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REMOVABLE SINGULARITIES FOR WEIGHTED BERGMAN SPACES

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Abstract. We develop a theory of removable singularities for the weighted Bergman space $\mathscr{A}^p_\mu(\Omega) = \{f \text{ analytic in } \Omega\colon \int_\Omega |f|^p \,\mathrm{d}\mu < \infty\}$, where μ is a Radon measure on $\mathbb{C}$. The set A is *weakly removable* for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if $\mathscr{A}^p_\mu(\Omega \setminus A) \subset \operatorname{Hol}(\Omega)$, and *strongly removable* for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if $\mathscr{A}^p_\mu(\Omega \setminus A) = \mathscr{A}^p_\mu(\Omega)$.

The general theory developed is in many ways similar to the theory of removable singularities for Hardy H^p spaces, BMO and locally Lipschitz spaces of analytic functions, including the existence of counterexamples to many plausible properties, e.g. the union of two compact removable singularities needs not be removable.

In the case when weak and strong removability are the same for all sets, in particular if μ is absolutely continuous with respect to the Lebesgue measure m, we are able to say more than in the general case. In this case we obtain a Dolzhenko type result saying that a countable union of compact removable singularities is removable.

When $\mathrm{d}\mu = w \,\mathrm{d}m$ and w is a Muckenhoupt A_p weight, $1 < p < \infty$, the removable singularities are characterized as the null sets of the weighted Sobolev space capacity with respect to the dual exponent $p' = p/(p-1)$ and the dual weight $w' = w^{1/(1-p)}$.

Keywords: analytic continuation, analytic function, Bergman space, capacity, exceptional set, holomorphic function, Muckenhoupt weight, removable singularity, singular set, Sobolev space, weight

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1. Introduction and background

Removable singularities for analytic functions are an old subject going back to Riemann's classification of isolated singularities. Characterizations of removable singularities have been given for many different spaces, see below, including unweighted Bergman spaces, see Carleson [10] and Hedberg [17].

In the preprint Björn [6] the author realized that the theory of removable singularities for weighted Bergman spaces and for Hardy H^p spaces have many similarities.

After having found more spaces with similar behaviour, the author developed an axiomatic theory for removable singularities in Björn [9].

This paper is an improved version of [6] containing all the results therein often in more general forms (the removability definition therein is more restrictive than the one used in this paper). It also shows that the axioms in [9] are fulfilled for weighted Bergman spaces and quotes all the relevant results obtained in [9]. The results for weighted Bergman spaces reported upon in Björn [8] are also included in this paper.

In this paper we develop the theory of removable singularities for quite general weighted Bergman spaces with respect to Radon measures. We give a number of results that hold in this general setting, and also give counterexamples showing the limitations of the theory.

In the case when the Radon measure is a weight ($d\mu = w\,dm$) we show that much more is true, including a Dolzhenko type result saying that a countable union of compact removable singularities is removable. We also generalize the characterization for unweighted Bergman spaces, giving a complete characterization for the removable singularities of Bergman spaces with respect to Muckenhoupt A_p weights w as null sets of the weighted Sobolev space capacity for the dual exponent $p' = p/(p-1)$ and dual weight $w' = w^{1/(1-p)}$.

Much attention has been given to find a characterization of the removable singularities for bounded analytic functions, a problem which was recently solved by Tolsa [30]. Other spaces of analytic functions for which removable singularities have been studied include: the Nevanlinna class N (Rudin [28]); the Smirnov class N^+ (Khavinson [22]); the Smirnov spaces E^p (Khavinson [21]); the Dirichlet spaces AD^p (Hedberg [17]); the John-Nirenberg class BMO (Král [26], Kaufman [20], Koskela [25] and Björn [9]); the Hölder classes C^α (Dolzhenko [12] and Koskela [25]); the Lipschitz space Lip (Nguyen [27] and Khrushchëv [23]); the Zygmund class ZC (Carmona-Donaire [11]); the spaces VMO, lip_α and Campanato spaces (Král [26] as special cases of the corresponding problem for more general partial differential operators); the locally Lipschitz classes locLip_α and loclip_α (Björn [9]); and let us also mention the paper by Ahlfors and Beurling [2]. In a sequence of papers [4], [5], [7], [8] the author built on older work in the study of removable singularities for H^p.

This paper is organised as follows. In Section 2 we define weak and strong removability, the Bergman spaces $\mathscr{A}^p_\mu$ and the auxiliary Bergman spaces B^p_μ used throughout this paper. In Section 3 we give a number of simple results that hold for $\mathscr{A}^p_\mu$. In Section 4 we show that the auxiliary Bergman spaces B^p_μ satisfy the main axioms in Björn [9], after which we quote all the relevant results obtained in [9]. In Section 5 we characterize removable singularities for $\mathscr{A}^\infty_\mu$, and in Section 6 we compare removability for different exponents. In Section 7 we introduce Bergman space capacities. In Section 8 we look at the case when weak and strong removability coincide for all sets, which, e.g., happens for B^p_w.

In Section 9 we give two characterizations of weakly removable singularities for $\mathscr{A}^p_\mu$. The first says that weakly removable singularities are the same for $\mathscr{A}^p_\mu$ and B^p_μ unless $\mathscr{A}^p_\mu(\Omega \setminus A) = \{0\}$. The second characterizes weakly removable singularities for $\mathscr{A}^p_\mu$ in terms of those for B^p_μ and some additional quantities under the weak assumption that there exists some n such that $\int_{\mathbf{C}\setminus\mathbf{D}} |z|^{-n}\, d\mu(z) < \infty$. Some criteria for the additional quantities in the second characterization are given in Section 11, which aims at simplicity, rather than generality, but includes the case of Muckenhoupt weights.

In Section 10 we introduce Muckenhoupt weights and associated capacities from non-linear potential theory. We also prove some lemmas that are used in Section 12, which is devoted to a complete characterization of removable singularities for $\mathscr{A}^p_w$, when w is a Muckenhoupt A_p weight, in terms of null sets of the weighted Sobolev space capacity for the dual exponent $p' = p/(p-1)$ and dual weight $w' = w^{1/(1-p)}$.

In Section 13 we take a look at the unweighted case. This is not new, see Carleson [10] and Hedberg [17]. We would like to direct the reader to Section 11.1 in Adams-Hedberg [1], which inspired much of the work in Section 12 in this paper. In Section 13 we also point out that the solution to the unweighted case is also a solution to the weighted case when the weight is locally bounded from above and below, as has often been the case when weighted Bergman spaces have been studied in the literature.

In Section 14 we give counterexamples to several plausible properties when weak and strong removability are different. A major reason for us to consider "weights" that are not weight functions, but Radon measures, is that we can find examples when the situation is fairly similar to the situation for removable singularities for H^p spaces and analytic functions in BMO, locLip_α and loclip_α (see Björn [9] for definitions of these spaces). A necessity for this is that weak and strong removability are different concepts, which never happens when μ is absolutely continuous with respect to the Lebesgue area measure m.

Many problems are easier to solve for Hardy spaces than for Bergman spaces, and a lot of work during the 1990s was done trying to develop the theory of Bergman spaces to the level of the theory of Hardy spaces. As we have seen, the problem of removable singularities is different in nature, since it is easier to solve for even quite general weighted Bergman spaces, than for Hardy spaces.

We close the paper by looking at the related problem of isometrically removable sets in Section 15.

The proofs in this paper are usually given for $p < \infty$. The omitted proofs for $p = \infty$ are either similar or easier.

2. Notation and definitions

Throughout this paper we assume, unless otherwise stated, that $0 < p \leqslant \infty$, that $\Omega \subset \mathbb{S} = \mathbb{C} \cup \{\infty\}$, the Riemann sphere, that $A, E \subset \Omega \cap \mathbb{C}$, that Ω and $\Omega \setminus E$ are domains, i.e. non-empty open connected sets, that $\mu|_{\mathbb{C}}$ is a positive complete Radon measure on $\mathbb{C}$, i.e. a positive complete Borel measure that is finite on all compact subsets of $\mathbb{C}$, and that $\mu(\{\infty\}) < \infty$.

We let $\operatorname{Hol}(\Omega) = \{f\colon f \text{ is analytic in } \Omega\}$. Because of the uniqueness theorem we will not distinguish between restrictions and extensions of analytic functions. We also let $L^p_\mu(\Omega)$ denote the weighted Lebesgue space (quasi)-normed by $\|f\|_{L^p_\mu(\Omega)} = \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{1/p}$, $0 < p < \infty$, and $\|f\|_{L^\infty_\mu(\Omega)} = \inf\{C \geqslant 0\colon \mu(\{z \in \Omega\colon |f(z)| > C\}) = 0\}$.

Definition 2.1. The *Bergman space* $\mathscr{A}^p_\mu(\Omega)$ is defined by

$$\mathscr{A}^p_\mu(\Omega) = \{f \in \operatorname{Hol}(\Omega)\colon \|f\|_{L^p_\mu(\Omega)} < \infty\}.$$

Remarks. The point at infinity is special since we do not require the existence of a neighbourhood of ∞ with finite measure. It will be helpful to include the point at infinity since $\operatorname{Hol}(\mathbb{S}) = \{f\colon f \text{ is constant}\}$ is a much simpler space than $\operatorname{Hol}(\mathbb{C})$.

These Bergman spaces are sometimes (quasi)-Banach spaces, but not always. The "norm" is in general only a (quasi)-seminorm, i.e. there may be several functions with "norm" zero. For $0 < p < 1$ the triangle inequality is replaced by a quasi-triangle inequality. In general these spaces are not complete. It is an interesting open problem (as far as the author knows) to characterize exactly when these Bergman spaces are (quasi)-Banach spaces. For $p = \infty$ such a characterization is given in Arcozzi-Björn [3], where also the case $p < \infty$ is studied briefly. It is interesting to note that for the results in this paper it does not matter if the Bergman space is (quasi)-Banach or not.

The case of infinite measure is sometimes quite different from the finite measure case. In order to develop the theory we shall use some auxiliary Bergman spaces. We first let $D(a, r) = \{z \in \mathbb{C}\colon |z - a| < r\}$ and $\mathbb{D} = D(0, 1)$.

Definition 2.2. The *auxiliary Bergman space* $B^p_\mu(\Omega)$, $0 < p \leqslant \infty$, is defined by

$$B^p_\mu(\Omega) = \operatorname{Hol}(\Omega) \cap \bigcap \mathscr{A}^p_\mu(\Omega'),$$

where the large intersection is taken over all domains $\Omega' \subset \Omega$ such that

$$\left\| \frac{1}{z} \right\|_{L^p_\mu(\Omega' \setminus \mathbf{D})}^p = \int_{\Omega' \setminus \mathbf{D}} \frac{1}{|z|^p} \, \mathrm{d}\mu(z) < \infty. \tag{2.1}$$

For $p = \infty$, we say that all domains satisfy condition (2.1).

We also define, for $0 < p \leqslant \infty$,

$$B^p_{\mu,\mathrm{fin}}(\Omega) = \mathrm{Hol}(\Omega) \cap \bigcap_{\substack{\Omega' \subset \Omega \text{ domain} \\ \mu(\Omega') < \infty}} \mathscr{A}^p_\mu(\Omega'),$$

$$B^p_{\mu,\mathrm{bdd}}(\Omega) = \mathrm{Hol}(\Omega) \cap \bigcap_{\substack{\Omega' \subset \Omega \text{ bounded} \\ \text{domain}}} \mathscr{A}^p_\mu(\Omega').$$

It is obvious that $\mathscr{A}^p_\mu(\Omega) \subset B^p_\mu(\Omega) \subset B^p_{\mu,\mathrm{fin}}(\Omega) \subset B^p_{\mu,\mathrm{bdd}}(\Omega)$ for any domain Ω, and that $\mathscr{A}^p_\mu(\Omega) = B^p_\mu(\Omega)$ if Ω satisfies condition (2.1), etc. It is also obvious that $H^\infty(\Omega) \subset \mathscr{A}^\infty_\mu(\Omega) \subset B^p_{\mu,\mathrm{fin}}(\Omega)$, with equality in the first inclusion if $\Omega \subset \operatorname{supp} \mu$. (The identity $H^\infty(\Omega) = \mathscr{A}^\infty_\mu(\Omega)$ is characterized by Theorem 2.1 in Arcozzi-Björn [3].)

If μ is absolutely continuous with respect to m, the Lebesgue area measure, we can write $\mathrm{d}\mu = w \, \mathrm{d}m$, where $w = \mathrm{d}\mu/\mathrm{d}m$ is the Radon-Nikodym derivative. In this case we will often write $\mathscr{A}^p_w(\Omega) = \mathscr{A}^p_\mu(\Omega)$ and $B^p_w(\Omega) = B^p_\mu(\Omega)$. If moreover $\mu = m$, or in other terms $w = 1$, we usually omit the subscript completely and write $\mathscr{A}^p(\Omega) = \mathscr{A}^p_m(\Omega)$ and $B^p(\Omega) = B^p_m(\Omega)$.

The theory of removable singularities for $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$ is essentially the same as for B^p_μ, with the same proofs. Some proofs are slightly simpler for $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$. We have chosen to develop the theory for B^p_μ, rather than for $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$, since $B^p_\mu(\Omega) = \mathscr{A}^p_\mu(\Omega)$ for more domains.

At this point it may be useful to see what the differences are between these Bergman spaces. Obviously, if Ω is bounded, then $\mathscr{A}^p_\mu(\Omega) = B^p_\mu(\Omega) = B^p_{\mu,\mathrm{fin}}(\Omega) = B^p_{\mu,\mathrm{bdd}}(\Omega)$. If $\infty \in \Omega$, then $B^p_{\mu,\mathrm{fin}}(\Omega) = B^p_{\mu,\mathrm{bdd}}(\Omega)$, moreover, $B^p_\mu(\Omega) = B^p_{\mu,\mathrm{fin}}(\Omega)$ if $1 \in B^p_\mu(\mathbb{C})$ (in particular if $\mu(\mathbb{C}) < \infty$), otherwise $B^p_\mu(\Omega) = \{f \in B^p_{\mu,\mathrm{fin}}(\Omega) \colon f(\infty) = 0\}$. The original Bergman space $\mathscr{A}^p_\mu(\Omega)$ depends much more on μ and p.

If $\infty \notin \Omega$, the picture is a little different. First of all $B^p_{\mu,\mathrm{bdd}}(\mathbb{C}) = \mathrm{Hol}(\mathbb{C})$, since all entire functions are bounded on bounded domains. We always have $1 \in B^p_{\mu,\mathrm{fin}}(\Omega) \subset B^p_{\mu,\mathrm{bdd}}(\Omega)$. If $\mu(\mathbb{C}) = \infty$ and $p \neq \infty$, then usually $1 \notin B^p_\mu(\mathbb{C})$ (but not always, see Remarks 7.12), there may however still be functions with essential singularities at infinity in $B^p_\mu(\mathbb{C}) \subset B^p_{\mu,\mathrm{fin}}(\mathbb{C})$. One always has $1/z \in B^p_\mu(\mathbb{S} \setminus \overline{\mathbf{D}})$.

In general, the difference between these Bergman spaces is the behaviour they allow at ∞. The spaces $B^p_{\mu,\mathrm{bdd}}$ allow any behaviour at infinity, whereas $B^p_{\mu,\mathrm{fin}}$ always allow functions bounded near infinity, and may allow more. The spaces B^p_μ always allow at least the behaviour similar to $1/z$ at infinity, whereas $\mathscr{A}^p_\mu$ may not allow any non-zero function in a neighbourhood of infinity, see Remarks 4.2.

We have so far defined (auxiliary) Bergman spaces over domains, we next extend the definition to non-domains. In our case we will have $X = \mathscr{A}^p_\mu$, $X = B^p_\mu$, $X = B^p_{\mu,\mathrm{fin}}$ or $X = B^p_{\mu,\mathrm{bdd}}$.

Definition 2.3. If $A \subset \mathbb{S}$, then we define

$$X(A) = \bigcup_{\Omega \supset A \text{ domain}} X(\Omega).$$

Note that

$$\mathscr{A}^p_\mu(A) = \{f\colon \|f\|_{L^p_\mu(A)} < \infty \text{ and there is a domain } \Omega \supset A \text{ such that } f \in \mathrm{Hol}(\Omega)\},$$

which is quite straightforward to show; we leave the proof to the interested reader.

It is easy to see that this definition is consistent with the definition for domains, e.g. by observing that Axiom A2 below holds.

We are now ready to define what removable singularities are.

Definition 2.4. The set A is *weakly removable* for $X(\Omega \setminus A)$ if $X(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$, and A is *strongly removable* for $X(\Omega \setminus A)$ if $X(\Omega \setminus A) = X(\Omega)$.

The requirement that Ω be a domain is to avoid pathological situations such as $\Omega \setminus A$ being connected, but Ω non-connected.

Remarks. It is obvious that strong removability implies weak removability. The converse is not true in general, but it is true if μ is absolutely continuous with respect to the Lebesgue area measure m, see Proposition 8.1.

We have made the general assumption that $\infty \notin A$. The point at infinity needs special attention, we refrain from this since it does not seem to be particularly interesting.

Let us end this section with some more notation: We let $\dim_H$ denote the Hausdorff dimension, δ_z denote the Dirac measure at z, $\lceil x \rceil$ denote the smallest integer $\geqslant x$, $\lfloor x \rfloor$ denote the largest integer $\leqslant x$ and let $\mathbb{N} = \{0, 1, \ldots\}$.

3. Removability results for $\mathscr{A}^p_\mu$

All the results in this section hold also if we replace $\mathscr{A}^p_\mu$ by B^p_μ, $B^p_{\mu,\mathrm{fin}}$ or $B^p_{\mu,\mathrm{bdd}}$ (and they will be quoted also in this setting), which follows just using the definitions of B^p_μ, $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$ and the corresponding results for $\mathscr{A}^p_\mu$.

Proposition 3.1. *Let $K \subset \Omega \cap \mathbb{C}$ be compact and such that $\Omega \setminus K$ is a domain. Then K is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus K)$ if and only if K is strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus K)$.*

Remark. Because of this result we will usually say that a compact set is removable, without specifying weak/strong removability.

