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ON COMPLEMENTED SUBGROUPS OF FINITE GROUPS

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Abstract. A subgroup H of a group G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In this paper we determine the structure of finite groups with some complemented primary subgroups, and obtain some new results about p-nilpotent groups.

Keywords: finite group, p-nilpotent group, primary subgroups, complemented subgroups

MSC 2000: 20D10, 20D20

1. Introduction

A subgroup H of a group G is said to be supplemented in G if there exists a subgroup K of G such that $G = HK$. Furthermore, a subgroup H of G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. It is obvious that the existence of supplements for some families of subgroups of a group gives a lot of information about its structure. For instance, Kegel [8], [9] showed that a group G is soluble if every maximal subgroup of G either has a cyclic supplement in G or if some nilpotent subgroup of G has a nilpotent supplement in G. Hall [6] proved that a group G is soluble if and only if every Sylow subgroup of G is complemented in G. Arad and Ward [1] proved that a group G is soluble if and only if every Sylow 2-subgroup and every Sylow 3-subgroup of G are complemented in G. More recently, A. Ballester-Bolinches and Guo Xiuyun [2] proved that the class of all finite supersoluble groups with elementary abelian Sylow subgroups is just the class of all finite groups for which every minimal subgroup is complemented.

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The aim of this paper is to take these studies further. In fact, we analyze the finite group for which some primary subgroups are complemented. We obtain a series of new results for the p-nilpotency of finite groups.

All groups considered in this paper will be finite. Our notation is standard and can be found in [4] and [10]. We denote the fact that G is the semi-product of subgroups H and K by $G = [H]K$, where H is normal in G.

A subgroup H is called the second maximal subgroup of G, if H is the maximal subgroup of some maximal subgroup of G. A subgroup H is called the third maximal subgroup of G, if H is the maximal subgroup of some second maximal subgroup of G.

Let π be a set of primes. We say that $G \in E_\pi$ if G has a Hall π-subgroup. We say that $G \in C_\pi$ if any two Hall π-subgroups of G are conjugate in G. We say that $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a Hall π-subgroup of G.

Let $\mathscr{F}$ be a class of groups. $\mathscr{F}$ is called Q-closed if G/N is in $\mathscr{F}$ for any normal subgroup N of G when $G \in \mathscr{F}$. $\mathscr{F}$ is called S-closed if any subgroup K of G is in $\mathscr{F}$ when $G \in \mathscr{F}$. We call $\mathscr{F}$ a formation provided that (1) if $G \in \mathscr{F}$ and $H \trianglelefteq G$, then $G/H \in \mathscr{F}$, and (2) if G/M and G/N are in $\mathscr{F}$, then $G/M \cap N$ is in $\mathscr{F}$ for normal subgroups M and N of G. Each group has a smallest normal subgroup N such that $G/N \in \mathscr{F}$. This uniquely determined normal subgroup of G is called the $\mathscr{F}$-residual subgroup of G and denoted by $G^{\mathscr{F}}$. A formation $\mathscr{F}$ is said to be saturated if $G/\Phi(G) \in \mathscr{F}$ implies that $G \in \mathscr{F}$. (cf. [4, Chapt. 2 and 3]). As we all know, the class of all p-nilpotent groups is a saturated formation.

2. Preliminaries

For the sake of easy reference, we first give some basic definitions and known results from the literature.

Lemma 2.1 ([2, Lemma 1]). *Let G be a group and N a normal subgroup of G. Then the following statements hold.*

(1) *If $H \leqslant K \leqslant G$ and H is complemented in G, then H is complemented in K.*
(2) *If N is contained in H and H is complemented in G, then H/N is complemented in G/N.*
(3) *Let π be a set of primes. If N is a π'-subgroup and A is a π-subgroup of G, then A is complemented in G if and only if AN/N is complemented in G/N.*

Lemma 2.2 (cf. [4, Theorem 1.8.17]). *Let N be a normal subgroup of a group G $(N \neq 1)$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in $F(N)$.*

