s,k} &\to \begin{cases} \mathscr{M}_{s,k+1} & \text{for } s \text{ even,}\\ \mathscr{M}_{s+2,k-1} & \text{for } s \text{ odd and } k \geqslant 1,\\ 0 & \text{for } s \text{ odd and } k = 0; \end{cases}\\
x_j^2\colon \mathscr{M}_{s,k} &\to \begin{cases} \mathscr{M}_{s-2,k+1} & \text{for } s \text{ even,}\\ \mathscr{M}_{s,k-1} & \text{for } s \text{ odd and } k \geqslant 1,\\ 0 & \text{for } s \text{ odd and } k = 0; \end{cases}\\
x_j^3\colon \mathscr{M}_{s,k} &\to \begin{cases} \mathscr{M}_{s,k-1} & \text{for } s \text{ even and } k \geqslant 1,\\ 0 & \text{for } s \text{ even and } k = 0,\\ \mathscr{M}_{s-2,k+1} & \text{for } s \text{ odd;} \end{cases}\\
x_j^4\colon \mathscr{M}_{s,k} &\to \mathscr{M}_{s+1,k};\\
x_j^5\colon \mathscr{M}_{s,k} &\to \mathscr{M}_{s-1,k}.
\end{aligned}
$$

5.4. The operators $f \to \partial_{x_j} f$, $j = 1, 2, \dots, m$

By means of

$$
\partial_{x_j} = -\frac{1}{2}(\partial_{\underline{x}} e_j + e_j \partial_{\underline{x}}),
$$

the CHM-decomposition of ∂_{x_j} follows at once from the CHM-decomposition of e_j and $\partial_{\underline{x}}$. The results are completely similar to those for the operators considered in the previous subsection.

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