P r o o f. It is clear that strong removability implies weak removability. Assume, conversely, that K is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus K)$ and consider a function $f \in \mathscr{A}^p_\mu(\Omega \setminus K) \subset \mathrm{Hol}(\Omega)$. Since f is continuous on K and K is compact, f is bounded on K. Since μ is a Radon measure $\mu(K) < \infty$. Hence

$$\|f\|^p_{L^p_\mu(\Omega)} = \|f\|^p_{L^p_\mu(\Omega\setminus K)} + \int_K |f|^p \,\mathrm{d}\mu < \infty.$$

Thus $f \in \mathscr{A}^p_\mu(\Omega)$ and since f was arbitrary, K is strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus K)$. □

Proposition 3.2. *Let $\Omega_1 \subset \Omega_2$ be domains and $A_1 \subset A_2 \subset \Omega_1 \cap \mathbb{C}$. If A_2 is weakly (strongly) removable for $\mathscr{A}^p_\mu(\Omega_1 \setminus A_2)$, then A_1 is weakly (strongly) removable for $\mathscr{A}^p_\mu(\Omega_2 \setminus A_1)$.*

P r o o f. For the weak part we have

$$\mathscr{A}^p_\mu(\Omega_2 \setminus A_1) \subset \mathscr{A}^p_\mu(\Omega_1 \setminus A_2) \cap \mathrm{Hol}(\Omega_2 \setminus A_1) \subset \mathrm{Hol}(\Omega_1) \cap \mathrm{Hol}(\Omega_2 \setminus A_1) = \mathrm{Hol}(\Omega_2).$$

Similarly, for the strong part we have

$$\mathscr{A}^p_\mu(\Omega_2 \setminus A_1) = \mathscr{A}^p_\mu(\Omega_1 \setminus A_2) \cap \mathscr{A}^p_\mu(\Omega_2 \setminus A_1) = \mathscr{A}^p_\mu(\Omega_1) \cap \mathscr{A}^p_\mu(\Omega_2 \setminus A_1) = \mathscr{A}^p_\mu(\Omega_2).$$

□

Proposition 3.3. *Let $E_k \subset \Omega \cap \mathbb{C}$ be pairwise disjoint sets such that $\Omega \setminus \bigcup_{j=1}^{k} E_j$ is a domain and E_k is strongly removable for $\mathscr{A}^p_\mu\big(\Omega \setminus \bigcup_{j=1}^{k} E_j\big)$, $k = 1, \dots, n$. Then $\bigcup_{j=1}^{n} E_j$ is strongly removable for $\mathscr{A}^p_\mu\big(\Omega \setminus \bigcup_{j=1}^{n} E_j\big)$.*

Proof. This is almost trivial, we have

$$\mathscr{A}^p_\mu\left(\Omega \setminus \bigcup_{j=1}^{n} E_j\right) = \mathscr{A}^p_\mu\left(\Omega \setminus \bigcup_{j=1}^{n-1} E_j\right) = \dots = \mathscr{A}^p_\mu(\Omega \setminus E_1) = \mathscr{A}^p_\mu(\Omega).$$

□

Proposition 3.4. *If A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ and $\mu(A) = 0$, then A is strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.*

Remark. In fact the assumptions imply isometric removability, see Proposition 15.3.

Proof. Let $f \in \mathscr{A}^p_\mu(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$. Since $\mu(A) = 0$, we have $\|f\|_{L^p_\mu(\Omega)} = \|f\|_{L^p_\mu(\Omega \setminus A)} < \infty$. Hence $f \in \mathscr{A}^p_\mu(\Omega)$, and thus A is strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$. □

Proposition 3.5. *The set A is weakly (strongly) removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if and only if E is weakly (strongly) removable for $\mathscr{A}^p_\mu(\Omega \setminus E)$ for all $E \subset A$ that are relatively closed in Ω.*

Proof. Let us start with the weak part. If A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ and $E \subset A$, then $\mathscr{A}^p_\mu(\Omega \setminus E) \subset \mathscr{A}^p_\mu(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$, which shows the necessity. As for the sufficiency, let $f \in \mathscr{A}^p_\mu(\Omega \setminus A)$, then there is a domain $\Omega' \supset \Omega \setminus A$ such that $f \in \mathscr{A}^p_\mu(\Omega' \setminus A)$. Let $E = \Omega \setminus \Omega'$. Then $f \in \mathscr{A}^p_\mu(\Omega \setminus E) \subset \mathrm{Hol}(\Omega)$.

The proof of the strong part is similar, we leave it to the interested reader. □

4. Axiomatic approach

In Björn [9] an axiomatic theory for removable singularities for spaces of analytic functions was developed that is well suited for Bergman spaces. It was developed for domains $\Omega \subset \mathbb{C}$, but it is trivial to rewrite the theory for domains $\Omega \subset \mathbb{S}$, as considered in this paper.

The following axioms are given.

(A1) For every domain $\Omega \subset \mathbb{S}$, $X(\Omega)$ is defined and $X(\Omega) \subset \mathrm{Hol}(\Omega)$.
(A2) If $\Omega_1 \subset \Omega_2 \subset \mathbb{S}$ are domains, then $X(\Omega_1) \supset X(\Omega_2)$.
(A3) If a compact set $K \subset \mathbb{C}$ is weakly removable for $X(\mathbb{S} \setminus K)$ and $\Omega \supset K$ is a domain, then K is strongly removable for $X(\Omega \setminus K)$.
(A4) If a compact set K is weakly removable for $X(\mathbb{S} \setminus K)$, then K is totally disconnected, i.e. no two different points in K can be connected by a curve in K.
(A5) If $K \subset K' \subset \mathbb{C}$ and K and K' are compact sets, then $\mathrm{cap}_X(K) \leqslant \mathrm{cap}_X(K')$.
(A6) If $K \subset \mathbb{C}$ is a compact set, then $\mathrm{cap}_X(K) = 0$ if and only if K is removable for X.
(A7) If Ω_1 and Ω_2 are domains and $\Omega_1 \cup \Omega_2$ is connected, then $X(\Omega_1) \cap X(\Omega_2) = X(\Omega_1 \cup \Omega_2)$.

Remark 4.1. In view of Axiom A3 and Proposition 3.2 we say that a compact set K is *removable* for X if there is one domain $\Omega \supset K$ such that K is weakly removable for $X(\Omega \setminus K)$, or equivalently, if K is strongly removable for $X(\Omega \setminus K)$ for all domains $\Omega \supset K$.

Remark 4.2. For $\mathscr{A}^p_\mu$, Axioms A1, A2 and A7 are always satisfied, whereas the others may not be satisfied. That Axiom A4 is not satisfied for $\mathscr{A}^p_\mu$ in general, can be seen by letting $w(z) = \mathrm{e}^{|z|}$, $\Omega = \mathbb{S}$ and $K = \overline{\mathbb{D}}$. If $f \in \mathscr{A}^p_w(\mathbb{S} \setminus \overline{\mathbb{D}})$, then $|f|^{p/2}$ is subharmonic. Letting $D = D\,(z, 1)$, $|z| > 2$, we obtain

$$\begin{aligned}|f(z)|^{p/2} &\leqslant \frac{1}{\pi}\int_D |f(\zeta)|^{p/2}\,\mathrm{d}m(\zeta) = \frac{1}{\pi}\int_D |f(\zeta)|^{p/2} w(\zeta)^{1/2} w(\zeta)^{-1/2}\,\mathrm{d}m(\zeta)\\ &\leqslant \frac{1}{\pi}\|f\|^{p/2}_{L^p_\mu(\mathbb{S}\setminus\overline{\mathbb{D}})}\left(\int_D \mathrm{e}^{-|\zeta|}\,\mathrm{d}m(\zeta)\right)^{1/2} \to 0,\end{aligned}$$

as $|z| \to \infty$, as fast as $\mathrm{e}^{-|z|/2}$. Hence $f(\infty) = f'(\infty) = f''(\infty) = \ldots = 0$, and thus $f \equiv 0$ and $\overline{\mathbb{D}}$ is removable for $\mathscr{A}^p_w(\mathbb{S} \setminus \overline{\mathbb{D}})$, but not totally disconnected.

That Axioms A3 and A4 are not satisfied for $\mathscr{A}^p_\mu$ in general is, of course, the reason for us to introduce the auxiliary Bergman spaces B^p_μ. It can be observed that if in (2.1) it was required that $\|z^{-\alpha}\|_{L^p_\mu(\Omega'\setminus\mathbb{D})} < \infty$ for some $\alpha > 1$, then B^p_μ would not satisfy Axiom A3 in general, cf. Theorem 13.3.

Note that Axioms A5 and A6 can always be satisfied, if Axioms A1–A4 are fulfilled, e.g., by defining

$$\mathrm{cap}_X(K) = \begin{cases} 0, & \text{if } K \text{ is removable for } X,\\ 1, & \text{if } K \text{ is not removable for } X.\end{cases}$$

We extend the definition of cap_X by the following definition.

Definition 4.3. Let $A \subset \mathbb{C}$ and define

$$\operatorname{cap}_X(A) = \sup\{\operatorname{cap}_X(K) \colon K \subset A \text{ is compact}\}.$$

In Section 7 we will define $\operatorname{cap}_{\mathscr{A}^p_\mu}(\,\cdot\,)$ which is suitable as cap_X for $X = B^p_\mu$, $X = B^p_{\mu,\mathrm{fin}}$ and $X = B^p_{\mu,\mathrm{bdd}}$, and for $X = \mathscr{A}^p_\mu$ when it satisfies Axioms A1–A7.

Before we quote the general results that follow from these axioms, we verify that the axioms are fulfilled for the auxiliary Bergman spaces.

Proposition 4.4. *Let $X = B^p_\mu$, $X = B^p_{\mu,\mathrm{fin}}$ or $X = B^p_{\mu,\mathrm{bdd}}$. Then Axioms* A1–A4 *and* A7 *are fulfilled.*

Proof. We prove this for $X = B^p_\mu$; the proofs for $X = B^p_{\mu,\mathrm{fin}}$ and $X = B^p_{\mu,\mathrm{bdd}}$ being similar. That Axioms A1 and A2 are fulfilled is immediate.

Axiom A3. Assume that $f \in B^p_\mu(\Omega \setminus K)$. Let Ω_1 and Ω_2 be smooth bounded domains with $K \subset \Omega_1 \Subset \Omega_2 \Subset \Omega$. Let $K_1 \supset K_2 \supset \ldots$ be compact smooth subsets of Ω_1 with $K = \bigcap\limits_{n=1}^{\infty} K_n$ and $\partial K_n \subset \Omega_1 \setminus K$ for all $n \geqslant 1$. Then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\partial K_n} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta =: g(z) + h_n(z), \quad z \in \Omega_2 \setminus K_n.$$

Since f is bounded on $\partial\Omega_2$, g is bounded on Ω_1 and $g \in B^p_\mu(\Omega_1)$. Moreover, $h_n \in \operatorname{Hol}(\mathbb{S} \setminus K_n)$ and

$$h_n(z) = f(z) - g(z), \quad z \in \Omega_2 \setminus K_n.$$

Thus $\{h_n(z)\}_{n=1}^{\infty}$ is constant when defined, so if

$$h(z) = \lim_{n\to\infty} h_n(z), \quad z \in \mathbb{S} \setminus K,$$

then $h \in \operatorname{Hol}(\mathbb{S} \setminus K)$. Furthermore, $h = f - g \in B^p_\mu(\Omega_1 \setminus K)$, $h(\infty) = 0$ and h is bounded in $\mathbb{S} \setminus \Omega_1$. Hence, for some constant C, $|h(z)| \leqslant C|z|^{-1}$ for all $z \in \mathbb{S} \setminus \Omega_1$. So, if $\Omega' \subset \mathbb{S} \setminus K$ is an arbitrary domain satisfying condition (2.1), then

$$\|h\|^p_{L^p_\mu(\Omega')} = \|h\|^p_{L^p_\mu(\Omega_1 \cap \Omega')} + \int_{(\Omega' \setminus \Omega_1) \cap \mathbb{D}} |h|^p \,\mathrm{d}\mu + \int_{(\Omega' \setminus \Omega_1) \setminus \mathbb{D}} |h|^p \,\mathrm{d}\mu < \infty.$$

The first term is bounded since $h \in B^p_\mu(\Omega_1 \setminus K)$. The second term is bounded since h is bounded, and the third term is bounded by condition (2.1). Hence, $h \in B^p_\mu(\mathbb{S} \setminus K) \subset \operatorname{Hol}(\mathbb{S})$, i.e. h is constant and $h \equiv h(\infty) = 0$.

So $f = g$ in $\Omega_1 \setminus K$ and f can be analytically continued to K. Since f was arbitrary, K is weakly removable for $B^p_\mu(\Omega \setminus K)$. Finally, it follows from Proposition 3.1 that K is strongly removable for $B^p_\mu(\Omega \setminus K)$.

Axiom A4. Assume that K is weakly removable for $B^p_\mu(\mathbf{C}\setminus K)$ and let $\Omega \supset K$ be a bounded domain. Then $H^\infty(\Omega \setminus K) \subset B^p_\mu(\Omega \setminus K) \subset \operatorname{Hol}(\Omega)$, and hence K is weakly removable for H^∞, from which it is well known that K is totally disconnected.

Axiom A7. This follows from the fact that $L^p_\mu(\Omega_1) \cap L^p_\mu(\Omega_2) = L^p_\mu(\Omega_1 \cup \Omega_2)$. $\square$

Next we are ready to quote the results proved under these axioms in Björn [9]. From now on we assume that Axioms A1–A7 are satisfied.

Proposition 4.5. *If A is weakly removable for $X(\Omega \setminus A)$, then A is totally disconnected.*

Proposition 4.6. *Assume that $E \subset \Omega \cap \mathbf{C}$ is relatively closed in Ω. Then the set E is weakly removable for $X(\Omega \setminus E)$ if and only if E can be written as a countable union of well-separated compact sets K_j removable for X, where by well-separated we mean that* $\operatorname{dist}\Big(K_k, \bigcup_{j=1, j\neq k}^{\infty} K_j\Big) > 0$ *for all $k = 1, 2, \ldots$.*

Proposition 4.7. *The set A is weakly removable for $X(\Omega \setminus A)$ if and only if* $\operatorname{cap}_X(A) = 0$.

Remark. Since the latter part is independent of Ω, we say that A is *weakly removable* for X if there is one domain $\Omega \supset A$ such that A is weakly removable for $X(\Omega \setminus A)$, or equivalently if A is weakly removable for $X(\Omega \setminus A)$ for all domains $\Omega \supset A$.

Proposition 4.8. *If $A \subset B$ and B is weakly removable for X, then A is weakly removable for X.*

Proposition 4.9. *Assume that $X(\Omega) \subset Y(\Omega)$ for all bounded domains Ω and that Axioms* A1–A6 *are satisfied also for Y. If $\operatorname{cap}_Y(A) = 0$, then $\operatorname{cap}_X(A) = 0$.*

Since $B^p_\mu(\Omega) = B^p_{\mu,\mathrm{fin}}(\Omega) = B^p_{\mu,\mathrm{bdd}}(\Omega)$ for bounded domains, it follows that B^p_μ, $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$ have the same capacities, and hence the same weakly removable singularities.

Proposition 4.10. *Let $K_1, K_2, \ldots, K_n \subset \mathbf{C}$ be pairwise disjoint compact sets removable for X. Then $\bigcup_{j=1}^{n} K_j$ is removable for X.*

Proposition 4.11. *Let $E_k \subset \Omega \cap \mathbf{C}$ be pairwise disjoint sets weakly removable for X and such that $\Omega \setminus E_k$ are domains, $k = 1, \ldots, n$. Then $\bigcup_{k=1}^{n} E_k$ is weakly removable for X.*

Proposition 4.12. *The set A is strongly removable for $X(\Omega \setminus A)$ if and only if E is strongly removable for $X(\Omega \setminus E)$ for all $E \subset A$ with $\Omega \setminus E$ being a domain.*

Proposition 4.13. *Assume that $E_1, E_2 \subset \Omega \cap \mathbb{C}$ are disjoint sets and such that $\Omega \setminus E_1$ and $\Omega \setminus E_2$ are domains. If E_1 and E_2 are strongly removable for $X(\Omega \setminus (E_1 \cup E_2))$, then $E_1 \cup E_2$ is strongly removable for $X(\Omega \setminus (E_1 \cup E_2))$.*

We end this section with a result not given in Björn [9].

Proposition 4.14. *The following are equivalent:*
(i) *A is weakly removable for X;*
(ii) $\operatorname{cap}_X(A) = 0$*;*
(iii) *for each $z \in A$, there exists a domain $\Omega_z \ni z$ with $\operatorname{cap}_X(A \cap \Omega_z) = 0$.*

Remark. The last part shows that weak removability for X is a local property of A.

P r o o f. (i) $\Leftrightarrow$ (ii) This is Proposition 4.7.

(ii) $\Rightarrow$ (iii) This follows directly from Definition 4.3.

(iii) $\Rightarrow$ (i) Let $f \in X(\mathbb{S} \setminus A) \subset X(\Omega_z \setminus A)$. Since $\operatorname{cap}_X(A \cap \Omega_z) = 0$, $A \cap \Omega_z$ is weakly removable for X, by Proposition 4.7, and is totally disconnected, by Proposition 4.5. Hence f can be continued analytically to $A \cap \Omega_z$. For $z, w \in A$ the continuations to the totally disconnected sets $A \cap \Omega_z$ and $A \cap \Omega_w$ must agree on their intersection. Hence f can be analytically continued to all of A, and A is weakly removable for X. □

5. A characterization of removability for $\mathscr{A}_\mu^\infty$

Proposition 5.1. *If A is weakly removable for B_μ^p, then $A \subset \operatorname{supp} \mu|_{\mathbb{C} \setminus A} \subset \overline{\operatorname{supp} \mu \setminus A}$.*

P r o o f. Assume that $z_0 \in A \setminus \operatorname{supp} \mu|_{\mathbb{C} \setminus A}$. Since the support is closed it follows that $f(z) := (z - z_0)^{-1} \in B_\mu^p(\Omega \setminus A)$, but clearly $f \notin \operatorname{Hol}(\Omega)$, and hence A is not weakly removable for B_μ^p. The latter inclusion is easy. □

Theorem 5.2. *The following are equivalent:*
(i) *A is weakly removable for $\mathscr{A}_\mu^\infty$;*
(ii) *A is strongly removable for $\mathscr{A}_\mu^\infty(\Omega \setminus A)$;*
(iii) *A is removable for H^∞ and there is no path $\gamma\colon [0, \infty) \to \mathbb{S} \setminus \operatorname{supp} \mu$ such that $\gamma(\infty) \subset A \setminus \gamma([0, \infty))$.*

Remarks. Here $\gamma(\infty) := \bigcap_{t>0} \overline{\gamma((t,\infty))}$ which is always a compact set. The condition in (iii) can be stated in many equivalent forms, see Theorem 2.1 in Arcozzi-Björn [3]. We will use Theorem 2.1 in [3] in the proof below, the main ingredient needed here is however Arakelyan's theorem.

This result is true also for $B^\infty_\mu = \mathscr{A}^\infty_\mu$, $B^\infty_{\mu,\mathrm{fin}}$ and $B^\infty_{\mu,\mathrm{bdd}}$, which follows directly from their definitions and this proposition.

Proof. (i) $\Rightarrow$ (ii) Let $f \in \mathscr{A}^\infty_\mu(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$, and assume that $\|f\|_{L^\infty_\mu(\Omega\setminus A)} = C$. By continuity, $|f(z)| \leqslant C$ for all $z \in \Omega \cap \operatorname{supp} \mu|_{\mathbb{C}\setminus A}$. By Proposition 5.1, $|f(z)| \leqslant C$ for $z \in A$. Hence $\|f\|_{L^\infty_\mu(\Omega)} = C$.

(ii) $\Rightarrow$ (i) This is obvious.

(i) $\Rightarrow$ (iii) We have $H^\infty(\Omega \setminus A) \subset \mathscr{A}^\infty_\mu(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$, which shows that A is weakly removable for H^∞. It is well known that A is then also strongly removable for H^∞ (this also follows from the already proved implication (i) $\Rightarrow$ (ii)).

