Hiroyuki Ishibashi
Involutions and semiinvolutions

Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 2, 533–541

Persistent URL: `http://dml.cz/dmlcz/128084`

Czechoslovak Mathematical Journal, 56 (131) (2006), 533–541

INVOLUTIONS AND SEMIINVOLUTIONS

Hiroyuki Ishibashi, Sakado

(Received September 26, 2003)

Abstract. We define a linear map called a semiinvolution as a generalization of an involution, and show that any nilpotent linear endomorphism is a product of an involution and a semiinvolution. We also give a new proof for Djocović's theorem on a product of two involutions.

Keywords: classical groups, vector spaces and linear maps, involutions, factorization of a linear map into a product of simple ones

MSC 2000: 15A04, 15A23, 15A33

1. Introduction

Let V be an n-dimensional vector space over a field k of any characteristic. The k-algebra of k-linear endmorphisms of V is denoted by $\mathrm{End}_k V$, and the unit group of $\mathrm{End}_k V$ is $\mathrm{Aut}_k V$. An element $\xi \in \mathrm{Aut}_k V$ is called an involution if $\xi^2 = 1$, and two elements $\eta, \eta' \in \mathrm{End}_k V$ are said to be similar if $\eta' = \varrho\eta\varrho^{-1}$ for some $\varrho \in \mathrm{Aut}_k V$. An element $\sigma \in \mathrm{End}_k V$ is nilpotent if $\sigma^n = 0$ for some integer $n \geqslant 1$.

Suppose that V is a direct sum of two subspaces, say, $V = L \oplus M$. Then we shall call a linear map $\sigma = 0_L \oplus \varrho \in \mathrm{End}_k V$ a semiinvolution if $0_L \in \mathrm{End}_k L$ is the zero map on L and $\varrho \in \mathrm{Aut}_k M$ is an involution on M. In case that L is spanned by a subset $S \subseteq V$, we may write 0_S for 0_L. Also 1_L or $1_S \in \mathrm{Aut}_k L$ denotes the identity map on L.

Let H be a subspace of V having a basis $Z = \{x_1, x_2, \ldots, x_m, y_m, \ldots, y_2, y_1\}$ of an even number of elements. Then an involution $\Delta_Z \in \mathrm{Aut}_k H$ is defined by

$$x_1 \rightleftarrows y_1, x_2 \rightleftarrows y_2, \ldots, x_m \rightleftarrows y_m.$$

We shall call Δ_Z the transpose of Z or H. Our purpose is to prove the following two theorems, Theorems A and B.

Theorem A. *For $\sigma \in \operatorname{End}_k V$, the following* (a) *and* (b) *are equivalent:*

(a) *σ is nilpotent.*

(b) *$\sigma = \theta\tau$ for an involution $\tau = 1_{Z_1} \oplus \Delta_{Z_2}$ and a semiinvolution $\theta = 0_{Z'_0} \oplus 1_{Z'_1} \oplus \Delta_{Z'_2}$, where $\{Z_1, Z_2\}$ and $\{Z'_0, Z'_1, Z'_2\}$ are two bases for V which satisfy the following condition* (C)*:*

(C) *Z_1 and Z_2 are expressed as*

$$Z_1 = \{x_{10}, x_{20}, \ldots, x_{r0}\},$$
$$Z_2 = \{X_{r+s}, \ldots, X_{r+1}, X_r, \ldots, X_2, X_1, Y_1, Y_2, \ldots, Y_r, Y_{r+1}, \ldots, Y_{r+s}\}$$

for $X_i = \{x_{im_i}, \ldots, x_{i2}, x_{i1}\}$, $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{im_i}\}$ and $1 \leqslant i \leqslant r+s$, and for which Z'_0, Z'_1, Z'_2 are expressed as