Lemma 2.3 ([3, Main Theorem]). *Suppose that a finite group G has a Hall π-subgroup where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.4. *Let G be a finite group and p be a prime divisor of $|G|$ such that $(|G|, p^2-1) = 1$. Assume that the order of G is not divisible by p^3. Then G is p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Since every proper subgroup and every proper quotient group also satisfy the hypothesis of the lemma, the minimal choice of G implies that G is a minimal non-p-nilpotent group but every proper subgroup and every proper quotient group of are p-nilpotent. Therefore $G = [P]Q$ with Q cyclic [10, 9.1.9]. Since both $\Phi(P)$ and $\Phi(G)$ are in $Z(G) = 1$, we see that P is an elemenary abelian Sylow p-subgroup and Q a cyclic group of order q. $Q \cong G/P$ and $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. Hence q divides $p(p+1)(p-1)$ [10, 3.2.7]. Since $p \neq q$ and $(|G|, p^2-1) = 1$, G is p-nilpotent by the Burnside p-nilpotent Theorem, a contradiction.

The final contradiction completes our proof. □

Lemma 2.5 ([7, IV, 5.4, P_{434}]). *Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.*

Lemma 2.6 ([7, 5.2, P_{281}]). *Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

1) *G has a normal Sylow p-subgroup P for some prime p and $G/P \cong Q$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*
2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
3) *If P is non-abelian and $p \neq 2$, then the exponent of P is p.*
4) *If P is non-abelian and $p = 2$, then the exponent of P is 4.*
5) *If P is abelian, then P is of exponent p.*

Lemma 2.7 ([12, P_{67}]). *Let K be a subgroup of G. If $|G : K| = p$, where p is the smallest prime divisor of $|G|$, then $K \trianglelefteq G$.*

Lemma 2.8. *Let G be a finite group and p be the prime divisor of $|G|$ such that $(|G|, p^2-1) = 1$. If G/L is p-nilpotent and $p^3 \nmid |L|$, then G is p-nilpotent.*

P r o o f. By hypothesis and Lemma 2.4, we know that L is p-nilpotent and L has a normal p-complement $L_{p'}$. Since $L_{p'}$ char L and L is normal in G, we have $L_{p'} \trianglelefteq G$. Therefore $G/L \cong (G/L_{p'})/(L/L_{p'})$ is p-nilpotent. There exists a Hall p'-subgroup $(H/L_{p'})/(L/L_{p'})$ of $(G/L_{p'})/(L/L_{p'})$ and $H/L_{p'} \trianglelefteq G/L_{p'}$. By the Schur-Zassenhaus Theorem, we have $H/L_{p'} = [L/L_{p'}]H_1/L_{p'}$, where $H_1/L_{p'}$ is a Hall p'-subgroup of $H/L_{p'}$. Then by Lemma 2.4 we have $H_1/L_{p'} \trianglelefteq H/L_{p'}$ and so $H_1/L_{p'}$ char $H/L_{p'} \trianglelefteq G/L_{p'}$. Therefore $H_1/L_{p'} \trianglelefteq G/L_{p'}$ and hence $G/L_{p'}$ is p-nilpotent. Thus G is p-nilpotent. □

Lemma 2.9 ([13]). *Let P be an elementary abelian p-group of order p^n, where p is a prime. Then $|\mathrm{Aut}(P)| = k_n \cdot p^{n(n-1)/2}$, here $k_n = \prod\limits_{i=1}^{n} (p^i - 1)$.*

Lemma 2.10 ([13]). *Let G be a group of order p^n, where p is a prime. Then $|\mathrm{Aut}(G)|$ is the factor of the order of $\mathrm{Aut}(P)$, where P is an elementary abelian p-group of order p^n.*

Lemma 2.11 ([7]). *Let G be a finite group and U any p-subgroup of G. If $N_G(U)/C_G(U)$ is a p-subgroup, then G is p-nilpotent.*

Lemma 2.12 ([5, Lemma 5]). *Let $\mathscr{F}$ be an S-closed local formation and H a subgroup of G. Then $H \cap Z_{\mathscr{F}}(G) \subseteq Z_{\mathscr{F}}(H)$.*

3. Main results

Theorem 3.1. *Let G be a finite group and p a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. If there exists a normal subgroup N in G such that G/N is p-nilpotent and each subgroup of N of order p^2 is complemented in G, then G is p-nilpotent.*

P r o o f. Assume that the claim is false and choose G to be a counterexample of minimal order.