Assume next that there is such a path γ and let $K = \gamma(\infty)$. Then condition (T1) in Theorem 2.1 in Arcozzi-Björn [3] is false with $E = \operatorname{supp} \mu \setminus K$ and $\Omega = \mathbb{S} \setminus K$. (The assumption therein that $\Omega \subset \mathbb{C}$ can be taken care of by applying a Möbius transformation mapping ∞ to a point in K.) This shows that also condition (A6) in Theorem 2.1 in [3] is false, i.e. that there exists an unbounded holomorphic function f in Ω which is bounded on E. But then $f \in \mathscr{A}^\infty_\mu(\mathbb{S} \setminus A)$ and clearly $f \notin \mathrm{Hol}(\mathbb{S})$, which shows that A is not weakly removable for $\mathscr{A}^\infty_\mu$, a contradiction. Hence there is no such path.

(iii) $\Rightarrow$ (i) Let $f \in \mathscr{A}^\infty_\mu(\mathbb{S} \setminus A)$, then, by definition, there is a compact set $K \subset A$, such that $f \in \mathscr{A}^\infty_\mu(\mathbb{S} \setminus K)$. Thus there is a constant C such that $|f(z)| < C$ for $z \in E := \operatorname{supp} \mu \setminus K$. By assumption, there is no path $\gamma\colon\ [0,\infty) \to \mathbb{S} \setminus (\operatorname{supp} \mu \cup K)$ such that $\gamma(\infty) \subset K$. By Theorem 2.1 in [3], f is bounded in $\mathbb{S} \setminus K$, i.e. $f \in H^\infty(\mathbb{S} \setminus A) \subset \mathrm{Hol}(\mathbb{S})$. Since f was arbitrary, A is weakly removable for $\mathscr{A}^\infty_\mu$. $\square$

6. Removability for different exponents

Proposition 6.1. *Let $0 < p \leqslant q \leqslant \infty$. If A is weakly removable for B^p_μ, then A is weakly removable for B^q_μ.*

Proof. This follows from Proposition 4.9 since $B^q_\mu(\Omega) \subset B^p_\mu(\Omega)$ for bounded domains. $\square$

Remark. The inclusions $B^q_{\mu,\mathrm{fin}}(\Omega) \subset B^p_{\mu,\mathrm{fin}}(\Omega)$ and $B^q_{\mu,\mathrm{bdd}}(\Omega) \subset B^p_{\mu,\mathrm{bdd}}(\Omega)$ are true for all domains Ω. On the other hand, the inclusion $B^q_\mu(\Omega) \subset B^p_\mu(\Omega)$ is not always true. Consider, e.g., $\Omega = \{r\mathrm{e}^{\mathrm{i}\theta}\colon\ r > 1, |\theta| < r^{-3/2}\}$ and $f(z) = z^{-1/2}$

(the principal branch). Then Ω satisfies condition (2.1) for $p = 1$ and $p = 2$, and $f \in \mathscr{A}^2(\Omega) = B^2(\Omega)$, but $f \notin \mathscr{A}^1(\Omega) = B^1(\Omega)$.

Corollary 6.2. *If A is weakly removable for B^p_μ, then A is removable for H^∞, and, in particular,* $\dim_H A \leqslant 1$.

Remarks. Recall that weak and strong removability are the same for H^∞, e.g. by Theorem 5.2.

As we saw in Remarks 4.2 this result is not true in general for $\mathscr{A}^p_\mu$.

P r o o f. This follows from Theorem 5.2 and Proposition 6.1. □

Proposition 6.3. *Let $0 < p \leqslant q \leqslant \infty$ and assume that $q/p \in \mathbb{N}$ or that $q = \infty$. Then the implication*

$$A \text{ is strongly removable for } B^p_{\mu,\mathrm{fin}}(\Omega \setminus A) \implies A \text{ is strongly removable for } B^q_{\mu,\mathrm{fin}}(\Omega \setminus A)$$

is true. The same is true if $B^p_{\mu,\mathrm{fin}}$ (and $B^q_{\mu,\mathrm{fin}}$) are replaced by $B^p_{\mu,\mathrm{bdd}}$ (and $B^q_{\mu,\mathrm{bdd}}$).

The corresponding result for B^p_μ is false, see Example 14.6. The implication is also false if q/p is a non-integer, see Example 14.7.

P r o o f. If $q = \infty$, the result follows directly from the corresponding result for weak removability, since weak and strong removability are the same for $B^\infty_{\mu,\mathrm{fin}}$.

Consider next the case when $N = q/p$ is an integer. Let $E \subset A$ be such that $\Omega \setminus E$ is a domain. Then E is strongly removable for $B^p_{\mu,\mathrm{fin}}(\Omega \setminus E)$, by Proposition 3.2 or 4.12. Hence E is weakly removable for $B^p_{\mu,\mathrm{fin}}(\Omega \setminus E)$, and thus weakly removable for $B^q_{\mu,\mathrm{fin}}(\Omega \setminus E)$, by Proposition 6.1. Let $f \in B^q_{\mu,\mathrm{fin}}(\Omega \setminus E) \subset \mathrm{Hol}(\Omega)$ and let $g = f^N \in \mathrm{Hol}(\Omega)$. It is straightforward to see that $g \in B^p_{\mu,\mathrm{fin}}(\Omega \setminus E) = B^p_{\mu,\mathrm{fin}}(\Omega)$. But, then it follows that $f \in B^q_{\mu,\mathrm{fin}}(\Omega)$. We have shown that E is strongly removable for $B^q_{\mu,\mathrm{fin}}(\Omega \setminus E)$. Since $E \subset A$ was arbitrary it follows from Proposition 4.12 that A is strongly removable for $B^q_{\mu,\mathrm{fin}}(\Omega \setminus A)$.

The proof is similar for $B^p_{\mu,\mathrm{bdd}}$. □

Lemma 7.1. *Let $K \subset \mathbb{C}$ be compact with $\mathbb{S} \setminus K$ connected. Then K is removable for B^p_μ if and only if there is no function $f \in B^p_\mu(\mathbb{S} \setminus K)$ with $f(\infty) = 0$ and $f'(\infty) \neq 0$.*

Remark. As usual $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

Proof. Assume first that K is removable for B^p_μ, then $f \in B^p_\mu(\mathbb{S} \setminus K) \subset \mathrm{Hol}(\mathbb{S}) = \{f\colon\ f \text{ is constant}\}$, so $f'(\infty) = 0$. This proves the sufficiency.

Assume, conversely, that K is not removable for B^p_μ, i.e. $B^p_\mu(\mathbb{S} \setminus K) \not\subset \mathrm{Hol}(\mathbb{S})$ and there is a non-constant $h \in B^p_\mu(\mathbb{S} \setminus K)$. Let $f(z) = h(z) - h(\infty)$, so that $f(\infty) = 0$. Since $|f| \leqslant |h|$ in some neighbourhood of ∞ and the complement of the neighbourhood has finite μ measure, we have $f \in B^p_\mu(\mathbb{S} \setminus K)$. Expand f in a Laurent series,

$$f(z) = \sum_{k=1}^{\infty} c_k z^{-k} \quad \text{for } |z| \text{ large.}$$

As f is non-constant there exists $k \geqslant 1$ with $c_k \neq 0$. Let k_0 be the least such k. Then $g(z) = z^{k_0-1} f(z)$, $z \in \mathbb{S} \setminus K$, is a well-defined analytic function with $g(\infty) = 0$ and $g'(\infty) = c_{k_0} \neq 0$.

It follows that there exists C such that $|g(z)| \leqslant C|z|^{-1}$ for all z with $|z| \geqslant C$. For $|z| \leqslant C$ we have $|g(z)| \leqslant C^{k_0-1}|f(z)|$. Let now $\Omega' \subset \mathbb{C} \setminus K$ be an arbitrary domain satisfying condition (2.1). Then

$$\|g\|^p_{L^p_\mu(\Omega')} \leqslant C^p \int_{\Omega' \setminus D(0,C)} \frac{\mathrm{d}\mu(z)}{|z|^p} + C^{p(k_0-1)} \int_{\Omega' \cap D(0,C)} |f|^p \, \mathrm{d}\mu < \infty.$$

Hence $g \in B^p_\mu(\mathbb{S} \setminus K)$. □

This leads us to making the following definition.

Definition 7.2. Let $K \subset \Omega \cap \mathbb{C}$ be compact. Let $\hat{K}$ be the complement of the component of $\mathbb{S} \setminus K$ containing ∞, i.e. $\hat{K}$ is K with all holes filled in. Let also γ be a smooth cycle in Ω with winding number $\mathrm{wind}_\gamma(z) = 1$ if $z \in K$ and $\mathrm{wind}_\gamma(z) = 0$ if $z \notin \Omega$. We then define

$$\mathrm{cap}_{\mathscr{A}^p_\mu}(K, \Omega) = \sup\left\{\frac{1}{2\pi}\left|\int_\gamma f(z)\,\mathrm{d}z\right| \colon\ f \in \mathrm{Hol}(\Omega \setminus K) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus K)} \leqslant 1\right\},$$
$$\mathrm{cap}_{B^p_\mu}(K, \Omega) = \sup\{|f'(\infty)|\colon\ f \in \mathrm{Hol}(\mathbb{S} \setminus \hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1\},$$
$$\mathrm{cap}^0_{B^p_\mu}(K, \Omega) = \sup\{|f'(\infty)|\colon\ f \in \mathrm{Hol}(\mathbb{S} \setminus \hat{K}),\ \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1 \text{ and } f(\infty) = 0\}.$$

Remarks 7.3. We do not require $\Omega \setminus K$ to be connected when we say that $f \in \operatorname{Hol}(\Omega \setminus K)$ in the definition of $\operatorname{cap}_{\mathscr{A}^p_\mu}$.

For B^p_μ the functional $f \mapsto |f'(\infty)|$ is not always bounded, hence it can happen that $\operatorname{cap}^0_{B^p_\mu}(K,\Omega) = \operatorname{cap}_{B^p_\mu}(K,\Omega) = \infty$, e.g., if $\mu = 0$ and $K \neq \emptyset$.

It is clear, by Cauchy's theorem, that $\operatorname{cap}^0_{B^p_\mu}(K,\Omega) \leqslant \operatorname{cap}_{B^p_\mu}(K,\Omega) \leqslant \operatorname{cap}_{\mathscr{A}^p_\mu}(K,\Omega)$.

If Ω is simply connected, then the integral over those parts of γ that are in the holes of K must be zero. It follows that the best choice is to let $f \equiv 0$ in all of its holes, and it is enough to let γ be a simple curve surrounding K. This is the way the (unweighted) capacity $\operatorname{cap}_{\mathscr{A}^p}$ was defined in Adams-Hedberg [1], before Proposition 11.1.10. Moreover, $\operatorname{cap}_{\mathscr{A}^p_\mu}(K,\Omega) = \operatorname{cap}_{\mathscr{A}^p_\mu}(\hat{K},\Omega)$ in this case.

We next extend the definition of the capacities to arbitrary sets.

Definition 7.4. Let cap be $\operatorname{cap}_{\mathscr{A}^p_\mu}$, $\operatorname{cap}_{B^p_\mu}$ or $\operatorname{cap}^0_{B^p_\mu}$. We then define

$$\operatorname{cap}(A,\Omega) = \sup\{\operatorname{cap}(K,\Omega) \colon K \subset A \text{ is compact}\}.$$

Remark. It follows from Proposition 7.5 that Definition 7.4 is consistent with Definition 7.2.

Proposition 7.5. *Let $\Omega \subset \Omega'$ be domains, $A \subset B \subset \Omega \cap \mathbb{C}$ be compact sets and* cap *be one of* $\operatorname{cap}_{\mathscr{A}^p_\mu}$, $\operatorname{cap}_{B^p_\mu}$ *and* $\operatorname{cap}^0_{B^p_\mu}$. *Then* $\operatorname{cap}(A,\Omega') \leqslant \operatorname{cap}(B,\Omega)$.

Proof. This follows from the fact that the L^p_μ norm increases with the domain. □

We next make a definition which abuses the notation a little.

Definition 7.6. We say that $\operatorname{cap}_{\mathscr{A}^p_\mu}(A) = 0$ if $\operatorname{cap}_{\mathscr{A}^p_\mu}(A \cap \Omega, \Omega) = 0$ for all domains Ω. If this is not true we write $\operatorname{cap}_{\mathscr{A}^p_\mu}(A) = 1$.

The main reasons for defining these capacities are of course the next two theorems.

Theorem 7.7. *If Ω satisfies condition* (2.1), *then the following are equivalent:*

(i) *A is weakly removable for B^p_μ;*
(ii) *A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$;*
(iii) $\operatorname{cap}_{\mathscr{A}^p_\mu}(A,\Omega) = 0$;
(iv) $\operatorname{cap}_{B^p_\mu}(A,\Omega) = 0$;
(v) $\operatorname{cap}^0_{B^p_\mu}(A,\Omega) = 0$;
(vi) $\operatorname{cap}_{\mathscr{A}^p_\mu}(A) = 0$.

Remark. Note that since (i) and (vi) are independent of the particular choice of Ω, also (ii)–(v) are independent of the choice of Ω.

P r o o f. (i) $\Leftrightarrow$ (ii) This follows directly from the fact that $\mathscr{A}^p_\mu(\Omega \setminus A) = B^p_\mu(\Omega \setminus A)$.

(i) $\Rightarrow$ (vi) Let Ω' be an arbitrary domain. Let $K \subset A \cap \Omega'$ be compact. Let Ω'' be a bounded domain with $K \subset \Omega'' \subset \Omega'$. Then K is removable for $\mathscr{A}^p_\mu(\Omega'' \setminus K) = B^p_\mu(\Omega'' \setminus K)$, by Proposition 3.2. Let $f \in \mathscr{A}^p_\mu(\Omega'' \setminus K) = \mathscr{A}^p_\mu(\Omega'')$. Cauchy's theorem shows that $\int_\gamma f(z)\,\mathrm{d}z = 0$, where γ is as in Definition 7.2, hence $\operatorname{cap}_{\mathscr{A}^p_\mu}(K, \Omega'') = 0$. By Proposition 7.5, $\operatorname{cap}_{\mathscr{A}^p_\mu}(K, \Omega') \leqslant \operatorname{cap}_{\mathscr{A}^p_\mu}(K, \Omega'') = 0$. Therefore it follows that $\operatorname{cap}_{\mathscr{A}^p_\mu}(A \cap \Omega', \Omega') = 0$.

That (vi) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follow directly from Definition 7.6 and Remarks 7.3.

$\neg$(i) $\Rightarrow$ $\neg$(v) There is a compact set $K \subset A$ not removable for B^p_μ. Thus $\hat{K}$ is not removable for B^p_μ either, where $\hat{K}$ is K with all holes filled in. By Lemma 7.1 there exists a function $f \in B^p_\mu(\mathbb{S} \setminus \hat{K})$ with $f(\infty) = 0$ and $f'(\infty) \neq 0$. Since $B^p_\mu(\mathbb{S}\setminus\hat{K}) \subset \mathscr{A}^p_\mu(\Omega\setminus\hat{K})$ we know that $\|f\|_{L^p_\mu(\Omega\setminus\hat{K})} < \infty$. It follows that $\operatorname{cap}^0_{B^p_\mu}(A, \Omega) \geqslant \operatorname{cap}^0_{B^p_\mu}(K, \Omega) > 0$. □

Theorem 7.8. *The following are equivalent:*

(i) *A is weakly removable for B^p_μ;*

(ii) $\operatorname{cap}_{\mathscr{A}^p_\mu}(A) = 0$*;*

(iii) $\operatorname{cap}_{\mathscr{A}^p_\mu}(A \cap \Omega, \Omega) = 0$ *for all domains Ω;*

(iv) *for each $z \in A$, there exists a bounded domain Ω_z with* $\operatorname{cap}_{\mathscr{A}^p_\mu}(A \cap \Omega_z, \Omega_z) = 0$ *such that $z \in \Omega_z$.*

Remarks. It follows that $\operatorname{cap}_{\mathscr{A}^p_\mu}(\,\cdot\,)$ satisfies Axioms A5 and A6 for $X = B^p_\mu$, $X = B^p_{\mu,\mathrm{fin}}$ and $X = B^p_{\mu,\mathrm{bdd}}$, and hence characterizes their weakly removable singularities.

Since the null sets are the same for $\operatorname{cap}_{\mathscr{A}^p_\mu}$, $\operatorname{cap}_{B^p_\mu}$ and $\operatorname{cap}^0_{B^p_\mu}$ we can replace $\operatorname{cap}_{\mathscr{A}^p_\mu}$ by $\operatorname{cap}_{B^p_\mu}$ or $\operatorname{cap}^0_{B^p_\mu}$ in (iii) and (iv).

P r o o f. (ii) $\Leftrightarrow$ (iii) This is Definition 7.6.

(i) $\Rightarrow$ (iii) Let Ω be an arbitrary domain and $K \subset A \cap \Omega$ be compact. Then K is weakly removable for B^p_μ. Since K is contained in a bounded domain, Theorem 7.7 shows that $\operatorname{cap}_{\mathscr{A}^p_\mu}(K) = 0$, and hence by Definition 7.6, $\operatorname{cap}_{\mathscr{A}^p_\mu}(K, \Omega) = 0$. Since K was arbitrary $\operatorname{cap}_{\mathscr{A}^p_\mu}(A \cap \Omega, \Omega) = 0$.

(iii) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (i) Let $f \in B^p_\mu(\mathbb{S} \setminus A) \subset B^p_\mu(\Omega_z \setminus A)$. Since $\operatorname{cap}_{\mathscr{A}^p_\mu}(A \cap \Omega_z, \Omega_z) = 0$, Theorem 7.7 shows that $A \cap \Omega_z$ is weakly removable for B^p_μ. Thus, f can be continued analytically to $A \cap \Omega_z$. For $z, w \in A$ the continuations to the totally disconnected sets $A \cap \Omega_z$ and $A \cap \Omega_w$ must agree on their intersection. Hence f can be analytically continued to all of A, and A is weakly removable for B^p_μ. □

We end this section with a few results about these capacities that will not be used in the sequel.

Proposition 7.9. *Assume that* Ω *satisfies condition* (2.1). *Let* cap *be* $\operatorname{cap}_{\mathscr{A}^p_\mu}$, $\operatorname{cap}_{B^p_\mu}$ *or* $\operatorname{cap}^0_{B^p_\mu}$. *Assume that* $\operatorname{cap}(A,\Omega) = \mu(A) = 0$ *(if* $p = \infty$ *it is enough to require that* $\operatorname{cap}(A,\Omega) = 0$*). Then* $\operatorname{cap}(E \cup A, \Omega) = \operatorname{cap}(E,\Omega)$.

Remarks. Note that by Theorem 7.7 the assumption $\operatorname{cap}(A,\Omega) = 0$ is the same for all three capacities.

Note also that it follows from Proposition 9.7 in Björn [8] that we cannot allow E to be an arbitrary set, not even for $p = \infty$.