(i) $Z'_0 = \{x_{im_i} \colon\ 1 \leqslant i \leqslant r+s\}$,

i.e., the first elements of $X_{r+s}, \ldots, X_2, X_1$,

(ii) $Z'_1 = \{y_{i1} \colon\ r+1 \leqslant i \leqslant r+s\}$,

i.e., the first elements of $Y_{r+1}, Y_{r+2}, \ldots, Y_{r+s}$,

and

(iii) $Z'_2 = \{X'_{r+s}, \ldots, X'_{r+1}, X'_r, \ldots, X'_2, X'_1, Y'_1, Y'_2, \ldots, Y'_r, Y'_{r+1}, \ldots, Y'_{r+s}\}$

for

$$X'_i = \begin{cases} \{x_{i(m_i-1)}, \ldots, x_{i1}, x_{i0}\} & \text{if } 1 \leqslant i \leqslant r, \\ \{x_{i(m_i-1)}, \ldots, x_{i1}\} & \text{if } r+1 \leqslant i \leqslant r+s, \end{cases}$$

and

$$Y'_i = \begin{cases} \{y_{i1}, y_{i2}, \ldots, y_{im_i}\} & \text{if } 1 \leqslant i \leqslant r, \\ \{y_{i2}, y_{i3}, \ldots, y_{im_i}\} & \text{if } r+1 \leqslant i \leqslant r+s. \end{cases}$$

Remark 1. Write $n_i = 2m_i + 1$ for $1 \leqslant i \leqslant r$ and $n_i = 2m_i$ for $r+1 \leqslant i \leqslant r+s$. By a rearrangement of $\{m_i\}$ we may assume that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_t$ for $t = r+s$. Then, by the definition of τ_i and θ_i in the proof for Theorem A, we shall see that $\{n_i\}$ are the invariants of σ. Thus, the involution τ and the semiinvolution θ in Theorem A are unique for σ up to similarity. Further, as we see in Theorem A, the relationship between τ and θ is given by the condition (C), more precisely τ determines θ.

Remark 2. τ is expressed in the basis $\{Z_1, Z_2\}$ for V as

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & \cdots & \\ & & & 1 & & \end{pmatrix}$$

and θ is expressed in the basis $\{Z_0', Z_1', Z_2'\}$ for V as

$$\begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & & & 1 \\ & & & & & & & \cdots & \\ & & & & & & 1 & & \end{pmatrix}.$$

Theorem B. *Any $\sigma \in \operatorname{Aut}_k V$ is a product of two involutions τ, θ if and only if σ and σ^{-1} are similar.*

Djocović' [1] proved Theorem B by applying the uniqueness of the elementary divisors, whereas we will do it by using the uniqueness of the system of invariants. As a result the proof will be shorter.

2. Proof of Theorem A

(I) (a) $\Rightarrow$ (b): We start our proof from the following well-known result on nilpotent linear endomorphisms of V.

Since $\sigma \in \operatorname{End}_k V$ is nilpotent, we may express V and σ as

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_t, \quad V_i = kv_{i1} \oplus kv_{i2} \oplus \ldots \oplus kv_{in_i} \quad \text{for } 1 \leqslant i \leqslant t,$$

and

$$\sigma = \sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_t, \quad \sigma_i \colon v_{i1} \to v_{i2} \to \ldots \to v_{in_i} \to 0 \quad \text{for } 1 \leqslant i \leqslant t,$$

where

$$\sigma_i = \sigma|_{V_i} \in \operatorname*{End}_k V_i$$

(see for example Herstein [5, Theorem 6.5.1]).

By the above result, for $1 \leqslant i \leqslant t$, if we define $\tau_i, \theta_i \in \operatorname{End}_k V_i$ by

(1) $$\tau_i\colon\ v_{ij} \to v_{i(n_i-j+1)} \quad \text{for } 1 \leqslant j \leqslant n_i,$$

and

(2) $$\theta_i = v_{i1} \to 0 \quad \text{and} \quad v_{ij} \to v_{i(n_i-j+2)} \quad \text{for } 2 \leqslant j \leqslant n_i,$$

we have

(3) $$\sigma_i = \theta_i \tau_i \quad \text{for } 1 \leqslant i \leqslant t,$$

and so

(4) $$\sigma = \theta_1\tau_1 \oplus \theta_2\tau_2 \oplus \ldots \oplus \theta_t\tau_t = (\theta_1 \oplus \theta_2 \oplus \ldots \theta_t)(\tau_1 \oplus \tau_2 \oplus \ldots \oplus \tau_t).$$