By Lemma 2.8 and the hypothesis, we have $|N|_p > p^2$. Let L be a proper subgroup of G. We prove that conditions of the Theorem are inherited by L. Clearly, $L/L\cap N \cong LN/N \leqslant G/N$ implies that $L/L \cap N$ is p-nilpotent. If $|L \cap N|_p \leqslant p^2$, then L is p-nilpotent by Lemma 2.8. If $|L \cap N|_p > p^2$, then each subgroup of $L \cap N$ of order p^2 is complemented in G by Lemma 2.1 and hence is complemented in L, thus L is p-nilpotent by induction. Thus G is a minimal non-p-nilpotent group. Now Lemma 2.5 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by Lemma 2.6, we have $G = PQ$, where P is normal in G and Q is a non-normal cyclic Sylow q-subgroup of G. It is clear that $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

Clearly, $P \leqslant N$. Let $A \leqslant N$ and $|A| = p^2$. By hypothesis, there exists a subgroup K of G such that $G = AK$ and $K \cap A = 1$. By hypothesis, K is nilpotent and $K = K_p \times K_{p'}$. If $K_p = 1$, then $P = A$ and hence G is p-nilpotent by Lemma 2.4, a contradiction. If $K_p \neq 1$, then K_p is the second maximal subgroup of P. Then we consider the subgroup $N_G(K_p)$. Since $K \leqslant N_G(K_p)$, we have $|G : N_G(K_p)| = p$ or $N_G(K_p) = G$. If $|G : N_G(K_p)| = p$, then we have $N_G(K_p) < G$. Let $P_1 = P \cap N_G(K_p)$. Then $P_1 \trianglelefteq G$. If $P_1 \leqslant \Phi(P)$, then $P = P \cap AN_G(K_p) = A(P \cap N_G(K_p)) = A$, a contradiction. If $P_1 \nleqslant \Phi(P)$, then $P_1\Phi(P)/\Phi(P) = P/\Phi(P)$ by Lemma 2.6. In this case, $P = P_1$ and hence $N_G(K_p) = G$, a contradiction. If $|G : N_G(K_p)| = 1$, then $K_p \trianglelefteq G$. Since G/K_p is p-nilpotent by Lemma 2.4, we have $P = K_p$, a contradiction.

The final contradiction completes our proof. $\square$

Corollary 3.2. *Let G be a finite group and p a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. If each subgroup of G of order p^2 is complemented in G, then G is p-nilpotent.*

Theorem 3.3. *Suppose that G is a group with a normal subgroup H such that G/H is p-nilpotent for some prime divisor p of $|G|$. If every cyclic subgroup of order 4 of H is complemented in G and every subgroup of H of order p is contained in $Z_{\mathscr{F}}(G)$, where $\mathscr{F}$ is the class of all p-nilpotent groups, then G is p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of the smallest order.

The hypothesis is inherited by all proper subgroups of G. Let K be a proper subgroup of G. Then $K/K \cap H \cong KH/H \leqslant G/H$ implies that $K/K \cap H$ is p-nilpotent. Every cyclic subgroup of $K \cap H$ of order 4 is complemented in G and hence is complemented in K by Lemma 2.1. Every subgroup of $H \cap K$ of order p is contained in $K \cap Z_{\mathscr{F}}(G) \leqslant Z_{\mathscr{F}}(K)$ by Lemma 2.12. Thus G is a minimal non-p-nilpotent group. Now Lemma 2.5 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Thus by Lemma 2.6, G has a normal Sylow p-subgroup P and $G/P \cong Q$, where Q is a non-normal cyclic Sylow q-subgroup of G, and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. We consider the following cases.

Case 1: P is abelian. By Lemma 2.6, P is an elementary abelian p-group. Since G/H is p-nilpotent, we have $P \leqslant H$. By hypothesis, every subgroup of H of order p is contained in $Z_{\mathscr{F}}(G)$, thus $P \leqslant Z_{\mathscr{F}}(G)$ and hence G is p-nilpotent, a contradiction.

Case 2: P is non-abelian and $p > 2$. By Lemma 2.6, the exponent of P is p and every subgroup of H of order p is contained in $Z_{\mathscr{F}}(G)$. Therefore $P \leqslant Z_{\mathscr{F}}(G)$ and we see that G is p-nilpotent, a contradiction.