P r o o f. Let $K \subset E \cup A$ be compact, and let $K' = K \cap E$ which is compact since E is relatively closed in Ω. Let Ω' be any component of $\Omega \setminus K'$. Since $K \cap \Omega' \subset A$, $K \cap \Omega'$ is weakly removable from $\mathscr{A}^p_\mu(\Omega' \setminus K) = B^p_\mu(\Omega' \setminus K)$, by Theorem 7.7. Since $\mu(K \cap \Omega') \leqslant \mu(A) = 0$, $\|f\|_{L^p_\mu(\Omega' \setminus K)} = \|f\|_{L^p_\mu(\Omega')}$ for $f \in \mathscr{A}^p_\mu(\Omega' \setminus K) = \mathscr{A}^p_\mu(\Omega')$. Hence the same functions compete in the suprema defining $\operatorname{cap}(K,\Omega)$ and $\operatorname{cap}(K',\Omega)$ and $\operatorname{cap}(K,\Omega) = \operatorname{cap}(K',\Omega) \leqslant \operatorname{cap}(E,\Omega)$. Taking supremum over all compact $K \subset E \cup A$ we find that $\operatorname{cap}(E \cup A,\Omega) \leqslant \operatorname{cap}(E,\Omega)$. The converse inequality is obvious. □

Proposition 7.10. *Let* $0 < p \leqslant q \leqslant \infty$. *If* $p < \infty$, *then assume also that* $\mu(\Omega) < \infty$. *Let* cap_p *be one of* $\operatorname{cap}_{\mathscr{A}^p_\mu}$, $\operatorname{cap}_{B^p_\mu}$ *and* $\operatorname{cap}^0_{B^p_\mu}$. *Then*

$$\operatorname{cap}_p(A,\Omega) \geqslant C \operatorname{cap}_q(A,\Omega),$$

where $C = \mu(\Omega)^{(p-q)/pq}$ *if* $p \leqslant q < \infty$, $C = \mu(\Omega)^{-1/p}$ *if* $p < q = \infty$, *and* $C = 1$ *if* $p = q = \infty$, *assuming that* $\mu(\Omega) > 0$. *If* $\mu(\Omega) = 0$, *both sides equal* ∞, *or* 0 *if* $A = \emptyset$, *and we may choose* $C = \infty$.

Remark. In the corresponding result for H^p, Proposition 5.5(ii) in Björn [4], the constant $C = 1$.

P r o o f. Let $K \subset A$ be compact. By Hölder's inequality we have

$$\|f\|_{L^p_\mu(\Omega \setminus \hat{K})} \leqslant \mu(\Omega \setminus \hat{K})^{(q-p)/qp} \|f\|_{L^q_\mu(\Omega \setminus \hat{K})} \leqslant \frac{1}{C} \|f\|_{L^q_\mu(\Omega \setminus \hat{K})},$$

where $\hat{K}$ is K with all holes filled in. This is enough to obtain the result. □

Proposition 7.11. *Let $K \subset \Omega \cap \mathbb{C}$ be compact, and let $\hat{K}$ be the complement of the component of $\mathbb{S} \setminus K$ containing ∞. Then*

$$\begin{aligned}\operatorname{cap}_{B^p_\mu}(K,\Omega) &= \sup\{|f'(\infty)|\colon\ f \in B^p_{\mu,\mathrm{bdd}}(\mathbb{S}\setminus\hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1\}\\ &= \sup\{|f'(\infty)|\colon\ f \in B^p_{\mu,\mathrm{fin}}(\mathbb{S}\setminus\hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1\}\end{aligned}$$

and

$$\begin{aligned}cap^0_{B^p_\mu}(K,\Omega) &= \sup\{|f'(\infty)|\colon\ f \in B^p_{\mu,\mathrm{bdd}}(\mathbb{S}\setminus\hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1, f(\infty)=0\}\\ &= \sup\{|f'(\infty)|\colon\ f \in B^p_{\mu,\mathrm{fin}}(\mathbb{S}\setminus\hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1, f(\infty)=0\}\\ &= \sup\{|f'(\infty)|\colon\ f \in B^p_{\mu}(\mathbb{S}\setminus\hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1, f(\infty)=0\}.\end{aligned}$$

Moreover,

$$\sup\{|f'(\infty)|\colon\ f \in B^p_\mu(\mathbb{S}\setminus\hat{K}) \text{ and } \|f\|_{L^p_\mu(\Omega\setminus\hat{K})} \leqslant 1\} = \begin{cases}\operatorname{cap}_{B^p_\mu}(K,\Omega), & \text{if } 1 \in B^p_\mu(\mathbb{S}),\\ \operatorname{cap}^0_{B^p_\mu}(K,\Omega), & \text{if } 1 \notin B^p_\mu(\mathbb{S}).\end{cases}$$

Remarks 7.12. In view of this proposition it would be more appropriate to call $\operatorname{cap}_{B^p_\mu}$, either $\operatorname{cap}_{B^p_{\mu,\mathrm{bdd}}}$ or $\operatorname{cap}_{B^p_{\mu,\mathrm{fin}}}$. We have refrained from this in order not to make the notation too cumbersome.

If $\mu(\Omega) = \infty$ and $p < \infty$, then usually $1 \notin B^p_\mu(\Omega)$, however, this is not always true. Consider, e.g., $\mu = \sum_{j=1}^{\infty} j\delta_j$. Then $1 \in B^1_\mu(\mathbb{S})$, since for any domain Ω' satisfying condition (2.1) we have $\operatorname{card}(\Omega' \cap \mathbb{N}) < \infty$.

Proof. Let first f be a function competing in the supremum defining $\operatorname{cap}_{B^p_\mu}(K,\Omega)$. Since $f \in \operatorname{Hol}(\mathbb{S}\setminus\hat{K})$, $|f|$ is bounded by a constant C in $\mathbb{S}\setminus\Omega$. Let $\Omega' \subset \mathbb{S}\setminus\hat{K}$ be a domain with $\mu(\Omega') < \infty$. Then

$$\|f\|^p_{L^p_\mu(\Omega')} \leqslant \|f\|^p_{L^p_\mu(\Omega\setminus\hat{K})} + \|f\|^p_{L^p_\mu(\Omega'\setminus\Omega)} \leqslant 1 + C^p\mu(\Omega') < \infty.$$

Thus the same functions compete in the different suprema in the first identity.

The proof of the second part is similar: Let f be a function competing in the supremum defining $\operatorname{cap}^0_{B^p_\mu}(K,\Omega)$. Since $f \in \operatorname{Hol}(\mathbb{S}\setminus\hat{K})$ and $f(\infty) = 0$ there is a constant $C > 1$ such that $|f(z)| \leqslant C|z|^{-1}$ for all z with $|z| \geqslant C$ and $|f(z)| \leqslant C$ for all $z \in \mathbb{S}\setminus\Omega$. Let $\Omega' \subset \mathbb{S}\setminus\hat{K}$ be a domain satisfying condition (2.1). Then

$$\begin{aligned}\|f\|^p_{L^p_\mu(\Omega')} &\leqslant \|f\|^p_{L^p_\mu(\Omega\setminus\hat{K})} + \|f\|^p_{L^p_\mu(\Omega'\setminus D(0,C))} + \|f\|^p_{L^p_\mu((\Omega'\setminus\Omega)\cap D(0,C))}\\ &\leqslant 1 + C^p \int_{\Omega'\setminus D(0,C)} \frac{\mathrm{d}\mu(z)}{|z|^p} + C^p\mu(D(0,C)) < \infty.\end{aligned}$$

Thus the same functions compete in the different suprema in the second identity.

If $1 \in B^p_\mu(\mathbb{S})$, the last part follows directly from the first part, since if $\Omega' \subset \mathbb{S} \setminus \hat{K}$ is a domain satisfying condition (2.1), then $\mu(\Omega') = \|1\|^p_{L^p_\mu(\Omega')} < \infty$. On the other hand, if $1 \notin B^p_\mu(\mathbb{S})$, then there is a domain $\Omega' \subset \mathbb{S} \setminus \hat{K}$ satisfying condition (2.1) with $\mu(\Omega') = \infty$. Let $f \in \operatorname{Hol}(\mathbb{S} \setminus \hat{K})$ with $f(\infty) \neq 0$, then $|f(z)| \geqslant \frac{1}{2}|f(\infty)|$ for $|z| \geqslant C$ for some constant C. Since $\mu(\Omega' \setminus D(0, C)) = \infty$ we find that $f \notin \mathscr{A}^p_\mu(\Omega') \supset B^p_\mu(\mathbb{S} \setminus \hat{K})$. Thus $f(\infty) = 0$ is no extra requirement in the left-hand side of the last part if $1 \notin B^p_\mu(\mathbb{S})$. $\square$

8. When weak and strong removability are the same

Proposition 8.1. *Assume that* $\mu = \nu + \sum_{j=1}^{m} c_j \delta_{z_j}$, *and that* $\nu(G) = 0$ *for all sets* $G \subset \mathbb{C}$ *with* $\dim_H G \leqslant 1$. *If* A *is weakly removable for* B^p_μ, *then* A *is strongly removable for* $B^p_\mu(\Omega \setminus A)$.

Remarks. The conclusion is that the two concepts, weak and strong removability, coincide for all sets and domains for B^p_μ. We will say that *weak and strong removability are the same for all sets.*

In particular, weak and strong removability are the same for all sets for B^p_w. Recall also that for $p = \infty$ weak and strong removability are always the same, by Theorem 5.2.

Proof. Let $f \in B^p_\mu(\Omega \setminus A) \subset \operatorname{Hol}(\Omega)$. By Corollary 6.2 we know that $\dim_H A \leqslant 1$, and hence $\nu(A) = 0$. Since μ is a Radon measure, we have $0 \leqslant c_j < \infty$, $1 \leqslant j \leqslant m$. Let $\Omega' \subset \Omega$ be any domain satisfying condition (2.1) and $J = \{j \in \mathbb{N} \colon 1 \leqslant j \leqslant m \text{ and } z_j \in \Omega' \cap A\}$, then

$$\|f\|^p_{L^p_\mu(\Omega')} = \|f\|^p_{L^p_\mu(\Omega' \setminus A)} + \int_{\Omega' \cap A} |f|^p \, \mathrm{d}\nu + \sum_{j \in J} c_j |f(z_j)|^p < \infty,$$

and hence $f \in B^p_\mu(\Omega)$. Since f was arbitrary, A is strongly removable for $B^p_\mu(\Omega \setminus A)$. $\square$

The following results were proved in Björn [9] under axiomatic assumptions.

Theorem 8.2. *Let* $E_j \subset \mathbb{C}$ *be removable for* B^p_μ *and assume that there exists a domain* $\Omega_j \supset E_j$ *with* $\Omega_j \setminus E_j$ *also being a domain,* $j = 1, 2, \ldots$. *Assume also that weak and strong removability for* B^p_μ *are the same for all subsets of* $\bigcup_{j=1}^{\infty} E_j$ *(which, in particular holds if* $\mu(E_j) = 0$ *for* $j = 1, 2, \ldots$*). Then* $\operatorname{cap}_{\mathscr{A}^p_\mu}\left(\bigcup_{j=1}^{\infty} E_j\right) = 0$.

Remark. This result is not true if we omit the assumption that $\Omega_j \setminus E_j$ be domains, which can be shown using the existence of non-measurable sets, see Proposition 9.7 in Björn [8].

Proposition 8.3. *Assume that weak and strong removability are the same for all sets and that all singleton sets are removable for B^p_μ. Assume also that $A \subset \Omega$ is not removable for B^p_μ, then* $\dim B^p_\mu(\Omega \setminus A)/B^p_\mu(\Omega) = \infty$.

Remark. The results in this section hold equally well for $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$.

9. Characterizations of removability for $\mathscr{A}^p_\mu$

Proposition 9.1. *If A is weakly removable for B^p_μ, then A is also weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.*

Let next $\nu(G) = \int_{G\setminus\mathbb{D}} |z|^{-p}\,\mathrm{d}\mu(z)$ for Borel sets $G \subset \mathbb{C}$, and extend ν to an outer measure. If $\nu(A) < \infty$ and A is strongly removable for $B^p_\mu(\Omega \setminus A)$, then A is strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.

Remark. Recall that this is trivial for $p = \infty$, since $\mathscr{A}^\infty_\mu(\Omega) = B^\infty_\mu(\Omega)$ for all domains Ω.

P r o o f. If A is weakly removable for B^p_μ, then $\mathscr{A}^p_\mu(\Omega \setminus A) \subset B^p_\mu(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$, and A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.

Since $\nu(A) < \infty$ and ν is an outer measure, there is an open set B with $A \subset B \subset \Omega$ and $\nu(B) < \infty$. The set B has countably (possibly finitely) many components B_1, $B_2, \ldots$. We can connect B_j and B_{j+1} with a bounded open connected set $B'_j \subset \Omega$, thus having finite ν measure. We can further split B'_j into enough pairwise disjoint pieces, each still connecting B_j and B_{j+1}, so that at least one piece has ν measure less than 2^{-j}, we forget about the rest of B'_j and assume B'_j to be this piece. Let $\Omega' = B \cup \bigcup_{j=1}^{\infty} B'_j$, a domain with $A \subset \Omega' \subset \Omega$ and $\nu(\Omega') < \infty$.

Let now $f \in \mathscr{A}^p_\mu(\Omega \setminus A) \subset \mathrm{Hol}(\Omega)$, by the first part. We also have $f \in \mathscr{A}^p_\mu(\Omega' \setminus A) = B^p_\mu(\Omega' \setminus A) = B^p_\mu(\Omega') = \mathscr{A}^p_\mu(\Omega')$, so

$$\|f\|^p_{L^p_\mu(\Omega)} \leqslant \|f\|^p_{L^p_\mu(\Omega\setminus A)} + \|f\|^p_{L^p_\mu(\Omega')} < \infty.$$

Since f was arbitrary, it follows that A is strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$. □

Remark. The condition $\nu(A) < \infty$ in the second part has to be changed to $\mu(A) < \infty$ for $B^p_{\mu,\mathrm{fin}}$, and to A being bounded for $B^p_{\mu,\mathrm{bdd}}$. The proof for $B^p_{\mu,\mathrm{bdd}}$ is simpler, but the proposition also becomes less powerful. See Example 14.4 for the necessity of these changes, and the necessity of the condition $\nu(A) < \infty$ in the second part of the proposition.

Theorem 9.2. *The set A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if and only if A is weakly removable for B^p_μ or $\mathscr{A}^p_\mu(\Omega \setminus A) = \{0\}$.*

Proof. The sufficiency is clear: if A is weakly removable for B^p_μ, then Proposition 9.1 shows that A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$, furthermore if $\mathscr{A}^p_\mu(\Omega \setminus A) = \{0\}$ then A is trivially removable.

We next turn to the necessity, we will actually prove the contrapositive statement. Assume that A is not weakly removable for B^p_μ and that $\mathscr{A}^p_\mu(\Omega \setminus A) \neq \{0\}$.

By Proposition 4.7 there is a compact set $K \subset A$ not removable for B^p_μ. Let $g \in B^p_\mu(\mathbb{S} \setminus K)$ be non-constant and let $z_0 \in K$ be a point where g has a (non-removable) singularity, not necessarily isolated.

Let $f \in \mathscr{A}^p_\mu(\Omega \setminus A)$, $f \not\equiv 0$. If $f \notin \operatorname{Hol}(\Omega)$, then A is not weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$, and we are done. We therefore assume that $f \in \operatorname{Hol}(\Omega)$. Since $f \not\equiv 0$, there exists $k \geqslant 0$ minimal with $f^{(k)}(z_0) \neq 0$. Let $\tilde{f}(z) = f(z)(z - z_0)^{-k}$. Then $\tilde{f}(z_0) \neq 0$. Moreover, there is $\delta > 0$ such that $\tilde{f}$ is bounded on $D(z_0, \delta)$ and $|\tilde{f}(z)| \leqslant \delta^{-k}|f(z)|$ on $\Omega \setminus D(z_0, \delta)$. Since $f \in \mathscr{A}^p_\mu(\Omega \setminus A)$, we obtain $\tilde{f} \in \mathscr{A}^p_\mu(\Omega \setminus A)$.

Let now $h = \tilde{f}g$, a function analytic in $\Omega \setminus K$ with a (non-removable) singularity at z_0. We shall show that $h \in \mathscr{A}^p_\mu(\Omega \setminus A)$, from which it directly follows that A is not weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.

Let Ω' be a bounded domain with $K \subset \Omega' \Subset \Omega$. Then there exists a constant C such that $|\tilde{f}(z)| \leqslant C$, $z \in \Omega'$, and $|g(z)| \leqslant C$, $z \in \mathbb{S} \setminus \Omega'$. Hence

$$\begin{aligned} \|h\|^p_{L^p_\mu(\Omega \setminus A)} &\leqslant \int_{\Omega' \cap (\Omega \setminus A)} C^p |g|^p \, \mathrm{d}\mu + \int_{(\Omega \setminus A) \setminus \Omega'} C^p |\tilde{f}|^p \, \mathrm{d}\mu \\ &\leqslant C^p (\|g\|^p_{L^p_\mu(\Omega' \setminus A)} + \|\tilde{f}\|^p_{L^p_\mu(\Omega \setminus A)}) < \infty, \end{aligned}$$

i.e. $h \in \mathscr{A}^p_\mu(\Omega \setminus A)$. □

Definition 9.3. For $z \neq \infty$, let $n_{z,\mu} = \infty$ if there is a function in $\mathscr{A}^p_\mu(D(z,1) \setminus \{z\})$ with an essential singularity at z, otherwise let $n_{z,\mu} = \sup\{n \in \mathbb{N}\colon\ (\zeta - z)^{-n} \in \mathscr{A}^p_\mu(D(z,1) \setminus \{z\})\}$.

Let $n_{\infty,\mu} = 0$ if $\infty \notin \Omega$ and there is a function in $\mathscr{A}^p_\mu(\mathbb{C})$ with an essential singularity at ∞, otherwise let $n_{\infty,\mu} = \inf\{n \in \mathbb{N}\colon\ z^{-n} \in \mathscr{A}^p_\mu(\mathbb{S} \setminus \overline{\mathbb{D}})\}$.

If $\mathrm{d}\mu = w\, \mathrm{d}m$, we write $n_{z,w} = n_{z,\mu}$, and if $w \equiv 1$, we write $n_z = n_{z,w}$.

Remarks 9.4. (i) Note that $n_{z,\mu}$ depends on p and whether or not $\infty \in \Omega$.

(ii) If $n_{\infty,\mu} = \inf\{n \in \mathbb{N}\colon\ z^{-n} \in \mathscr{A}^p_\mu(\mathbb{S} \setminus \overline{\mathbb{D}})\}$, then it is easy to see that $\sum\limits_{k=0}^{\infty} a_k \zeta^{-k} \in \operatorname{Hol}(\mathbb{S} \setminus \overline{\mathbb{D}})$ belongs to $\mathscr{A}^p_\mu(\mathbb{S} \setminus \overline{D(0,2)})$ if and only if $a_0 = a_1 = \ldots = a_{n_{\infty,\mu}-1} = 0$.

(iii) Similarly, if $n_{z,\mu} < \infty$, $z \neq \infty$, then $f \in \operatorname{Hol}(D(z,2) \setminus \{z\})$ belongs to $\mathscr{A}^p_\mu(D(z,1) \setminus \{z\})$ if and only if f has a pole of order at most $n_{z,\mu}$ or a removable singularity at z. In particular, $\{z\}$ is removable for B^p_μ if and only if $n_{z,\mu} = 0$.