To construct an involution τ and a semiinvolution θ as in the theorem, we will rearrange the basis elements $\{v_{ij}\}$ for V. To do so we will renumber the suffixes of the subspaces $\{V_1, V_2, \ldots, V_t\}$ so that their dimensions $\{n_1, n_2, \ldots, n_r\}$ are all odd numbers with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$, and $\{n_{r+1}, n_{r+2}, \ldots, n_{r+s}\}$ are all even with $n_{r+1} \geqslant n_{r+2} \geqslant \ldots \geqslant n_{r+s}$ and $t = r + s$. Moreover, we rewrite the basis elements in $S_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ for V_i as

(5) $$S_i = \begin{cases} \{x_{im_i}, \ldots, x_{i2}, x_{i1}, x_{i0}, y_{i1}, y_{i2}, \ldots, y_{im_i}\} & \text{for } 1 \leqslant i \leqslant r, \\ \{x_{im_i}, \ldots, x_{i2}, x_{i1}, y_{i1}, y_{i2}, \ldots, y_{im_i}\} & \text{for } r+1 \leqslant i \leqslant r+s, \end{cases}$$

where $n_i = 2m_i + 1$ for $1 \leqslant i \leqslant r$, and $2m_i$ for $r + 1 \leqslant i \leqslant r + s$.

This is equivalent to saying that for $1 \leqslant i \leqslant r + s$, setting

$$X_i = \{x_{im_i}, \ldots, x_{i2}, x_{i1}\} \quad \text{and} \quad Y_i = \{y_{i1}, y_{i2}, \ldots, y_{im_i}\},$$

we then have

$$S_i = \{X_i, x_{i0}, Y_i\} \ \text{for } 1 \leqslant i \leqslant r, \quad \text{and} \quad S_i = \{X_i, Y_i\} \ \text{for } r+1 \leqslant i \leqslant r+s.$$

Hence, if we define

$$Z_1 = \{x_{10}, x_{20}, \ldots, x_{r0}\},$$
$$Z_2 = \{X_{r+s}, \ldots, X_{r+1}, X_r, \ldots, X_1, Y_1, \ldots, Y_r, Y_{r+1}, \ldots, Y_{r+s}\},$$

and

$$\tau = 1_{Z_1} \oplus \Delta_{Z_2},$$

then by (1) we find that

$$\tau = \tau_1 \oplus \tau_2 \oplus \ldots \oplus \tau_t. \tag{7}$$

Similarly, setting

$$X'_i = \{x_{i(m_i-1)}, \ldots, x_{i0}\},\ \ Y'_i = \{y_{i1}, \ldots, y_{im_i}\} \quad \text{for } 1 \leqslant i \leqslant r,$$

and

$$X'_i = \{x_{i(m_i-1)}, \ldots, x_{i1}\},\ \ Y'_i = \{y_{i2}, \ldots, y_{im_i}\} \quad \text{for } r+1 \leqslant i \leqslant r+s,$$

we get

$$S_i = \{x_{im_i}, X'_i, Y'_i\} \text{ for } 1 \leqslant i \leqslant r, \quad \text{and} \quad \{x_{im_i}, X'_i, y_{i1}, Y'_i\} \text{ for } r+1 \leqslant i \leqslant r+s.$$

Therefore, if we define

$$Z'_0 = \{x_{1m_1}, x_{2m_2}, \ldots, x_{(r+s)m_{(r+s)}}\},$$
$$Z'_1 = \{y_{(r+1)1}, y_{(r+2)1}, \ldots, y_{(r+s)1}\},$$
$$Z'_2 = \{X'_{r+s}, \ldots, X'_{r+1}, X'_r, \ldots, X'_1, Y'_1, \ldots, Y'_r, Y'_{r+1}, \ldots, Y'_{r+s}\},$$

and

$$\theta = 0_{Z'_0} \oplus 1_{Z'_1} \oplus \Delta_{Z'_2}, \tag{8}$$

we have

$$\theta = \theta_1 \oplus \theta_2 \oplus \ldots \oplus \theta_t. \tag{9}$$

This shows that $\sigma = \theta\tau$ by (4), which gives us (b).