Case 3: P is non-abelian and $p = 2$. Let A be a cyclic subgroup of H of order 4. By hypothesis, A is complemented in G and there exists a subgroup L of G such that $G = AL$ and $L \cap A = 1$. Since L is nilpotent, we have $L = L_p \times L_{p'}$. If $L_p = 1$, then $P = A$, a contradiction. Clearly, L_p is the 2-maximal subgroup of P. We consider $N_G(L_p)$. Since $L \leqslant N_G(L_p)$, we have $|G : N_G(L_p)| = 2$ or $|G : N_G(L_p)| = 1$. If $|G : N_G(L_p)| = 2$, then $N_G(L_p) \trianglelefteq G$ by Lemma 2.7 and hence G is 2-nilpotent, a contradiction. If $|G : N_G(L_p)| = 1$, then $L_p \trianglelefteq G$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, we have $P = L_p$ or $L_p \leqslant \Phi(P)$. It is clear that $P = L_p$ is impossible. If $L_p \leqslant \Phi(P)$, then $P = AL_p = A$, a contradiction.

The final contradiction completes our proof. □

Corollary 3.4. *Let G be a group and p be the prime divisor of $|G|$. If every cyclic subgroup of order 4 is complemented in G and every subgroup of G of order p is contained in $Z_{\mathscr{F}}(G)$, where $\mathscr{F}$ is the class of all p-nilpotent groups, then G is p-nilpotent.*

Theorem 3.5. *Let G be a group and $(|G|, 21) = 1$. If each subgroup of G of order 8 (if it exists) is complemented in G, then G is 2-nilpotent.*

P r o o f. Assume that the theorem is false and choose G to be a counterexample of the smallest order. Let 2^α be the order of a Sylow 2-subgroup P of G.

If $2 \nmid |G|$, then G is 2-nilpotent. If P is cyclic, then G is 2-nilpotent by [10, 10.1.9]. So we can suppose that P is not cyclic. Let L be a proper subgroup of G. We prove that L inherits the condition of the theorem. If $8 \nmid |L|$, then L is 2-nilpotent by Lemma 2.4. If $8 \mid |L|$, then each subgroup of L of order 8 is complemented in G and hence is complemented in L by Lemma 2.1, so L is 2-nilpotent by induction. Thus we may assume that G is a minimal non-2-nilpotent group. Now Lemma 2.5 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Thus by Lemma 2.6, we have $G = PQ$, where P is normal in G and Q is a non-normal cyclic Sylow q-subgroup of G ($q \neq p$), and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

Let H be a subgroup of G of order 8. By hypothesis, there exists a subgroup K of G such that $G = HK$ and $K \cap H = 1$.

First, we claim that $H \neq P$. Otherwise $P = H \trianglelefteq G$. If $\Phi(P) \neq 1$, then $G/\Phi(P)$ is 2-nilpotent by Lemma 2.4 and hence G is 2-nilpotent, a contradiction. It follows from $\Phi(P) = 1$ and Lemma 2.6 that P is an elementary abelian 2-group. Next we will consider $N_G(P)/C_G(P)$. It is clear that $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. By Lemma 2.9 we have $|\mathrm{Aut}(G)| = 168$. Since $(|G|, 21) = 1$, we see that $N_G(P) = C_G(P)$. Then, by the Burnside Theorem (cf. [10]), we see that G is 2-nilpotent, a contradiction.

Since $H \cap K = 1$ and K is nilpotent, K_2 is the third maximal subgroup of P. We claim that $K_2 \ntrianglelefteq G$. Otherwise if $K_2 \trianglelefteq G$, then $P = K_2$ or $K_2 \leqslant \Phi(P)$. It is clear that $P = K_2$ is impossible. If $K_2 \leqslant \Phi(P)$, then $P = HK_2 = H$, a contradiction. Consider the subgroup $N_G(K_2)$. If $|G : N_G(K_2)| = 2$, then $N_G(K_2)$ is a nilpotent normal subgroup of G by Lemma 2.7 and so G is 2-nilpotent, a contradiction. If $|G : N_G(K_2)| = 4$, then we consider $N_G((N_G(K_2))_2)$. It is clear that $(N_G(K_2))_2$ is the second maximal subgroup of P. If $(N_G(K_2))_2 \trianglelefteq G$, then $G/(N_G(K_2))_2$ is 2-nilpotent by Lemma 2.4 and we have $P \leqslant (N_G(K_2))_2$, a contradiction. Thereby we may assume that $|G : N_G((N_G(K_2))_2)| = 2$. By Lemma 2.7 we know that $N_G((N_G(K_2))_2) \trianglelefteq G$. Since $N_G((N_G(K_2))_2)$ is a nilpotent normal subgroup of G, we know that G is 2-nilpotent, a contradiction.