(iv) If $\infty \notin \Omega$, then $n_{\infty,\mu} = 0$ if and only if there is a non-zero function in $\mathscr{A}^p_\mu(\mathbb{C})$. Why? If $n_{\infty,\mu} = 0$, then either there is $f \in \mathscr{A}^p_\mu(\mathbb{C})$ with an essential singularity at ∞, or $1 \in \mathscr{A}^p_\mu(\mathbb{C})$. On the other hand, if $f \in \mathscr{A}^p_\mu(\mathbb{C})$, $f \not\equiv 0$, does not have an essential singularity at ∞, then $z^n \in \mathscr{A}^p_\mu(\mathbb{C})$ for some $n \geqslant 0$. It follows directly that $1 \in \mathscr{A}^p_\mu(\mathbb{C})$ and $n_{\infty,\mu} = 0$, moreover this happens exactly when $\mu(\mathbb{C}) < \infty$.

(v) If $\infty \notin \Omega$ and there exists $f \in \mathscr{A}^p_\mu(\Omega' \setminus \{\infty\})$ with an essential singularity at ∞ for some domain $\Omega' \ni \infty$, and $N = \inf\{n \in \mathbb{N}\colon\ z^{-n} \in \mathscr{A}^p_\mu(\mathbb{S} \setminus \overline{\mathbb{D}})\} < \infty$, then $n_{\infty,\mu} = 0$. Why? Without loss of generality we can assume that $f \in \mathscr{A}^p_\mu(\mathbb{C} \setminus \overline{\mathbb{D}})$. Then also $f(z)z^{-j} \in \mathscr{A}^p_\mu(\mathbb{C} \setminus \overline{\mathbb{D}})$ for $0 \leqslant j \leqslant N$. By taking a non-trivial linear combination of these functions we find a function $g \in \mathscr{A}^p_\mu(\mathbb{C} \setminus \overline{\mathbb{D}})$, with Laurent series

$$g(z) = \sum_{j=0}^{\infty} a_{-j} z^j + \sum_{k=N}^{\infty} a_k z^{-k}, \quad |z| > 1,$$

i.e. the linear combination is chosen to make $a_1 = \ldots = a_{N-1} = 0$. By the choice of N we directly have $h(z) := \sum_{k=N}^{\infty} a_k z^{-k} \in \mathscr{A}^p_\mu(\mathbb{S} \setminus \overline{\mathbb{D}})$, and hence $g - h \in \mathscr{A}^p_\mu(\mathbb{C})$. If g did not have an essential singularity at ∞, then f would not have an essential singularity at ∞ either, a contradiction. Hence $g - h$ has an essential singularity at ∞.

(vi) Note also that if $n_{z,\mu} = \infty$, $z \neq \infty$, then there is a function in $\mathscr{A}^p_\mu(D(z,1) \setminus \{z\})$ with an essential singularity at z. If not, then $(\zeta - z)^{-n} \in \mathscr{A}^p_\mu(D(z,1) \setminus \{z\})$ for all $n \in \mathbb{N}$. Hence coefficients $a_j \neq 0$ can be found so that $\sum_{j=0}^{\infty} a_j(\zeta - z)^{-n} \in \mathscr{A}^p_\mu(D(z,1) \setminus \{z\})$, a function with an essential singularity at z.

Theorem 9.5. *Assume that $n_{\infty,\mu} < \infty$. Then A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if A is weakly removable for B^p_μ, or $\mathbb{C} \setminus (\Omega \setminus A) = A_1 \cup A_2$, where A_1 is strongly removable for $B^p_\mu(\Omega \setminus A)$, $A_2 = \{z_1, \ldots, z_m\}$ and $\sum_{k=1}^{m} n_{z_k,\mu} < n_{\infty,\mu}$.*

Furthermore, if A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$, then A is weakly removable for B^p_μ, or $\mathbb{C} \setminus (\Omega \setminus A) = A_1 \cup A_2$ for some sets A_1 and A_2 with A_1 weakly removable for B^p_μ, $A_2 = \{z_1, \ldots, z_m\}$ and $\sum_{k=1}^{m} n_{z_k,\mu} < n_{\infty,\mu}$.

Remarks. Note that it is *not* assumed that $A_1 \subset A$. Nor is it assumed that A_1 and A_2 are disjoint, though this can always be achieved by replacing A_2 by $A_2 \setminus A_1$.

Note, also, that if $n_{\infty,\mu} = 0$, then A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if and only if A is weakly removable for B^p_μ.

Remarks 4.2 show that the situation can be quite different when $n_{\infty,\mu} = \infty$.

The proof below works equally well if we replace B^p_μ by $B^p_{\mu,\mathrm{fin}}$ or $B^p_{\mu,\mathrm{bdd}}$.

P r o o f. We start with the first part. If A is weakly removable for B^p_μ, then A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$, by Proposition 9.1. We therefore assume that $\mathbb{C} \setminus (\Omega \setminus A) = A_1 \cup A_2$, where A_1 is strongly removable for $B^p_\mu(\Omega \setminus A)$, $A_2 = \{z_1, \ldots, z_m\}$ and $\sum_{k=1}^m n_{z_k,\mu} < n_{\infty,\mu}$ (note that this is never possible if $n_{\infty,\mu} = 0$). Let $f \in \mathscr{A}^p_\mu(\Omega \setminus A) \subset B^p_\mu(\Omega \setminus A) = B^p_\mu((\Omega \setminus A) \cup A_1) \subset B^p_\mu(\mathbb{C} \setminus A_2)$. The function f can have a pole of order up to $n_{z_k,\mu}$ at the point z_k. This means that $g(z) = f(z) \prod_{k=1}^m (z - z_k)^{n_{z_k,\mu}}$ is an entire function. Furthermore, $f(z) = \sum_{k=n_{\infty,\mu}}^{\infty} c_k z^{-k}$ for $|z|$ large, and since $\sum_{k=1}^m n_{z_k,\mu} < n_{\infty,\mu}$, we see that $g(z) \to 0$, as $z \to \infty$. Liouville's theorem shows that $g \equiv 0$, and hence $f \equiv 0 \in \mathscr{A}^p_\mu(\Omega)$.

We next turn to the second part and assume that A is not weakly removable for B^p_μ. Let $\Omega' = \bigcup\{\Omega''\colon\ B^p_\mu(\Omega \setminus A) \subset \operatorname{Hol}(\Omega'')$ and Ω'' is a domain$\} \cap \mathbb{C} = \mathbb{C} \setminus A_2$. Since A is not weakly removable for B^p_μ, by assumption, $A_2 \neq \emptyset$. Let further $A' = \mathbb{C} \setminus (\Omega \setminus A)$.

We split the rest of the proof into the following cases:

(a) $n_{\infty,\mu} = 0$;

(b) $n_{\infty,\mu} > 0$ and A' not totally disconnected;

(c) $n_{\infty,\mu} > 0$, A' totally disconnected and A_2 infinite;

(d) $n_{\infty,\mu} > 0$ and there is $z_0 \in A_2$ with $n_{z_0,\mu} = \infty$;

(e) $n_{\infty,\mu} > 0$, $A_2 = \{z_1, \ldots, z_m\}$ and $n_{\infty,\mu} \leqslant \sum_{k=1}^m n_{z_k,\mu} < \infty$.

If none of (a)–(e) holds, then A_2 is finite and it follows that $A_1 := \Omega' \setminus (\Omega \setminus A)$ is weakly removable for $B^p_\mu(\Omega' \cap (\Omega \setminus A))$, and hence for $B^p_\mu(\Omega \setminus A)$. Moreover, $\sum_{z \in A_2} n_{z,\mu} < n_{\infty,\mu}$.

Thus, by Theorem 9.2, it is enough to show that in each case (a)–(e) there is a non-zero function in $\mathscr{A}^p_\mu(\Omega \setminus A)$, to conclude that A is not weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$, and thus finish the proof.

(a) There is a non-zero function in $\mathscr{A}^p_\mu(\Omega \setminus A)$, either 1 or one with an essential singularity at ∞, see Remarks 9.4 (iv).

(b) There exist $n_{\infty,\mu}$ pairwise disjoint compact continua $K_1, \ldots, K_{n_{\infty,\mu}} \subset A'$. Since K_j is not totally disconnected, by Axiom A4, it is not removable for B^p_μ, so there is a non-constant function $f_j \in B^p_\mu(\mathbb{S} \setminus K_j)$ with $f_j(\infty) = 0$. Let $g = \prod_{j=1}^{n_{\infty,\mu}} f_j$. Let also $\Omega_0, \ldots, \Omega_{n_{\infty,\mu}}$ be pairwise disjoint neighbourhoods of $\{\infty\}$, $K_1, \ldots, K_{n_{\infty,\mu}}$, respectively. There is a constant C, such that $|g(z)| \leqslant C|f_j(z)|$ for $z \in \Omega_j \setminus K_j$ and $|f_j(z)| \leqslant C|z|^{-1}$ for $z \in \Omega_0$, $1 \leqslant j \leqslant n_{\infty,\mu}$. Thus $g \in \mathscr{A}^p_\mu(\Omega_j \setminus K_j)$, $1 \leqslant j \leqslant n_{\infty,\mu}$, and since $|g(z)| \leqslant C^{n_{\infty,\mu}} |z|^{-n_{\infty,\mu}}$ for $z \in \Omega_0$, also $g \in \mathscr{A}^p_\mu(\Omega_0)$. Finally, g is bounded on the bounded set $\mathbb{S} \setminus \bigcup_{j=0}^{n_{\infty,\mu}} \Omega_j$, and hence $g \in \mathscr{A}_\mu^p\Big(\mathbb{S} \setminus \bigcup_{j=1}^{n_{\infty,\mu}} K_j\Big) \subset \mathscr{A}_\mu^p(\Omega \setminus A)$. Moreover g is non-constant.

(c) We can find $z_1, \dots, z_{n_{\infty,\mu}} \in A_2$. Find pairwise disjoint neighbourhoods G_j of z_j. By the maximality of Ω', $G_j \cap A'$ is not weakly removable for B_μ^p. Hence there is a compact set $K_j \subset G_j \cap A'$ which is not removable for B_μ^p. We can now proceed as we did in (b).

(d) As we have observed there is $f_0(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^{-k} \in \mathscr{A}_\mu^p(D(z_0, 1) \setminus \{z_0\})$ with an essential singularity at z_0. For $j \geqslant 1$ let recursively $f_j(z) = (z-z_0) f_{j-1}(z_0) - a_j = \sum_{k=1}^{\infty} a_{k+j}(z - z_0)^{-k} \in \mathscr{A}_\mu^p(D(z_0, 1) \setminus \{z_0\})$, and let $g(z) = \sum_{k=1}^{\infty} c_k (z - z_0)^{-k}$ be a non-trivial linear combination of $f_0, \dots, f_{n_{\infty,\mu}}$ such that $c_1 = c_2 = \dots = c_{n_{\infty,\mu}} = 0$. By Remarks 9.4 (ii) we see that $g \in \mathscr{A}_\mu^p\big(\mathbb{S} \setminus \overline{D(z_0, \frac{1}{2})}\big)$, and hence $g \in \mathscr{A}_\mu^p(\mathbb{S} \setminus \{z_0\}) \subset \mathscr{A}_\mu^p(\Omega \setminus A)$. If g were constant, then f_0 would be a rational function, a contradiction.

(e) Let $f_k(z) = (z - z_k)^{-n_{z_k,\mu}} \in \mathscr{A}_\mu^p(D(z_k, 1) \setminus \{z_k\})$ and $g = \prod_{k=1}^{m} f_k$, then also $g \in \mathscr{A}_\mu^p(D(z_k, 1) \setminus A_2)$, $1 \leqslant k \leqslant m$. Since there is a constant C such that $|g(z)| \leqslant C|z|^{-n_{\infty,\mu}}$ for $|z| > C$, we also have $g \in \mathscr{A}_\mu^p(\{z \in \mathbb{S} \colon |z| > C\})$. Since g is bounded on $\overline{D(0, C)} \setminus \bigcup_{k=1}^{m} D(z_k, 1)$, it follows that $g \in \mathscr{A}_\mu^p(\mathbb{S} \setminus A_2) \subset \mathscr{A}_\mu^p(\Omega \setminus A)$. Since g is non-constant we are done. □

As a corollary we obtain the following characterization of weak removability for $\mathscr{A}_\mu^p$.

Theorem 9.6. *Let $\nu = \mu|_{\Omega \setminus A}$ and assume that $n_{\infty,\nu} < \infty$. Then A is weakly removable for $\mathscr{A}_\mu^p(\Omega \setminus A)$ if and only if A is weakly removable for B_μ^p, or $\mathbb{C} \setminus (\Omega \setminus A) = A_1 \cup A_2$, where A_1 is weakly removable for B_μ^p, $A_2 = \{z_1, \dots, z_m\}$ and $\sum_{k=1}^{m} n_{z_k,\nu} < n_{\infty,\nu}$.*

Remarks 9.7. In this corollary we can make the requirement that $n_{z_k,\nu} \geqslant 1$ for $z_k \in A_2$, since if, e.g., $n_{z_1,\nu} = 0$, then, as $\nu(A_1) = \nu(\{z_1\}) = 0$, we have A_1 and $\{z_1\}$ both being strongly removable for B_ν^p, by Proposition 3.4, independently of the domain. Hence $A_1 \cup \{z_1\}$ is also strongly removable for B_ν^p and thus weakly removable for B_μ^p, and z_1 can be moved from A_2 to A_1. It is possible to require that $n_{z_k,\mu} \geqslant 1$ for $z_k \in A_2$ in the first part, but not in the second part, of Theorem 9.5.

It is obvious that $n_{z,\nu} \geqslant n_{z,\mu}$ for $z \neq \infty$ and that $n_{\infty,\nu} \leqslant n_{\infty,\mu}$. It is easy to construct examples with at least one strict inequality. In view of Theorem 9.6, this shows that it is not possible to find a necessary and sufficient condition using $n_{z,\mu}$. The reason behind this is that weak removability for $\mathscr{A}_\mu^p(\Omega \setminus A)$ is independent of μ outside of $\Omega \setminus A$, whereas $n_{z,\mu}$ depends on μ outside of $\Omega \setminus A$. The number $n_{z,\nu}$, on the other hand, is independent of μ outside of $\Omega \setminus A$. The drawback is obvious, instead $n_{z,\nu}$ has to be made dependent on $\Omega \setminus A$. Recall that $n_{z,\mu}$ is independent of Ω and A, apart from depending on whether or not $\infty \in \Omega$.

Proof. Since $\mathscr{A}^p_\mu(\Omega \setminus A) = \mathscr{A}^p_\nu(\Omega \setminus A)$, A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if and only if A is weakly removable for $\mathscr{A}^p_\nu(\Omega \setminus A)$. Similarly, A is weakly removable for B^p_μ if and only if A is weakly removable for B^p_ν. Moreover for any domain $\Omega' \supset A_1$, Proposition 3.4 shows that A_1 is weakly removable for B^p_ν if and only if A_1 is strongly removable for $B^p_\nu(\Omega' \setminus A_1)$. By applying Theorem 9.5 to $\mathscr{A}^p_\nu(\Omega \setminus A)$ we obtain the result. □

10. Muckenhoupt A_p weights

Definition 10.1. A Radon measure μ on $\mathbb{C}$ is *doubling* if there exists a constant C such that $\mu(D(z, 2r)) \leqslant C\mu(D(z, r))$ for all $z \in \mathbb{C}$ and $r > 0$.

A non-negative function w is *doubling* if the corresponding measure μ, defined by $\mathrm{d}\mu = w\, \mathrm{d}m$, is doubling.

Definition 10.2. Let $1 < p < \infty$. An A_p *weight* w is a non-negative function such that there exists a constant C so that

$$(10.1)\quad \left(\frac{1}{m(D)}\int_D w\, \mathrm{d}m\right)\left(\frac{1}{m(D)}\int_D w^{1/(1-p)}\, \mathrm{d}m\right)^{p-1} < C \quad \text{for all discs } D \subset \mathbb{C}.$$

An A_1 *weight* is a non-negative function w such that there exists a constant C so that

$$(10.2)\qquad \frac{1}{m(D)}\int_D w\, \mathrm{d}m < C \operatorname*{ess\,inf}_D w \quad \text{for all discs } D \subset \mathbb{C}.$$

Remarks 10.3. In particular, $0 < w < \infty$ a.e., w is doubling and w is a p-admissible weight, see Chapter 15 in Heinonen-Kilpeläinen-Martio [18].

It is easy to see from the definition that if $1 < p < \infty$, $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$, then w' is an $A_{p'}$ weight if and only if w is an A_p weight.

If $p < q$ and w is an A_p weight, then w is an A_q weight. Moreover, if $p_0 = \inf\{p\colon\ w \text{ is an } A_p \text{ weight}\} > 1$, then w is not an A_{p_0} weight, this being the so called open-end property, see e.g. [18], Section 15.13.

We want to make our results more general and therefore make the following definition.

Definition 10.4. Let $1 \leqslant p < \infty$. A *local* A_p *weight* w is a function such that for each $R > 0$ there exists an A_p weight v such that $w|_{D(0,R)} = v|_{D(0,R)}$.

Remarks. It follows directly that if $1 < p < \infty$, $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$, then w' is a local $A_{p'}$ weight if and only if w is a local A_p weight.

It is not true in general that local A_p weights are doubling, consider, e.g., $w(z) = \mathrm{e}^{|z|}$.

Proposition 10.5. *Let $1 \leqslant p < \infty$ and let w be a non-negative function. Then w is a local A_p weight if and only if for each $R > 0$ there exists a constant C_R such that for all discs $D \subset D\,(0, R)$,*

$$\left(\frac{1}{m(D)} \int_D w \,\mathrm{d}m\right)\left(\frac{1}{m(D)} \int_D w^{1/(1-p)} \,\mathrm{d}m\right)^{p-1} < C_R, \quad \text{if } 1 < p < \infty,$$

and

$$\frac{1}{m(D)} \int_D w \,\mathrm{d}m < C_R \operatorname*{ess\,inf}_D w, \quad \text{if } p = 1.$$

Proof. The necessity is clear, we want to prove the sufficiency, without loss of generality we can assume that $R = 1$. We also assume that $1 < p < \infty$.

Let $\sim$ be the equivalence class on $\mathbb{C}$ defined by saying that $\pm 1 + x + y\mathrm{i} \sim \pm 1 - x + y\mathrm{i}$ and $x \pm \mathrm{i} + y\mathrm{i} \sim x \pm \mathrm{i} - y\mathrm{i}$, $x, y \in \mathbb{R}$, i.e. invariance under reflections in the sides of $Q = [-1, 1] \times [-1, 1]$. Let $v = w$ on Q and continue v using reflections in the sides, i.e. $v(z) = v(\zeta)$ if $z \sim \zeta$. We have $v|_{\mathbb{D}} = w|_{\mathbb{D}}$ and need only prove that v is an A_p weight.