(II) (b) $\Rightarrow$ (a): By the definition of τ and θ, we have for $1 \leqslant i \leqslant r+s$,

(10)

$$\begin{array}{lccccccccc}
 & & & y_{im_i} & & & & y_{i(m_i-1)} & & \\
 & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & \\
\sigma^{m_i-1}\colon & x_{im_i} & & \xrightarrow{\quad\sigma\quad} & & x_{i(m_i-1)} & & \xrightarrow{\quad\sigma\quad} & & x_{i(m_i-2)} \longrightarrow
\end{array}$$

$$\begin{array}{cccccc}
 & & & y_{i2} & & \\
 & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & \\
\cdots \longrightarrow & x_{i2} & & \xrightarrow{\quad\sigma\quad} & & x_{i1}.
\end{array}$$

Further for $1 \leqslant i \leqslant r$,

(11)

$$\begin{array}{lccccccccccccc}
 & & & y_{i1} & & & & x_{i0} & & & & x_{i1} & & \\
 & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & \\
\sigma^{m_i+2}\colon & x_{i1} & & \xrightarrow{\quad\sigma\quad} & & x_{i0} & & \xrightarrow{\quad\sigma\quad} & & y_{i1} & & \xrightarrow{\quad\sigma\quad} & & y_{i2} \longrightarrow
\end{array}$$

$$\begin{array}{cccccc}
 & & & x_{im_i} & & \\
 & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & \\
\cdots \longrightarrow & y_{im_i} & & \xrightarrow{\quad\sigma\quad} & & 0.
\end{array}$$

and for $r+1 \leqslant i \leqslant r+s$,

(12)

$$\begin{array}{lccccccccc}
 & & & y_{i1} & & & & x_{i1} & & \\
 & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & \\
\sigma^{m_i+1}\colon & x_{i1} & & \xrightarrow{\quad\sigma\quad} & & y_{i1} & & \xrightarrow{\quad\sigma\quad} & & y_{i2} \longrightarrow
\end{array}$$

$$\begin{array}{cccccc}
 & & & x_{im_i} & & \\
 & & \overset{\tau}{\nearrow} & & \overset{\theta}{\searrow} & \\
\cdots \longrightarrow & y_{im_i} & & \xrightarrow{\quad\sigma\quad} & & 0.
\end{array}$$

Therefore,

$$\sigma^{2m_i+1}\{X_i, x_{i0}, Y_i\} = 0 \quad \text{for } 1 \leqslant i \leqslant r,$$

and

$$\sigma^{2m_i}\{X_i, Y_i\} = 0 \quad \text{for } r+1 \leqslant i \leqslant r+s.$$

Hence, for $l = \max\{\{2m_i+1\colon\ 1 \leqslant i \leqslant r\} \cup \{2m_i\colon\ r+1 \leqslant i \leqslant r+s\}\}$, we conclude that $\sigma^l V = 0$. Thus σ is nilpotent and we have proved Theorem A.

3. Proof of Theorem B

If $\sigma = \tau\theta$ with $\tau^2 = \theta^2 = 1$, then $\sigma^{-1} = \theta\tau = \theta\tau\theta\theta^{-1} = \theta\sigma\theta^{-1}$ is similar to σ. So, all what we have to do is to show the converse.

Let $k[x]$ be the polynomial ring in x over k. Then, since the correspondence

$$\pi_\sigma\colon\ k[x] \longrightarrow \operatorname*{End}_k V$$

defined by $\pi_\sigma(f(x))(v) = f(\sigma)(v)$ for $v \in V$ and $f(x) \in k[x]$ is a ring homomorphism, if we define $f(x)v = f(\sigma)(v)$, V is endowed a module structure over the principal ideal domain $k[x]$. In particular, since $\dim V < \infty$, V is a finitely generated torsion $k[x]$-module. Therefore by [10, XIV, Theorem 2.1, p. 557] there is a finite number of monic polynomials $f_1(x), f_2(x), \ldots, f_n(x)$ in $k[x]$ such that

$$V \simeq k[x]/(f_1(x)) \oplus \ldots \oplus k[x]/(f_n(x)) \quad \text{with} \quad f_1 \mid \ldots \mid f_n$$

as $k[x]$-modules. Further the sequence of ideals $(f_1), \ldots, (f_n)$ is an invariant for V and π_σ, which is called the system of invariants.