The final contradiction completes the proof. □

Lemma 3.6. *Let G be a finite group with $(|G|, 21) = 1$. Assume that every third maximal subgroup (if it exists) of a Sylow 2-subgroup of G is complemented in G. Then $G/O_2(G)$ is 2-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Let P be a Sylow 2-subgroup of G. Furthermore, we have:

(1) $O_2(G) = 1$.

If $O_2(G) = P$, then $G/O_2(G)$ is a $2'$-group and of course it is 2-nilpotent, a contradiction. If $O_2(G) = P_1$, where P_1 is the maximal subgroup of P, then $G/O_2(G)$ is 2-nilpotent since $|G/O_2(G)|_2 = 2$, a contradiction. If $O_2(G) = P_2$ where P_2 is the second maximal subgroup of P, then $2^3 \nmid |G/O_2(G)|$. Hence $G/O_2(G)$ is 2-nilpotent by Lemma 2.4, a contradiction. If $1 < O_2(G) < P_2$, then $G/O_2(G)$ satisfies the hypothesis and the minimal choice of G implies that $G/O_2(G) \cong G/O_2(G)/O_2(G/O_2(G))$ is 2-nilpotent, a contradiction.

(2) *$|G|$ is divisible by 2^4.*

If $2^3 \nmid |G|$ and $(|G|, 21) = 1$, then G is 2-nilpotent by Lemma 2.4, a contradiction. If $2^3 \mid |G|$ and $2^4 \nmid |G|$, then $|G_2| = 2^3$. Next we consider $N_G(U)/C_G(U)$, where U is any 2-subgroup of G. If $U = P$, then $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. By Lemma 2.9 and Lemma 2.10, $N_G(P)/C_G(P)$ is a 2-subgroup. If $U \neq P$, it is easy to see that $N_G(U)/C_G(U)$ is also a 2-group according to Lemma 2.9 and Lemma 2.10. Then by Lemma 2.11 it is clear that G is 2-nilpotent in this case, a contradiction.

(3) *For every third maximal subgroup P_3 of a Sylow 2-subgroup P of G, the complement of P_3 in G is 2-nilpotent.*

By the hypothesis, P_3 is complemented in G. There exists a subgroup K_3 of G such that $G = P_3K_3$ and $K_3 \cap P_3 = 1$. By (2), we know that K_3 is 2-nilpotent.

(4) *G is 2-nilpotent.*

Let $N = N_G((K_3)_{2'})$ and $K_3 = (K_3)_2(K_3)_{2'}$. By (3), $K_3 \leqslant N$. So we have $G = P_3K_3 = P_3N$. If $N = G$, then G is 2-nilpotent, a contradiction. Let $P_3 \leqslant P_2 \leqslant P_1 \leqslant P$ where P_2 is the second maximal subgroup of P and P_1 is the maximal subgroup of P. Hence $G = P_3K_3 = P_2K_3 = P_2N$. If $P_2 \leqslant N$, then G is 2-nilpotent, a contradiction. So we may assume $P_2 \cap N < P_2$. We may choose a maximal subgroup ${P_3}^*$ of P_2 such that $P_2 \cap N \leqslant {P_3}^*$. It is clear that ${P_3}^*$ is the third maximal subgroup of P. By (3), P_3^* is complemented in G and the complement K_3^* of P_3^* is 2-nilpotent. We denote ${K_3}^* = ({K_3}^*)_2({K_3}^*)_{2'}$. Lemma 2.3 implies that $G \in C_{2'}$. Now both $(K_3)_{2'}$ and $({K_3}^*)_{2'}$ are the Hall $2'$-subgroups of G, these two subgroups are conjugate in G. Let $(K_3)_{2'} = (({K_3}^*)_{2'})^g$. Since $G = {P_3}^*{K_3}^*$ and ${K_3}^* \leqslant N_G(({K_3}^*)_{2'})$, we may choose $g \in {P_3}^*$. We also note that $({K_3}^*)^g$ normalizes $(({K_3}^*)_{2'})^g = (K_3)_{2'}$ and hence $({K_3}^*)^g \leqslant N$. Now $G = G^g = ({P_3}^*{K_3}^*)^g = {P_3}^*N$. Therefore $P_2 = P_2 \cap {P_3}^*N = {P_3}^*(P_2 \cap N) = {P_3}^*$, contrary to the choice of G.