Let $D = D\,(z_0, r)$, without loss of generality we can assume that $z_0 \in Q$. We see that D intersects at most $(r+2)^2$ squares of the form $Q + 2(j + k\mathrm{i})$, $j, k \in \mathbb{Z}$. Note also that for each $z \in \mathbb{C}$ there is a unique $\zeta \in Q$ with $z \sim \zeta$, and moreover, if $z \in D$, then $\zeta \in D$. We see that

$$\int_D v \,\mathrm{d}m \leqslant (r+2)^2 \int_{D \cap Q} v \,\mathrm{d}m = (r+2)^2 \int_{D \cap Q} w \,\mathrm{d}m \leqslant (r+2)^2 \int_D w \,\mathrm{d}m,$$

and similarly for $v' := v^{1/(1-p)}$. Let also $w' := w^{1/(1-p)}$.

If $r \leqslant 1$, then $D \subset D\big(0, 1 + \sqrt{2}\big)$ and we have

$$\begin{aligned}\left(\frac{1}{m(D)} \int_D v \,\mathrm{d}m\right)&\left(\frac{1}{m(D)} \int_D v' \,\mathrm{d}m\right)^{p-1} \\ &\leqslant \left(\frac{9}{m(D)} \int_D w \,\mathrm{d}m\right)\left(\frac{9}{m(D)} \int_D w' \,\mathrm{d}m\right)^{p-1} < 9^p C_{1+\sqrt{2}}.\end{aligned}$$

On the other hand, if $r \geqslant 1$, then $D \cap Q \subset D' = D\left(0, \sqrt{2}\right)$, and we get

$$\left(\frac{1}{m(D)}\int_D v \ \mathrm{d}m\right)\left(\frac{1}{m(D)}\int_D v' \ \mathrm{d}m\right)^{p-1} \leqslant \left(\frac{2(r+2)^2}{r^2 m(D')}\int_{D'} w \ \mathrm{d}m\right)\left(\frac{2(r+2)^2}{r^2 m(D')}\int_{D'} w' \ \mathrm{d}m\right)^{p-1} < 18^p C_{\sqrt{2}}.$$

The proof for $p = 1$ is easier, we leave it to the interested reader. □

Remark. With obvious modification of constants this proof characterizes local A^p weights on $\mathbb{R}^n$ also when $n > 2$.

Definition 10.6. Let $1 < p < \infty$ and let $\mathrm{d}\mu = w \ \mathrm{d}m$. Let $K \subset \Omega$ be compact. Then we define

$$\operatorname{cap}_{p,w}(K,\Omega) = \inf\{\|\nabla\varphi\|^p_{L^p_w(\Omega)} \colon \varphi \in \mathscr{C}_0^\infty(\Omega) \text{ and } \varphi = 1 \text{ in an open set containing } K\},$$

where $\mathscr{C}_0^\infty(\Omega)$ denotes the set of infinitely differentiable functions with compact support in Ω. For an arbitrary set $A \subset \Omega$ we define

$$\operatorname{cap}_{p,w}(A,\Omega) = \sup\{\operatorname{cap}_{p,w}(K,\Omega) \colon K \subset A \text{ is compact}\}.$$

Remarks. In the unweighted case, when $w = 1$, we usually drop w from the notation.

Note first, that since $\operatorname{cap}_{p,w}(\,\cdot\,,\Omega)$ is increasing there is no ambiguity in defining $\operatorname{cap}_{p,w}(K,\Omega)$ twice for compact K. Note also that as elsewhere in this paper our functions are complex-valued, but in the definition of $\operatorname{cap}_{p,w}$ the optimal is to have $\operatorname{Im}\varphi \equiv 0$.

For p-admissible weights, in particular for A_p weights, the capacity is the same as the one defined in Heinonen-Kilpeläinen-Martio [18], Chapter 2, p. 27, when G is compact or open, see the discussion on pp. 27–28 in [18]. In fact the definitions coincide for Suslin sets, see Theorem 2.5 in [18]. All Borel sets are Suslin sets. Suslin sets are sometimes called analytic sets, despite the fact that analytic set has a different meaning in the theory of functions of several complex variables.

Definition 10.7. Let $1 < p < \infty$ and let w be a local A_p weight. For a complex-valued $\mathscr{C}^\infty$ function, i.e. a complex-valued infinitely differentiable function, $\varphi\colon \mathbb{C} \to \mathbb{C}$ we let the *Sobolev norm* of φ be

$$\|\varphi\|_{W^{1,p}_w} = \left(\int_{\mathbb{C}} (|\varphi|^p + |\nabla\varphi|^p) w \ \mathrm{d}m\right)^{1/p}.$$

We let the *Sobolev space* $W^{1,p}_w(\mathbb{C})$ be defined by

$$W^{1,p}_w(\mathbb{C}) = \overline{\{\varphi \in \mathscr{C}^\infty(\mathbb{C})\colon \ \|\varphi\|_{W^{1,p}_w} < \infty\}},$$

where the closure is taken in the $\|\cdot\|_{W^{1,p}_w}$ norm. We further define the *Sobolev space* $\mathring{W}^{1,p}_w(\Omega) = \overline{\mathscr{C}^\infty_0(\Omega)}$, where the closure is also taken in the $\|\cdot\|_{W^{1,p}_w}$ norm.

Remark. Sobolev spaces defined in this way are often denoted by the letter H instead of W, since we use H for Hardy spaces we will use W instead. In fact for A_p weights this definition is equivalent to the definition of Sobolev spaces usually denoted by W, see Kilpeläinen [24]. We prefer this definition since it follows our main source, Heinonen-Kilpeläinen-Martio [18], on the theory of weighted Sobolev spaces.

Definition 10.8. Let $1 < p < \infty$ and let w be a local A_p weight. For a compact set $K \subset \mathbb{C}$ we define the *Sobolev (p,w)-capacity* by

$$\operatorname{cap}_{W^{1,p}_w}(K) = \inf\{\|\varphi\|^p_{W^{1,p}_w}\colon \ \varphi \in W^{1,p}_w(\mathbb{C}) \text{ and } \varphi = 1 \text{ in an open set containing } K\}.$$

For an arbitrary set $A \subset \mathbb{C}$ we define the *Sobolev (p,w)-capacity* by

$$\operatorname{cap}_{W^{1,p}_w}(A) = \sup\{\operatorname{cap}_{W^{1,p}_w}(K)\colon \ K \subset A \text{ is compact}\}.$$

Remarks 10.9. In the unweighted case, when $w = 1$, we usually drop w from the notation.

This definition is a little different from the definition in Section 2.35 in Heinonen-Kilpeläinen-Martio [18], where they are only concerned with the case when w is an A_p weight. The two definitions coincide when A is a Suslin set and w is an A_p weight, see Theorems 2.5 and 2.37 in [18].

If K is compact, $\Omega \supset K$ is a bounded domain and w is a local A_p weight, $1 < p < \infty$, then $\operatorname{cap}_{W^{1,p}_w}(K) = 0$ if and only if $\operatorname{cap}_{p,w}(K,\Omega) = 0$, the proof of this fact on p. 49 in [18] directly extends to local A_p weights.

Theorem 10.10. *Let $1 < p < \infty$ and let w be an A_p weight. Let also $p_0 = \inf\{q\colon \ w \text{ is an } A_q \text{ weight}\}$. If $\operatorname{cap}_{W^{1,p}_w}(A) = 0$ for some non-empty $A \subset \mathbb{C}$, then $p \leqslant 2p_0$ and $\operatorname{cap}_{W^{1,p/p_0}}(A) = 0$. In particular, $\dim_H A \leqslant 2 - p/p_0$.*

Remarks. This is Corollary 2.33 in Heinonen-Kilpeläinen-Martio [18]. Recall also that $p_0 < p$, see Remarks 10.3.

This theorem is sharp in the sense that given $p_0 < p < 2p_0$ there is a weight w with $p_0 = \inf\{q\colon \ w \text{ is an } A_q \text{ weight}\}$, and a set A with $\operatorname{cap}_{W^{1,p}_w}(A) = 0$ and

$\dim_H A = 2 - p/p_0$, and hence $\operatorname{cap}_{W^{1,q}}(A) > 0$ for all $q < p/p_0$, see Theorem 13.1. One such example is obtained by letting $w(z) = \operatorname{dist}(z, A)^{p(p_0-1)/p_0}$, where $A \subset \mathbb{R}$ is an unbounded self-similar Cantor set with $\dim_H A = 2 - p/p_0$. In higher dimensions similar examples can be obtained with A being an unbounded self-similar Cantor set in some hyperplane.

The theorem is not sharp for all weights. Consider for instance a power weight $w(z) = |z|^\beta$, $\beta > 0$, an A_p weight for $p > 1 + \frac{1}{2}\beta$, and let K be a compact set. Since w and 1 are comparable away from the origin we see that if $\operatorname{cap}_{W^{1,p}_w}(K) = 0$, then $\operatorname{cap}_{W^{1,p}}(K \setminus \{0\}) = 0$, which is stronger than the theorem above.

Lemma 10.11. *Let $1 < p < \infty$ and let w be an A_p weight. Then there is a constant $C > 0$ such that*

$$C\|\nabla\varphi\|_{L^p_w} \leqslant \|\overline{\partial}\varphi\|_{L^p_w} \leqslant \frac{1}{\sqrt{2}}\|\nabla\varphi\|_{L^p_w} \quad \textit{for all } \varphi \in \mathscr{C}_0^\infty(\mathbb{C}).$$

Remark. Here, as usual, ∂_1 and ∂_2 denote the partial derivative operators with respect to the real and imaginary variables, respectively, and $\overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2)$.

Proof. The second inequality follows directly from $|\overline{\partial}\varphi(z)| \leqslant |\nabla\varphi(z)|/\sqrt{2}$ and therefore holds much more generally.

Let $\hat{f}(\zeta) := \int_{\mathbb{C}} f(z)\mathrm{e}^{-\mathrm{i}\operatorname{Re} z\overline{\zeta}}\,\mathrm{d}m(z)$ denote the Fourier transform of f. Let also $\zeta = \xi + \mathrm{i}\eta$. Then $\widehat{\partial_1\varphi}(\zeta) = \mathrm{i}\xi\widehat{\varphi}(\zeta)$ and $\widehat{\partial_2\varphi}(\zeta) = \mathrm{i}\eta\widehat{\varphi}(\zeta)$. It follows that

$$2\frac{\xi^2 - \mathrm{i}\xi\eta}{|\zeta|^2}\widehat{\overline{\partial}\varphi}(\zeta) = \frac{\xi^2 - \mathrm{i}\xi\eta}{|\zeta|^2}\mathrm{i}(\xi + \mathrm{i}\eta)\widehat{\varphi}(\zeta) = \mathrm{i}\xi\widehat{\varphi}(\zeta) = \widehat{\partial_1\varphi}(\zeta)$$

and similarly

$$2\frac{\xi\eta - \mathrm{i}\eta^2}{|\zeta|^2}\widehat{\overline{\partial}\varphi}(\zeta) = \frac{\xi\eta - \mathrm{i}\eta^2}{|\zeta|^2}\mathrm{i}(\xi + \mathrm{i}\eta)\widehat{\varphi}(\zeta) = \mathrm{i}\eta\widehat{\varphi}(\zeta) = \widehat{\partial_2\varphi}(\zeta).$$

The Riesz transforms are defined by their Fourier transforms,

$$\widehat{\mathscr{R}_1\varphi}(\zeta) = -\mathrm{i}\frac{\xi}{|\zeta|}\widehat{\varphi}(\zeta) \quad \text{and} \quad \widehat{\mathscr{R}_2\varphi}(\zeta) = -\mathrm{i}\frac{\eta}{|\zeta|}\widehat{\varphi}(\zeta).$$

So we get

$$\partial_1\varphi = -2(\mathscr{R}_1^2 - i\mathscr{R}_1\mathscr{R}_2)\overline{\partial}\varphi \quad \text{and} \quad \partial_2\varphi = -2(\mathscr{R}_1\mathscr{R}_2 - i\mathscr{R}_2^2)\overline{\partial}\varphi.$$

The crucial point now is that since w is an A_p weight, the Riesz transforms are bounded operators on $L^p_w(\mathbb{C})$, see Theorem IV.3.1 in García-Cuerva-Rubio de Francia [13]. (In fact, this is only true for A_p weights, see Theorem IV.3.7 in [13].) Thus there exists a constant C', independent of φ, such that

$$\|\partial_1\varphi\|_{L^p_w} \leqslant C'\|\overline{\partial}\varphi\|_{L^p_w} \quad \text{and} \quad \|\partial_2\varphi\|_{L^p_w} \leqslant C'\|\overline{\partial}\varphi\|_{L^p_w}.$$

Hence there exists $C > 0$ such that

$$C\|\nabla\varphi\|_{L^p_w} \leqslant \|\overline{\partial}\varphi\|_{L^p_w} \quad \text{for all } \varphi \in \mathscr{C}_0^\infty(\mathbb{C}).$$

□

The following corollary may be of independent interest, although we do not need it.

Corollary 10.12. *Let $1 < p < \infty$ and let w be a local A_p weight. Let Ω be a bounded domain. Then there exists a constant $C > 0$ such that*

$$C\|\varphi\|_{W^{1,p}_w} \leqslant \|\overline{\partial}\varphi\|_{L^p_w} \leqslant \frac{1}{\sqrt{2}}\|\varphi\|_{W^{1,p}_w} \quad \textit{for all } \varphi \in \overset{\circ}{W}{}^{1,p}_w(\Omega).$$

P r o o f. The Poincaré inequality, see Heinonen-Kilpeläinen-Martio [18], Section 1.4, says that there exists a constant C', independent of φ, such that

$$\|\varphi\|_{L^p_w} \leqslant C'\|\nabla\varphi\|_{L^p_w} \quad \text{for all } \varphi \in \mathscr{C}_0^\infty(\Omega).$$

Combining this with Lemma 10.11 we see that there is a constant C such that

$$C\|\varphi\|_{W^{1,p}_w} \leqslant \|\overline{\partial}\varphi\|_{L^p_w} \leqslant \frac{1}{\sqrt{2}}\|\varphi\|_{W^{1,p}_w}$$

for all $\varphi \in \mathscr{C}_0^\infty(\Omega)$, and hence by continuity for all $\varphi \in \overset{\circ}{W}{}^{1,p}_w(\Omega)$. □

Proposition 11.1. *Let* $\mathrm{d}\mu = w\ \mathrm{d}m$ *and assume that there exists* $\varepsilon > 0$ *such that* $w^{-\varepsilon}$ *is integrable in a neighbourhood of* $z_0 \neq \infty$. *Then* $n_{z_0,w} < 2(1+1/\varepsilon)/p$.

If there exists $C > 0$ *such that* $w > C$ *a.e. in a neighbourhood of* z_0, *then* $n_{z_0,w} < 2/p$.

In particular, if w *is a local* A_q *weight,* $1 \leqslant q < \infty$, *then* $n_{z_0,w} < 2q/p$.

Remarks. Recall that $n_{z_0,w} = 0$ if and only if $\{z_0\}$ is removable for B^p_w.

A direct consequence is that if $q_0 = \inf\{q\colon\ w \text{ is an } A_q \text{ weight}\}$, then $n_{z_0,w} \leqslant \lfloor 2q_0/p \rfloor$. If w is a local A_1 weight we can improve this slightly obtaining $n_{z_0,w} \leqslant \lceil 2/p \rceil - 1$.

P r o o f. Without loss of generality we can assume that $z_0 = 0$ and that $w^{-\varepsilon} \in L^1(\mathbb{D})$. Let $q = 1 + 1/\varepsilon$, $1/q + 1/q' = 1$, $w' = w^{1/(1-q)} = w^{-\varepsilon}$, $f \in \mathscr{A}^p_w(\mathbb{D} \setminus \{0\})$ and $z \in D\big(0, \frac{1}{2}\big)$. Since $|f|^{p/q}$ is subharmonic we have

$$\begin{aligned} |f(z)|^{p/q} &\leqslant \frac{1}{\pi|z|^2} \int_{D(z,|z|)} |f(\zeta)|^{p/q}\ \mathrm{d}m(\zeta) \\ &= \frac{1}{\pi|z|^2} \int_{D(z,|z|)} |f(\zeta)|^{p/q} w(\zeta)^{1/q} w'(\zeta)^{1/q'}\ \mathrm{d}m(\zeta) \\ &\leqslant \frac{1}{\pi|z|^2} \|f\|^{p/q}_{L^p_w(D(z,|z|))} \bigg(\int_{D(z,|z|)} w'\ \mathrm{d}m \bigg)^{1/q'}. \end{aligned}$$

Both the middle and the last factors on the right-hand side tend to 0, as $z \to 0$. Hence f cannot have an essential singularity or a pole of order $\geqslant 2q/p = 2(1+1/\varepsilon)/p$ at the origin. This concludes the first part.

For the second part, we can assume without loss of generality that $z_0 = 0$ and that $w(z) > C$ for $z \in \mathbb{D}$. Since $|f|^p$ is subharmonic we have

$$\begin{aligned} |f(z)|^p &\leqslant \frac{1}{\pi|z|^2} \int_{D(z,|z|)} |f(\zeta)|^p\ \mathrm{d}m(\zeta) = \frac{1}{\pi|z|^2} \int_{D(z,|z|)} |f(\zeta)|^p w(\zeta) \frac{1}{w(\zeta)}\ \mathrm{d}m(\zeta) \\ &\leqslant \frac{1}{C\pi|z|^2} \|f\|^p_{L^p_w(D(z,|z|))}. \end{aligned}$$

The last factor in the right-hand side tends to 0, as $z \to 0$. Hence f cannot have an essential singularity or a pole of order $\geqslant 2/p$ at the origin.

The last part follows directly from the A_1 condition (10.2), if $q = 1$, and from the A_q condition (10.1) which, in particular, requires $w^{1/(1-q)}$ to be integrable, if $1 < q < \infty$. □

Proposition 11.2. *Let μ be a doubling measure with doubling constant C. Then $n_{\infty,\mu} \leqslant \lfloor(\log_2 C)/p\rfloor + 1$.*

Proof. Let $n = \lfloor(\log_2 C)/p\rfloor + 1$ and $f(z) = z^{-n}$. Then

$$\int_{\mathbb{C}\setminus\overline{\mathbb{D}}} |f|^p \,\mathrm{d}\mu \leqslant \sum_{j=1}^{\infty} 2^{(1-j)np}\mu(D\left(0, 2^j\right)) \leqslant 2^{np}\mu(\mathbb{D}) \sum_{j=1}^{\infty} (2^{-np}C)^j < \infty,$$

since $2^{-np}C < 1$. □

Lemma 11.3. *If μ is a doubling measure with doubling constant C, then*

$$\mu(D\,(z, 2r)) \geqslant (1 + C^{-3})\mu(D\,(z, r)) \quad \textit{for all } z \in \mathbb{C} \textit{ and } r > 0.$$

Proof. Without loss of generality we can assume that $z = 0$ and $r = 1$. We find that

$$\mu(D\,(0, 2)) \geqslant \mu(\mathbb{D}) + \mu\big(D\big(\tfrac{3}{2}, \tfrac{1}{2}\big)\big) \geqslant \mu(\mathbb{D}) + C^{-3}\mu\big(D\big(\tfrac{3}{2}, 4\big)\big) \geqslant \mu(\mathbb{D}) + C^{-3}\mu(\mathbb{D}).$$

□

Proposition 11.4. *Let w be an A_q weight, $1 < q < \infty$. Then $1 \leqslant n_{\infty,w} \leqslant \lceil 2q/p\rceil$.*

Remark. A direct consequence is that if $q_0 = \inf\{q\colon w \text{ is an } A_q \text{ weight}\}$, then $1 \leqslant n_{\infty,w} \leqslant \lfloor 2q_0/p\rfloor + 1$. As we will see in Proposition 11.5 this is best possible (also when w is an A_1 weight).