Since $k[x]/(f_i(x)) = k[x](1 + (f_i(x)))$ is a cyclic $k[x]$-submodule generated by one element $1 + (f_i(x))$, if we write

$$\text{(1)} \qquad f_i(x) = a_{i0} + a_{i1}x + \ldots + a_{i(m_i-1)}x^{m_i-1} + x^{m_i}, \quad a_{ij} \in k,$$

for $i = 1, 2, \ldots, n$, we will find n elements $v_1, v_2, \ldots, v_n \in V$ which satisfy for $i = 1, 2, \ldots, n$,

(i) $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ where $V_i = kv_i \oplus k\sigma v_i \oplus \ldots \oplus k\sigma^{m_i-1}v_i \simeq k[x]/(f_i(x))$,
(ii) $\sigma = \sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_n$, $\sigma_i = \sigma|_{V_i}$, and
(iii) $f_i(x)$ is the minimal polynomial of σ_i.

Here we note that $\sigma_i \in \operatorname{Aut}_k V_i$, or equivalently $a_{i0} \neq 0$, since $\sigma \in \operatorname{Aut}_k V$. This implies that for $i = 1, 2, \ldots, n$

$$V_i = (\sigma_i^{-1})^{m_i-1} V_i = kv_i \oplus k\sigma_i^{-1}v_i \oplus \ldots \oplus k(\sigma_i^{-1})^{m_i-1}v_i,$$
$$\sigma^{-1} = \sigma_1^{-1} \oplus \sigma_2^{-1} \oplus \ldots \oplus \sigma_n^{-1}$$

and

$$\text{(2)} \qquad \begin{aligned} g_i(x) &= a_{i0}^{-1}x^{m_i} f_i(x^{-1}) \\ &= a_{i0}^{-1} + a_{i0}^{-1}a_{i(m_i-1)}x + \ldots + a_{i0}^{-1}a_{i1}x^{m_i-1} + x^{m_i} \end{aligned}$$

is the minimal polynomial of σ_i^{-1}. Accordingly if we give V another $k[x]$-module structure by a ring homomophism

$$\pi_{\sigma^{-1}}\colon\, k[x] \longrightarrow \operatorname*{End}_k V \quad \text{defined by} \quad \pi_{\sigma^{-1}}(f(x))(v) = f(\sigma^{-1})(v)$$

and write it V' for V, we have

$$V' \simeq k[x]/(g_1(x)) \oplus \ldots \oplus k[x]/(g_n(x)) \quad \text{with} \quad g_1 \mid \ldots \mid g_n.$$

As for $g_i \mid g_{i+1}$, since $f_i \mid f_{i+1}$, if we set $f_{i+1} = f_i h_i$ with $m_i = \dim f_i$ and $r_i = \dim h_i$, we get $g_{i+1}(x) = g_i(x)q_i(x)$ for $q_i(x) = h_i(0)^{-1}x_i^{r_i}h_i(x^{-1}) \in k[x]$. Hence $g_1 \mid \ldots \mid g_n$.

On the other hand, since σ and σ^{-1} are similar, we have $\sigma^{-1} = \varrho\sigma\varrho^{-1}$ for some $\varrho \in \operatorname{Aut}_k V$. Hence $\varrho\pi_\sigma(f(x))(v) = \varrho f(\sigma)(v) = \pi_{\sigma^{-1}}(f(x))\varrho(v)$, since $\sigma^{-1}\varrho = \varrho\sigma$. This shows that ϱ is a $k[x]$-module isomorphism of V to V'. Therefore the uniqueness of the system of invariants gives us $(f_i) = (g_i)$ and so $f_i = g_i$, since they are monic. Thus (1), (2) imply that