The final contradiction completes our proof. $\square$

Theorem 3.7. *Let G be a finite group with $(|G|, 21) = 1$. Assume that there exists a normal subgroup N of G such that G/N is 2-nilpotent and every third maximal subgroup of a Sylow subgroup of N is complemented in G. Then G is 2-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Then

(1) *G is soluble, G has a minimal normal subgroup $L \leqslant N$ and L is an elementary abelian r-group for some prime r.*

By hypothesis, every third maximal subgroup of every Sylow subgroup of N is complemented in G, thus is complemented in N by Lemma 2.1. By the choice of G and Lemma 3.6, N is soluble and hence G is soluble. Let L be a minimal normal subgroup of G which is contained in N. Then L is an elementary abelian r-group for some prime r.

(2) *G/L is 2-nilpotent and L is the unique minimal normal subgroup of G which is contained in N. Furthermore, $L = F(N) = C_N(L)$.*

In fact, $(G/L)/(N/L) \cong G/N$ is 2-nilpotent and $(|G/L|, 21) = 1$. Let R_1/L be a third maximal subgroup of the Sylow r-subgroup of N/L. Then R_1 is a third maximal subgroup of the Sylow r-subgroup R of N. By hypothesis of the theorem R_1 is complemented in G. By Lemma 2.1, R_1/L is complemented in G/L. Set Q_1/L be a third maximal subgroup of the Sylow q-subgroup of N/L, where $q \neq r$. It is clear that $Q_1 = Q_1^*L$, where Q_1^* is a third maximal subgroup of the Sylow q-subgroup of N. By the hypothesis Q_1^* is complemented in G. Hence Q_1^*L/L is complemented in G/L by Lemma 2.1. We have proved that G/L satisfies the hypothesis of the theorem. Hence G/L is 2-nilpotent by the minimal choice of G. Since the class of all 2-nilpotent groups is a saturated formation, we can easily prove that L is the unique minimal normal subgroup of G which is contained in N, $L \nleqslant \Phi(G)$. By Lemma 2.2, $F(N) = L$. The solubility of N implies that $L \leqslant C_N(F(N)) \leqslant F(N)$ and so $C_N(L) = F(N) = L$.

(3) *L is a Sylow 2-subgroup of N.*

By (1), we know that G is soluble. If $2 \nmid |N|$, then G is 2-nilpotent since G/N is 2-nilpotent, a contradiction. If $2 \neq r$, then G is 2-nilpotent by (2), a contradiction. Therefore L is an elementary abelian 2-subgroup of G which is contained in N. Let D be a Hall $2'$-subgroup of N; then it is clear that LD/L is a Hall $2'$-subgroup of N/L. We have $LD/L \trianglelefteq N/L$ since N/L is 2-nilpotent. So $LD \trianglelefteq N$. Let P be a Sylow 2-subgroup of N. Then $L < P$ and $PD = PLD$ is a subgroup of N. Note that every third maximal subgroup of a Sylow subgroup of PD is complemented in G and hence is complemented in PD by Lemma 2.1. Therefore PD satisfies the hypothesis for G. If $PD < G$, the minimal choice of G implies that PD is 2-nilpotent, in particular, $D \trianglelefteq PD$. Hence $LD = L \times D$ and $D \leqslant C_N(L) = L$, a contradiction.