Proof. Let $1/q + 1/q' = 1$ and $w' = w^{1/(1-q)}$. Since w' is an $A_{q'}$ weight it is doubling. Let C' be the doubling constant of w', and also $\mathrm{d}\mu' = w' \,\mathrm{d}m$.

Consider first $f \in \mathscr{A}^p_w(\mathbb{C})$ and let $D = D\,(z, r)$. Since $|f|^{p/q}$ is subharmonic we have, using the A_q condition (10.1) with A_q constant C,

$$\begin{aligned}|f(z)|^{p/q} &\leqslant \frac{1}{m(D)} \int_D |f(\zeta)|^{p/q} \,\mathrm{d}m(\zeta) = \frac{1}{m(D)} \int_D |f(\zeta)|^{p/q} w(\zeta)^{1/q} w'(\zeta)^{1/q'} \,\mathrm{d}m(\zeta)\\ &\leqslant \frac{\|f\|^{p/q}_{L^p_w(D)}}{m(D)} \left(\int_D w' \,\mathrm{d}m\right)^{1/q'} \leqslant C^{1/q} \|f\|^{p/q}_{L^p_w(\mathbb{C})} \left(\int_{D(z,r)} w \,\mathrm{d}m\right)^{-1/q} \to 0,\end{aligned}$$

as $r \to \infty$, since it is a consequence of Lemma 11.3 that $\int_{D(z,r)} w \,\mathrm{d}m \to \infty$, as $r \to \infty$. Hence $f \equiv 0$, and since this also shows that $1 \notin \mathscr{A}^p_w(\mathbb{S} \setminus \overline{\mathbb{D}})$, we see that $n_{\infty,w} \geqslant 1$.

From the A_q condition (10.1) and Lemma 11.3, it follows that

$$\mu(D\left(0,2^j\right)) < \frac{Cm(D\left(0,2^j\right))^q}{\mu'(D\left(0,2^j\right))^{q-1}} < \frac{C\pi^q}{\mu'(\mathbb{D})^{q-1}}\Big(\frac{2^{2q}}{(1+(C')^{-3})^{q-1}}\Big)^j.$$

Let $f(z) = z^{-n}$, $n = \lceil 2q/p \rceil$. Then

$$\int_{\mathbb{C}\setminus\overline{\mathbb{D}}} |f|^p \,\mathrm{d}\mu \leqslant \sum_{j=1}^{\infty} 2^{(1-j)np}\mu(D\left(0,2^j\right)) \leqslant \frac{C2^{np}\pi^q}{\mu'(\mathbb{D})^{q-1}} \sum_{j=1}^{\infty}\Big(\frac{2^{2q-np}}{(1+(C')^{-3})^{q-1}}\Big)^j < \infty.$$

Hence $n_{\infty,w} \leqslant \lceil 2q/p \rceil$. $\square$

Proposition 11.5. *Let* $w(z) = |z|^\beta$, $\beta > -2$. *Then* $n_{0,w} = \lceil (2+\beta)/p \rceil - 1$, $n_{z,w} = \lceil 2/p \rceil - 1$, $z \neq 0$, $z \neq \infty$, *and* $n_{\infty,w} = \lfloor (2+\beta)/p \rfloor + 1$.

Remarks. The condition $\beta > -2$ is needed for $w\,\mathrm{d}m$ to be a Radon measure.

It is well known, and easy to check, that w is an A_q weight exactly when $q > 1+\frac{1}{2}\beta$, $\beta \geqslant 0$, and $q \geqslant 1$, $-2 < \beta \leqslant 0$. This shows that the upper bound on $n_{\infty,w}$ is the best possible in Proposition 11.4, since for $\beta \geqslant 0$ we have $n_{\infty,w}$ equal to the lowest upper bound obtainable from Proposition 11.4, when varying q. This also shows that $n_{0,w}$ equals the lowest upper bound obtainable from Proposition 11.1, when varying q, when $\beta \geqslant 0$, unless $(2+\beta)/p$ is an integer, in which case $n_{0,w}$ is one less.

P r o o f. Propositions 11.1 and 11.4 rule out essential singularities and it is easy to check which negative integer powers belong to $\mathscr{A}^p_w(D\left(z,1\right)\setminus\{z\})$ and $\mathscr{A}^p_w(\mathbb{S}\setminus\overline{\mathbb{D}})$. $\square$

12. Removability for Muckenhoupt A_p weights

Lemma 12.1. *Let* $1 < p < \infty$ *and* $\mathrm{d}\mu = w\,\mathrm{d}m$. *Let also* $w' = w^{1/(1-p)}$ *and* $1/p + 1/p' = 1$. *Let* $K \subset \Omega \subset \mathbb{C}$ *be compact. Assume that* $w' \in L^1_{\mathrm{loc}}(\Omega)$. *Then the annihilator of* $\mathscr{A}^p_w(\Omega\setminus K) \subset L^p_w(\Omega)$ *is* $\overline{\{\overline{\partial}\varphi\colon\ \varphi \in \mathscr{C}^\infty_0(\Omega\setminus K)\}}$, *where the closure is taken in the* $L^{p'}_{w'}(\Omega)$ *norm.*

If $p = 1$ *and* $w^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$, *then the same is true with the closure taken in the norm* $g \mapsto \|g/w\|_{L^\infty(\Omega)}$.

Remarks. In this lemma we do not require $\Omega\setminus K$ to be connected, $\mathscr{A}^p_w(\Omega\setminus K)$ being defined in the obvious way.

In the unweighted case, with $1 < p < \infty$, this is Lemma 1 in Havin-Maz'ya [15].

P r o o f. First we need some clarification. We will say that an equivalence class $[f]$ in $L^p_w(\Omega)$ is in $\mathscr{A}^p_w(\Omega\setminus K)$ if there is a representative $\tilde{f}\in[f]$ which is analytic in $\Omega\setminus K$.

If $m(K)=0$, then each function in $\mathscr{A}^p_w(\Omega\setminus K)$ corresponds to just one equivalence class in $L^p_w(\Omega)$. On the other hand, if $m(K)>0$, then there are infinitely many equivalence classes in $L^p_w(\Omega)$ corresponding to each function in $\mathscr{A}^p_w(\Omega\setminus K)$.

We want to find the dual space $L^p_w(\Omega)^*$ with respect to the pairing $\langle f,g\rangle := \int_\Omega fg\ \mathrm{d}m$. Since $L^p_w(\Omega)^* = L^{p'}_w(\Omega)$ with pairing $\int_\Omega fgw\ \mathrm{d}m = \langle f,\tilde{g}\rangle$, where $\tilde{g}=gw\in L^{p'}_{w'}(\Omega)$ if and only if $g\in L^{p'}_w(\Omega)$ (with equal norms), we find that $L^p_w(\Omega)^* = L^{p'}_{w'}(\Omega)$. For $p=1$ we see that $L^1_w(\Omega)^* = \{g\colon\ gw^{-1}\in L^\infty(\Omega)\}$ with $\|g\|_{L^1_w(\Omega)^*} = \|g/w\|_{L^\infty(\Omega)}$.

Denote the *annihilator* of $\mathscr{A}^p_w(\Omega\setminus K)\subset L^p_w(\Omega)$ by $\mathscr{A}^p_w(\Omega\setminus K)^\perp$. By definition

$$\mathscr{A}^p_w(\Omega\setminus K)^\perp = \{g\in L^p_w(\Omega)^*\colon\ \langle f,g\rangle = 0 \text{ for all } f\in\mathscr{A}^p_w(\Omega\setminus K)\}.$$

If $f\in L^p_w(\Omega)$ and $D\subset\Omega$ is a disc then

$$\int_D |f|\ \mathrm{d}m = \int_D |f| w^{1/p}(w')^{1/p'}\ \mathrm{d}m \leqslant \Big(\int_D |f|^p w\ \mathrm{d}m\Big)^{1/p}\Big(\int_D w'\ \mathrm{d}m\Big)^{1/p'} < \infty.$$

(For $p=1$ the last integral should be understood as $\|w^{-1}\|_{L^\infty(D)}$.) Thus $L^p_w(\Omega)\subset L^1_{\mathrm{loc}}(\Omega)\subset\mathscr{D}'(\mathbb{C})$, where $\mathscr{D}'(\mathbb{C})$ is the set of all distributions on $\mathbb{C}$, and we consider functions in $L^1_{\mathrm{loc}}(\Omega)$ to be 0 on $\mathbb{C}\setminus\Omega$.

Using Weyl's lemma (see, e.g., Hörmander [19], Theorem 4.4.1) we find that $f\in L^p_w(\Omega)$ is analytic in $\Omega\setminus K$ if and only if $\overline{\partial}f=0$ in $\Omega\setminus K$ in the sense of distributions, i.e.

$$\langle f,\overline{\partial}\varphi\rangle = \int_\Omega f\overline{\partial}\varphi\ \mathrm{d}m = 0 \quad \text{for all } \varphi\in\mathscr{C}^\infty_0(\Omega\setminus K).$$

Therefore

$$\{\overline{\partial}\varphi\colon\ \varphi\in\mathscr{C}^\infty_0(\Omega\setminus K)\}\subset\mathscr{A}^p_w(\Omega\setminus K)^\perp.$$

Moreover, since Weyl's lemma only requires these functionals to be 0, and $\mathscr{A}^p_w(\Omega\setminus K)^\perp$ is closed, Hahn-Banach's theorem shows that

$$\mathscr{A}^p_w(\Omega\setminus K)^\perp = \overline{\{\overline{\partial}\varphi\colon\ \varphi\in\mathscr{C}^\infty_0(\Omega\setminus K)\}},$$

where the closure is taken in the $L^{p'}_{w'}$ norm. □

Theorem 12.2. *Let $1 < p < \infty$ and $\mathrm{d}\mu = w\,\mathrm{d}m$. Let also $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$. Let $K \subset \Omega \subset \mathbb{C}$ be compact. Assume that $w' \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$cap_{\mathscr{A}^p_w}(K,\Omega) = \frac{1}{\pi} \inf\big\{\|\overline{\partial}\varphi\|_{L^{p'}_{w'}(\Omega\setminus K)}\colon\ \varphi \in \mathscr{C}^\infty_0(\Omega),\\ \varphi = 1 \text{ in an open set containing } K\big\}.$$

If $w^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$, then

$$cap_{\mathscr{A}^1_w}(K,\Omega) = \frac{1}{\pi} \inf\left\{\Big\|\frac{\overline{\partial}\varphi}{w}\Big\|_{L^\infty(\Omega\setminus K)}\colon\ \varphi \in \mathscr{C}^\infty_0(\Omega),\\ \varphi = 1 \text{ in an open set containing } K\right\}.$$

Remarks. As in Lemma 12.1 we do not require $\Omega \setminus K$ to be connected.

In the unweighted case, with $1 < p < \infty$, this is part of Proposition 11.1.10 in Adams-Hedberg [1], which comes from the proof of Lemma 1 in Hedberg [16].

Proof. Let $h \in \mathscr{C}^\infty_0(\Omega)$ be equal to one in an open set containing K, and let γ be a smooth cycle in $\{z\colon\ h(z) = 1\} \setminus K$ with winding number $\mathrm{wind}_\gamma(z) = 1$ if $z \in K$ and $\mathrm{wind}_\gamma(z) = 0$ if $z \notin \Omega$. Let $f \in \mathscr{A}^p_w(\Omega \setminus K)$. Then by Stokes' theorem we have

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma f(z)\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_\gamma f(z)h(z)\,\mathrm{d}z = -\frac{1}{2\pi\mathrm{i}} \int_{\Omega\setminus K} \overline{\partial}(fh)(z)\,\mathrm{d}\overline{z} \wedge \mathrm{d}z \\ = -\frac{1}{\pi} \int_{\Omega\setminus K} f(z)\overline{\partial}h(z)\ \mathrm{d}m(z).$$

Hence the functional $f \mapsto (1/2\pi\mathrm{i}) \int_\gamma f(z)\,\mathrm{d}z$ is represented by the function $-(1/\pi)\overline{\partial}h$. We obtain, using the Hahn-Banach theorem (cf. Exercise 4.19 in Rudin [29]) and Lemma 12.1,

$$\begin{aligned} \mathrm{cap}_{\mathscr{A}^p_w}(K,\Omega) &= \frac{1}{\pi} \sup\{|\langle f, \overline{\partial}h\rangle|\colon\ f \in \mathscr{A}^p_w(\Omega \setminus K) \text{ and } \|f\|_{L^p_w(\Omega\setminus K)} \leqslant 1\} \\ &= \frac{1}{\pi} \inf\big\{\|\overline{\partial}h + g\|_{L^{p'}_{w'}(\Omega\setminus K)}\colon\ g \in \mathscr{A}^p_w(\Omega \setminus K)^\perp\big\} \\ &= \frac{1}{\pi} \inf\big\{\|\overline{\partial}(h+\varphi)\|_{L^{p'}_{w'}(\Omega\setminus K)}\colon\ \varphi \in \mathscr{C}^\infty_0(\Omega \setminus K)\big\} \\ &= \frac{1}{\pi} \inf\big\{\|\overline{\partial}\varphi\|_{L^{p'}_{w'}(\Omega\setminus K)}\colon\ \varphi \in \mathscr{C}^\infty_0(\Omega), \\ &\qquad \varphi = 1 \text{ in an open set containing } K\big\}. \end{aligned}$$

This concludes the proof of the first part. The proof of the last part is similar. □

Corollary 12.3. *Let $1 < p < \infty$ and $\mathrm{d}\mu = w\,\mathrm{d}m$. Let also $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$. Let $A \subset \Omega \subset \mathbb{C}$. Assume that $w' \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\operatorname{cap}_{\mathscr{A}^p_w}(A, \Omega) \leqslant \frac{1}{\pi\sqrt{2}} \operatorname{cap}_{p',w'}(A, \Omega)^{1/p'}.$$

P r o o f. Assume first A to be compact, since $|\overline{\partial}\varphi| \leqslant |\nabla\varphi|/\sqrt{2}$ the inequality directly follows from the previous theorem. For general A the inequality follows after taking suprema on both sides over $K \subset A$ compact. □

Theorem 12.4. *Let $1 < p < \infty$ and let w be an A_p weight. Let also $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$. Then there exists a constant C such that if $A \subset \Omega \subset \mathbb{C}$, then*

$$\frac{1}{C} \operatorname{cap}_{p',w'}(A, \Omega)^{1/p'} \leqslant \operatorname{cap}_{\mathscr{A}^p_w}(A, \Omega) \leqslant \frac{1}{\pi\sqrt{2}} \operatorname{cap}_{p',w'}(A, \Omega)^{1/p'}.$$

Remark. An immediate consequence of this together with Theorem 7.7 is that if Ω satisfies condition (2.1), then A is removable for B^p_w if and only if $\operatorname{cap}_{p',w'}(A, \Omega) = 0$.

P r o o f. Let $K \subset A$ be compact. By Theorem 12.2 together with Lemma 10.11 we see that there exists C such that

$$\frac{1}{C}\operatorname{cap}_{p',w'}(K, \Omega)^{1/p'} \leqslant \operatorname{cap}_{\mathscr{A}^p_w}(K, \Omega) \leqslant \frac{1}{\pi\sqrt{2}}\operatorname{cap}_{p',w'}(K, \Omega)^{1/p'}.$$

In fact we can choose $C = \pi C'$, where C' is the constant given by Lemma 10.11 for p' and w', so C is independent of K and Ω.

Taking suprema over compact $K \subset A$ yields the desired inequalities for A. □

Theorem 12.5. *Let $1 < p < \infty$ and let w be a local A_p weight. Let also $w' = w^{1/(1-p)}$ and $1/p+1/p' = 1$. Then $\operatorname{cap}_{\mathscr{A}^p_w}(A) = 0$ if and only if $\operatorname{cap}_{W^{1,p'}_{w'}}(A) = 0$.*

Remark. Recall that Theorem 10.10 gives an upper bound on $\dim_H A$ when $\operatorname{cap}_{W^{1,p'}_{w'}}(A) = 0$.

P r o o f. Let first $K \subset A$ be compact, and let $\Omega \supset K$ be a bounded domain. Let v be an A_p weight that coincides with w on Ω, and let $v' = v^{1/(1-p)}$. Obviously $\operatorname{cap}_{p',w'}(K, \Omega) = \operatorname{cap}_{p',v'}(K, \Omega)$ and $\operatorname{cap}_{\mathscr{A}^p_v}(K, \Omega) = \operatorname{cap}_{\mathscr{A}^p_w}(K, \Omega)$. By Theorems 7.7 and 12.4, $\operatorname{cap}_{\mathscr{A}^p_w}(K) = 0$ if and only if $\operatorname{cap}_{p',w'}(K, \Omega) = 0$, which is equivalent to $\operatorname{cap}_{W^{1,p'}_{w'}}(K) = 0$, see Remarks 10.9.

The full result follows directly by taking suprema over $K \subset A$ compact. □

A consequence of Theorems 9.5 and 12.5 and Propositions 11.1 and 11.4 is the following result.

Theorem 12.6. *Let $1 < p < \infty$ and let w be an A_p weight. Let also $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$. Then A is removable for $\mathscr{A}^p_w(\Omega \setminus A)$ if and only if either $\operatorname{cap}_{W^{1,p'}_{w'}}(A) = 0$, or there is z such that $\operatorname{cap}_{W^{1,p'}_{w'}}((\mathbb{C} \setminus (\Omega \setminus A)) \setminus \{z\}) = 0$ and $\int_{\mathbb{C}\setminus\mathbb{D}} |z|^{-p} w(z)\, \mathrm{d}m(z) = \infty$.*

Remark. Recall that weak and strong removability are the same in this case, by Proposition 8.1.

P r o o f. Proposition 11.4 shows that $n_{\infty,w} = 1$ or $n_{\infty,w} = 2$, and thus $n_{\infty,w} = 2$ is equivalent to $\int_{\mathbb{C}\setminus\mathbb{D}} |z|^{-p} w(z)\, \mathrm{d}m(z) = \infty$. Let A_2 be as in Theorem 9.5. We may assume that if $z \in A_2$, then $n_{z,w} \geqslant 1$, see Remarks 9.7. If $n_{\infty,w} = 1$, then we have $A_2 = \emptyset$, and hence the result follows from Theorems 9.5 and 12.5.

On the other hand, if $n_{\infty,w} = 2$, then $\operatorname{card} A_2 \leqslant 1$. The necessity follows directly from Theorems 9.5 and 12.5. As for the sufficiency, let $A_2 = \{z\}$. Proposition 11.1 shows that $n_{z,w} \leqslant 1$, and hence removability follows from Theorems 9.5 and 12.5, regardless of whether $n_{z,w} = 0$ or $n_{z,w} = 1$. □

Corollary 12.7. *Let $1 < p < \infty$ and let w be an A_p weight. If A is removable for $\mathscr{A}^p_w(\Omega \setminus A)$, then $\dim_H A < 1$.*

It is an open problem if there exists a set A with $\dim_H A = 1$, weakly removable for B^p_μ for some Radon measure μ and some $p < \infty$.