(3) $$a_{i0} = a_{i0}^{-1}, \quad a_{ij} = a_{i0}^{-1}a_{i(m_i-j)} \quad \text{for } j = 1, 2, \ldots, m_i - 1.$$

Now for $i = 1, 2, \ldots, n$, we define $\tau_i, \theta_i \in \operatorname{Aut}_k V_i$ by

$$\begin{aligned} \tau_i\colon \sigma_i^j v_i &\longrightarrow \sigma_i^{m_j-j-1}v_i \quad \text{for } 0 \leqslant j \leqslant m_i - 1, \\ \theta_i\colon \sigma_i^j v_i &\longrightarrow \sigma_i^{m_i-j}v_i \qquad \text{for } 0 \leqslant j \leqslant m_i - 1. \end{aligned}$$

Then, for $i = 1, 2, \ldots, n$, we have

$$\sigma_i = \theta_i\tau_i \quad \text{and} \quad \tau_i^2 = 1 \ \text{ on } V_i, \quad \text{and} \quad \theta_i^2 = 1 \ \text{ on } \{\sigma_i v_i, \ldots, \sigma_i^{m_i-1}v_i\}.$$

However, using (3), an easy calculation gives us $\theta_i^2 v_i = v_i$ and so $\theta_i^2 = 1$ on V_i.

Thus, setting

$$\tau = \bigoplus_{i=1}^n \tau_i \quad \text{and} \quad \theta = \bigoplus_{i=1}^n \theta_i,$$

we obtain $\sigma = \tau\theta$ and $\tau^2 = \theta^2 = 1$, which completes the proof of Theorem B.

References

[1] *D. Ž. Djocović*: Product of two involutions. Arch. Math. *XVIII* (1967), 582–584. Zbl 0153.35502

[2] *E. W. Ellers, H. Ishibashi*: Bireflectionality of the orthogonal group over a valuation domain. J. Algebra *149* (1992), 322–325. Zbl 0779.20028

[3] *W. H. Gustafson, P. R. Halmos, and H. Radjavi*: Products of involutions. Linear Algebra Appl. *13* (1976), 157–162. Zbl 0325.15009

[4] *A. J. Hahn, O. T. O'Meara*: The Classical Groups and K-Theory. Springer-Verlag, Berlin-Tokyo, 1989. Zbl 0683.20033

[5] *R. Henstock*: The General Theory of Integration. Clarendon Press, Oxford, 1991. Zbl 0745.26006

[6] *I. N. Herstein*: Topics in Algebra (2nd ed.). John Wiley and Sons, New York, 1964. Zbl 0122.01301

[7] *H. Ishibashi*: Decomposition of isometries of $U_n(V)$ over finite fields into simple isometries. Czechoslovak Math. J. *31* (1981), 301–305. Zbl 0464.20032

[8] *H. Ishibashi*: Involutary expressions for elements in $GL_n(Z)$ and $SL_n(Z)$. Linear Algebra Appl. *219* (1995), 165–177. Zbl 0823.20048

[9] *H. Ishibashi*: Groups generated by symplectic transvections over local rings. J. Algebra *218* (1999), 26–80. Zbl 0984.20031

[10] *T. J. Laffey*: Products of matrices. In: Generators and Relations in Groups and Geometries. Proc. NATO ASI (C) (A. Barlotti et al., eds.). Kluwer Academic, Dordrecht-London, 1991, pp. 95–123. Zbl 0729.15012

[11] *S. Lang*: Algebra (3rd ed.). Addison Wesley, Tokyo, 1993. Zbl 0848.13001

[12] *A. R. Sourour*: A factorization theorem for matrices. Linear Multilinear Alg. *19* (1986), 141–147. Zbl 0591.15008

[13] *B. Zheng*: Decomposition of matrices into commutators of involutions. Linear Algebra Appl. *347* (2002), 1–7. Zbl 1004.20026

Author's address: H. Ishibashi, Department of Mathematics, Josai University, Sakado, Saitama 350-02, Japan, e-mail: `hishi@math.josai.ac.jp`.