Now we assume that $G = PD = N$ and $L < P$. Since N/L is 2-nilpotent, $LD \trianglelefteq N = G$. By the Frattini argument, $G = LN_G(D)$. Note that $L \cap N_G(D) = 1$ since L is the unique minimal normal subgroup of G which is contained in N and D is not normal in G in this case. Therefore $G = [L]N_G(D)$. Let P_2 be a Sylow 2-subgroup of $N_G(D)$. Then LP_2 is a Sylow 2-subgroup of G. Choose a third maximal subgroup P_3 of LP_2 such that $P_2 \leqslant P_3$. Otherwise, if P_2 is the maximal subgroup of LP_2, then $|L| = 2$ and hence G is 2-nilpotent by Lemma 2.8, a contradiction. If P_2 is the second maximal subgroup of LP_2, then $|L| = 2^2$ and hence G is 2-nilpotent by Lemma 2.8 and (2), a contradiction. Clearly, $L \nleqslant P_3$ and hence $(P_3)_G = 1$. By our hypothesis, P_3 is complemented in G. There exists a subgroup K of G such that $G = P_3K$ and $K \cap P_3 \leqslant (P_3)_G = 1$. It follows that K has a normal 2-complement which is in fact a Hall $2'$-subgroup D_1 of G in this case. By the hypothesis and Lemma 2.3, there exists an element $g \in L$ such that $D_1^g = D$. Since $P_2 \leqslant P_3 < P_2^* < P_1 < LP_2$, where P_1 is a maximal subgroup of LP_2 which contains P_2^*, and P_2^* is a second maximal subgroup of LP_2 which contains P_3, we have $G = P_3K = P_1K = (P_1K)^g = P_1K^g$. Since $K^g \cong K$ has a normal 2-complement D and $D = D_1^g \leqslant K^g$, it follows that $K^g \leqslant N_G(D)$. Since $LP_2 = LP_2 \cap G = LP_2 \cap P_1K^g = P_1(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \nleqslant P_2$. Otherwise $LP_2 \leqslant P_1P_2 = P_1$, a contradiction. Therefore P_2 is a proper subgroup of $P_4 = \langle P_2, LP_2 \cap K^g \rangle$ where P_4 is a subgroup of the Sylow 2-subgroup LP_2. Now both P_2 and K^g are contained in $N_G(D)$ and we see that P_4 is a 2-subgroup of $N_G(D)$

which contains a Sylow subgroup P_2 as a proper subgroup, a contradiction. Hence L is a Sylow 2-subgroup of N.

(4) *G is 2-nilpotent.*

If $|L| \leqslant 4$, then G is 2-nilpotent by Lemma 2.8 and (2), a contradiction. If $|L| = 2^3$, then G is 2-nilpotent by Lemma 3.6. So we may assume $|L| > 8$. Let L_1 be a nontrivial third maximal subgroup of L. Then L_1 is complemented in G. There exists a subgroup K of G such that $L_1K = G$ and $K \cap L_1 = 1$. Therefore $L = L_1(L \cap K)$ and $L \cap K \trianglelefteq G$. We have $L \cap K = L$, otherwise, $L \cap K = 1$ and $L = L_1$, a contradiction. Therefore $K = G$, a contradiction.

The final contradiction completes our proof. □

Corollary 3.8. *Let G a finite group with $(|G|, 21) = 1$. If every third maximal subgroup of every Sylow subgroup of G is complemented in G, then G is 2-nilpotent.*

References

[1] *Z. Arad, M. B. Ward*: New criteria for the solvability of finite groups. J. Algebra *77* (1982), 234–246.
[2] *A. Ballester-Bolinches, X. Guo*: On complemented subgroups of finite groups. Arch. Math. *72* (1999), 161–166.
[3] *F. Gross*: Conjugacy of odd order Hall subgroup. Bull. London Math. Soc. *19* (1987), 311–319.
[4] *W. Guo*: The Theory of Classes of Groups. Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.
[5] *W. Guo*: The influence of minimal subgroups on the structure of finite groups. Southeast Asian Bulletin of Mathematics *22* (1998), 287–290.
[6] *P. Hall*: A characteristic property of soluble groups. J. London Math. Soc. *12* (1937), 188–200.
[7] *B. Huppert*: Endliche Gruppen I. Springer-Verlag, Berlin-Heidelberg-New York, 1967.
[8] *O. H. Kegel*: On Huppert's characterization of finite supersoluble groups. In: Proc. Internat. Conf. Theory Groups, Canberra, 1965. New York, 1967, pp. 209–215.
[9] *O. H. Kegel*: Produkte nilpotenter gruppen. Arch. Math. *12* (1961), 90–93.
[10] *D. J. Robinson*: A Course in the Theory of Groups. Springer-Verlag, Berlin-New York, 1993.
[11] *Y. Wang*: Finite groups with some subgroups of Sylow subgroups c-supplemented. J. Algebra *224* (2000), 467–478.
[12] *M. Xu*: An Introduction to Finite Groups. Science Press, Beijing, 1999. (In Chinese.)
[13] *Y. Zhang*: The Structure of Finite Groups. Science Press, Beijing, 1982. (In Chinese.)

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