P r o o f. Let $w' = w^{1/(1-p)}$ and $1/p + 1/p' = 1$. Let also $p'_0 = \inf\{q\colon w' \text{ is an } A_q \text{ weight}\}$. Recall that the open-end property says that $p'_0 < p'$, see Remarks 10.3. Then by the above theorem, there is z such that $\operatorname{cap}_{W^{1,p'}_{w'}}(A \setminus \{z\}) = 0$. By Theorem 10.10 and the open-end property

$$\dim_H A = \dim_H A \setminus \{z\} \leqslant 2 - \frac{p'}{p'_0} < 1.$$

□

13. The unweighted case

For the unweighted Sobolev capacity we want to recall the following well-known theorem.

Theorem 13.1. *Let $1 < p < \infty$, Λ_d denote the d-dimensional Hausdorff measure and* $\operatorname{cap}_{\log}$ *denote the logarithmic capacity. Then, the following are true:*

$$\begin{aligned}
\dim_H A > 2 - p &\Longrightarrow \operatorname{cap}_{W^{1,p}}(A) > 0, \quad 1 < p < 2;\\
\Lambda_{2-p}(A) < \infty &\Longrightarrow \operatorname{cap}_{W^{1,p}}(A) = 0, \quad 1 < p < 2;\\
\operatorname{cap}_{\log}(A) = 0 &\Longleftrightarrow \operatorname{cap}_{W^{1,2}}(A) = 0;\\
A = \emptyset &\Longleftrightarrow \operatorname{cap}_{W^{1,p}}(A) = 0, \quad p > 2.
\end{aligned}$$

Theorem 13.2. *Let* $\operatorname{cap}_{\log}$ *denote the logarithmic capacity and γ denote the analytic capacity. Then A is removable for B^p if and only if*

$$\begin{aligned}
A = \emptyset, &\quad 0 < p < 2,\\
\operatorname{cap}_{\log}(A) = 0, &\quad p = 2,\\
\operatorname{cap}_{W^{1,p'}}(A) = 0, &\quad 2 \leqslant p < \infty \text{ and } 1/p + 1/p' = 1,\\
\gamma(A) = 0, &\quad p = \infty.
\end{aligned}$$

Remark. Recall that weak and strong removability are the same in the unweighted case, by Proposition 8.1. We assume that $\gamma(A) = \sup\{\gamma(K) \colon K \subset A \text{ is compact}\}$.

P r o o f. For $0 < p < 2$ this follows directly from the fact that $z^{-1} \in \mathscr{A}^p(\mathbb{D} \setminus \{0\})$. For $1 < p < \infty$ it follows directly using Theorems 7.8, 12.5 and 13.1.

For $p = \infty$ we know that $\mathscr{A}^\infty(\Omega) = B^\infty(\Omega) = H^\infty(\Omega)$ for all domains, and that γ characterizes the removable singularities for H^∞, see e.g. Garnett [14]. □

Theorem 13.3. *Let* $\operatorname{cap}_{\log}$ *denote the logarithmic capacity and γ denote the analytic capacity. Then A is removable for $\mathscr{A}^p(\Omega \setminus A)$ if and only if*

$$\begin{aligned}
\operatorname{card}(\mathbb{C} \setminus (\Omega \setminus A)) \leqslant 1, &\quad 0 < p < 2 \text{ and } p \neq 1,\\
\operatorname{card}(\mathbb{C} \setminus (\Omega \setminus A)) \leqslant 2, &\quad p = 1,\\
\operatorname{cap}_{\log}(A) = 0, &\quad p = 2,\\
\operatorname{cap}_{W^{1,p'}}(A) = 0, &\quad 2 \leqslant p < \infty \text{ and } 1/p + 1/p' = 1,\\
\gamma(A) = 0, &\quad p = \infty.
\end{aligned}$$

This result is not new, see Carleson [10], Theorem 6.1, Hedberg [17] and Adams-Hedberg [1], Section 11.1.

P r o o f. For $2 < p \leqslant \infty$ condition (2.1) holds for all domains, so A is removable for $\mathscr{A}^p(\Omega \setminus A)$ if and only if A is removable for B^p, and Theorem 13.2 yields the result.

For $p = 2$ we have $n_z = 0$ for $z \neq \infty$ by Proposition 11.1. So if $\mathbb{C} \setminus (\Omega \setminus A) = A_1 \cup A_2$, where A_1 is removable for B^2 and $A_2 = \{z_1, \ldots, z_m\}$, then A is removable for B^2, by Theorem 8.2. Hence Theorem 9.5 shows that A is removable for $\mathscr{A}^2(\Omega \setminus A)$ if and only if A is removable for B^2, and Theorem 13.2 yields the result.

For $0 < p < 2$ only the empty set is removable for B^p, by Theorem 13.2. By Theorem 9.5 we see that A is removable for $\mathscr{A}^p(\Omega \setminus A)$ if and only if $\mathbb{C} \setminus (\Omega \setminus A) = A_2 = \{z_1, \ldots, z_m\}$ and $\sum_{k=1}^{m} n_{z_k} < n_\infty$. By Proposition 11.5, $n_{z_k} = \lceil 2/p \rceil - 1$, $z_k \neq \infty$, and $n_\infty = \lfloor 2/p \rfloor + 1$. In particular, $n_{z_1} < 2/p < n_\infty$, and hence A is removable for $\mathscr{A}^p(\Omega \setminus A)$ if $\operatorname{card} A_2 \leqslant 1$. Furthermore, $n_{z_1} + n_{z_2} - n_\infty = \lceil 2/p \rceil + (\lceil 2/p \rceil - \lfloor 2/p \rfloor) - 3 < 0$ if and only if $\lceil 2/p \rceil = 2$ and $2/p \in \mathbb{Z}$, i.e. $p = 1$. So, if $p \neq 1$ and $\operatorname{card} A_2 = 2$, then A is not removable for $\mathscr{A}^p(\Omega \setminus A)$.

Finally, for $p = 1$ we have $n_z = 1$ for $z \neq \infty$ and $n_\infty = 3$. Hence A is removable for $\mathscr{A}^1(\Omega \setminus A)$ if and only if $\operatorname{card} A_2 \leqslant 2$. □

Proposition 13.4. *Let* $\mathrm{d}\mu = w \, \mathrm{d}m$ *and assume that for every* $z \in A$ *there is a neighbourhood of* z *in which* w *is bounded from above and below (away from zero). Then* A *is removable for* B^p_w *if and only if* A *is removable for* B^p.

Remarks. Recall that weak and strong removability are the same in this case, by Proposition 8.1.

If Ω satisfies condition (2.1), a direct consequence is that A is removable for $\mathscr{A}^p_w(\Omega \setminus A)$ if and only if A is removable for $\mathscr{A}^p(\Omega \setminus A)$.

Much of the theory of weighted Bergman spaces has been developed with weights locally bounded from above and below, and the problem of removable singularities has the same solution as in the unweighted case.

P r o o f. Assume that A is removable for B^p_w. Let $z \in A$ be arbitrary, and let $\Omega_z \subset \Omega$ be a domain in which w is bounded from above and below. Then $\Omega_z \cap A$ is removable for B^p_w. Thus $B^p(\Omega_z \setminus A) = B^p_w(\Omega_z \setminus A) \subset \operatorname{Hol}(\Omega_z)$, and $\operatorname{cap}_{B^p}(A \cap \Omega_z) = 0$. It follows from Proposition 4.14 that A is removable for B^p.

The proof of the other direction is similar. □

In this section we also want to observe that the spaces $\mathscr{A}^p(\,\cdot\,)$ are not conformally invariant (not even for bounded domains). However, despite this, removability is conformally invariant for B^p. For $1 < p < \infty$ this can also be concluded from the conformal invariance of $\operatorname{cap}_{p'}$, see, e.g., Väisälä [31], together with Theorem 12.4, but as we shall see below it is much easier to prove than that.

Example 13.5. Let $0 < \theta < 2\pi$, $D_\theta = \{z \in \mathbb{D}\colon\ 0 < \arg z < \theta\}$ and $f_\alpha(z) = z^{-\alpha} = \mathrm{e}^{-\alpha \log z}$, where we choose any branch of log containing $D_{2\pi}$. Then $f_\alpha \in \mathscr{A}^p(D_\theta)$ if and only if

$$\infty > \int_{D_\theta} |z|^{-\alpha p}\, \mathrm{d}m(z) = \theta \int_0^1 r^{1-\alpha p}\, \mathrm{d}r,$$

i.e. if and only if $\alpha < 2/p$.

Let now $0 < \theta < \pi$ and $\varphi\colon D_\theta \to D_{2\theta}$, $\varphi(z) = z^2$. As already observed $f_\alpha \in \mathscr{A}^p(D_{2\theta})$ if and only if $\alpha < 2/p$. However, $f_\alpha \circ \varphi(z) = z^{-2\alpha}$, so $f_\alpha \circ \varphi \in \mathscr{A}^p(D_\theta)$ if and only $2\alpha < 2/p$, i.e. $\alpha < 1/p$. The conclusion is that $\mathscr{A}^p$ is not conformally invariant for any p, $0 < p < \infty$.

Proposition 13.6. *Let $\Omega \subset \mathbb{C}$ be a domain and let $\varphi\colon \Omega \to \mathbb{C}$ be a conformal mapping. Then A is removable for B^p if and only if $\varphi(A)$ is removable for B^p.*

Proof. Since φ^{-1} is also a conformal mapping it is enough to show that if A is removable for B^p, then so is $\varphi(A)$. Assume therefore that A is removable for B^p.

Let $K' \subset \varphi(A)$ be compact. Since φ is conformal $K := \varphi^{-1}(K')$ is also compact. We can therefore find a bounded domain Ω' with $K \subset \Omega' \Subset \Omega$. As φ' is continuous and non-zero on $\overline{\Omega'}$ there exists $C > 0$ such that $|\varphi'(z)| \geqslant C$ for all $z \in \Omega'$.

Now φ gives a one-to-one correspondence between $\mathrm{Hol}(\Omega' \setminus K)$ and $\mathrm{Hol}(\varphi(\Omega') \setminus K')$. Let $f \circ \varphi^{-1}$ be an arbitrary function in $\mathscr{A}^p(\varphi(\Omega') \setminus K') = \mathscr{A}^p(\varphi(\Omega' \setminus K))$, then

$$\infty > \int_{\varphi(\Omega' \setminus K)} |f \circ \varphi^{-1}(w)|^p\, \mathrm{d}m(w) = \int_{\Omega' \setminus K} |f(z)|^p |\varphi'(z)|^2\, \mathrm{d}m(z) \geqslant C^2 \|f\|_{L^p(\Omega' \setminus K)}.$$

Thus $f \in \mathscr{A}^p(\Omega' \setminus K) \subset \mathrm{Hol}(\Omega')$, and hence $f \circ \varphi^{-1} \in \mathrm{Hol}(\varphi(\Omega'))$. Since f was arbitrary this shows that K' is removable for B^p. We conclude from Proposition 4.7 that $\varphi(A)$ is removable for B^p. □

14. Counterexamples

In the following example we will show what can happen when weak and strong removability do not coincide. Before giving the example we give a lemma that will be useful in this section.

Lemma 14.1. *Let $0 < p < \infty$. Let $\mathrm{d}\mu = w\ \mathrm{d}m + d\sigma$, where $w(z) = |\mathrm{Im}\, z|^{p-1}\tilde{w}(z)$ and for every $a \in \mathbb{R} \setminus \{0\}$, $\tilde{w}$ is bounded from above and below (away from zero) in a neighbourhood of a. (The measure σ is an arbitrary positive Radon measure.) Then $\{a\}$, $a \in \mathbb{R} \setminus \{0\}$, is removable for B^p_μ. Moreover, if $E \subset \mathbb{R} \setminus \{0\}$ is a countable set with no finite non-zero limit point, then E is weakly removable for B^p_μ.*

Proof. Proposition 11.1 and a simple calculation show that $n_{a,w} = 0$, $a \in \mathbb{R} \setminus \{0\}$, and hence $\{a\}$ is removable for B^p_w. Since $B^p_\mu(\Omega \setminus \{a\}) \subset B^p_w(\Omega \setminus \{a\}) \subset \mathrm{Hol}(\Omega)$, it follows directly that $\{a\}$ is removable for B^p_μ. Proposition 4.6 then shows that E is weakly removable for B^p_μ. □

Example 14.2. Let $0 < p < \infty$, $w(z) = \min\{|\mathrm{Im}\, z|^{p-1}|\mathrm{Re}\, z|^{p-1}, |z|^{-3}\}$ and

$$\mathrm{d}\mu = w\ \mathrm{d}m + \sum_{n=0}^{\infty} \frac{1}{2^{np}} (\delta_{2^{-n}} + \delta_{-2^{-n}}).$$

Let $E_1 = \{2^{-n}\colon\ n \in \mathbb{N}\}$, $E_2 = -E_1$, $K_1 = E_1 \cup \{0\}$, $K_2 = E_2 \cup \{0\}$ and $K = K_1 \cup K_2$.

Lemma 14.1 shows that the sets E_1, E_2 and $E_1 \cup E_2$ all are weakly removable for B^p_μ. However, $z^{-1} \in B^p_w(\mathbb{D} \setminus \{0\})$, so $\{0\}$ is not removable for B^p_w.

Let $f \in B^p_\mu(\mathbb{S} \setminus K) \subset \mathrm{Hol}(\mathbb{S} \setminus \{0\})$. Since $\|f\|_{L^p_\mu(\mathbb{S}\setminus K)} = \|f\|_{L^p_w(\mathbb{S}\setminus K)} = \|f\|_{L^p_w(\mathbb{S}\setminus\{0\})}$, we can use Proposition 11.1 to show that $B^p_\mu(\mathbb{S} \setminus K) = \{b + cz^{-1}\colon\ b, c \in \mathbb{C}\}$. It is now easy to see that $z^{-1} \notin B^p_\mu(\mathbb{S} \setminus K_1)$ and $z^{-1} \notin B^p_\mu(\mathbb{S} \setminus K_2)$, so that $B^p_\mu(\mathbb{S} \setminus K_1) = B^p_\mu(\mathbb{S} \setminus K_2) = B^p_\mu(\mathbb{S} \setminus \{0\}) = B^p_\mu(\mathbb{S}) = \{f\colon\ f \text{ is constant}\}$. This shows that none of E_1, E_2 and $E_1 \cup E_2$ is strongly removable for $B^p_\mu(\mathbb{S} \setminus K)$, despite all of them being weakly removable for B^p_μ.

This also shows that K_1 and K_2 are both removable for B^p_μ, but $K = K_1 \cup K_2$ is not removable for B^p_μ, cf. Proposition 4.10 and Theorem 8.2. Since $E_1 \subset K_1$, we also see that it is not possible to replace weak removability by strong removability in Proposition 4.8. Furthermore, E_1 is strongly removable for $B^p_\mu(\mathbb{S} \setminus K_1)$, so strong removability is dependent on the domain, cf. Remark 4.1.

Finally, E_1 is weakly removable for $B^p_\mu(\mathbb{S} \setminus K)$ and K_2 is weakly removable for $B^p_\mu(\mathbb{S} \setminus K_2)$, but $E_1 \cup K_2 = K$ is not weakly removable for $B^p_\mu(\mathbb{S} \setminus K)$, which shows that strong removability cannot be replaced by weak removability in Proposition 3.3 for B^p_μ.

Since $\mu(\mathbb{S}) < \infty$ we have $\mathscr{A}^p_\mu(\Omega) = B^p_\mu(\Omega) = B^p_{\mu,\mathrm{fin}}(\Omega)$ for all domains $\Omega \subset \mathbb{S}$. Hence all of the above discussion also applies to $\mathscr{A}^p_\mu$ and $B^p_{\mu,\mathrm{fin}}$. Since $B^p_{\mu,\mathrm{bdd}}(\Omega) = B^p_\mu(\Omega)$ when Ω is bounded or $\infty \in \Omega$, the above discussion also applies to $B^p_{\mu,\mathrm{bdd}}$.

We next want to show that it is possible for some of the above properties to fail, without all of them failing. It follows, however, from Theorem 8.2 that if any of these properties fail, then weak removability is different from strong removability.

Example 14.3. Let $0 < p < \infty$, $w(z) = \min\{|\operatorname{Im} z|^{p-1}|\operatorname{Re} z|^{2p-1}, |z|^{-3}\}$ and

$$\mathrm{d}\mu = w\ \mathrm{d}m + \sum_{n=0}^{\infty} \frac{1}{2^{2np}}(\delta_{2^{-n}} + \delta_{-2^{-n}}).$$

Let $E_1 = \{2^{-n} \colon n \in \mathbb{N}\}$, $E_2 = -E_1$, $K_1 = E_1 \cup \{0\}$, $K_2 = E_2 \cup \{0\}$ and $K = K_1 \cup K_2$. Lemma 14.1 shows that the sets E_1, E_2 and $E_1 \cup E_2$ all are weakly removable for B^p_μ.

Let $f \in B^p_\mu(\mathbb{S} \setminus K) \subset \operatorname{Hol}(\mathbb{S} \setminus \{0\})$. Since $\|f\|_{L^p_\mu(\mathbb{S}\setminus K)} = \|f\|_{L^p_w(\mathbb{S}\setminus K)} = \|f\|_{L^p_w(\mathbb{S}\setminus\{0\})}$, we can use Proposition 11.1 to show that $B^p_\mu(\mathbb{S} \setminus K) = \{b + cz^{-1} + \mathrm{d}z^{-2} \colon b, c, d \in \mathbb{C}\}$. After that it is easy to see that $B^p_\mu(\mathbb{S} \setminus K_1) = B^p_\mu(\mathbb{S} \setminus K_2) = B^p_\mu(\mathbb{S} \setminus \{0\}) = \{b + cz^{-1} \colon b, c \in \mathbb{C}\}$. This shows that none of E_1, E_2 and $E_1 \cup E_2$ is strongly removable for $B^p_\mu(\mathbb{S} \setminus K)$, despite all of them being weakly removable for B^p_μ. Moreover, E_1 is strongly removable for $B^p_\mu(\mathbb{S} \setminus K_1)$, so strong removability is dependent on the domain.

Let now $K' \subset \mathbb{C}$ be a totally disconnected compact set. If $0 \in K'$, then $z^{-1} \in B^p_\mu(\mathbb{S} \setminus K')$, so K' is not removable for B^p_μ. On the other hand, if $0 \notin K'$, then we can find a bounded domain $\Omega \supset K'$, with $\operatorname{dist}(0, \Omega) > 0$. It follows that $B^p_\mu(\Omega \setminus K') = B^p_w(\Omega \setminus K')$ and similarly $B^p_\mu(\Omega) = B^p_w(\Omega)$. Hence K' is removable for B^p_μ if and only if K' is removable for B^p_w and $0 \notin K'$.

Let now $K'_1, K'_2 \subset \mathbb{C}$ be compact sets removable for B^p_μ, and hence totally disconnected sets removable for B^p_w and not containing 0. It follows from Theorem 8.2 that $K'_1 \cup K'_2$ is removable for B^p_w, and hence for B^p_μ, by the discussion above.

As in the previous example the above discussion also applies to $\mathscr{A}^p_\mu$, $B^p_{\mu,\mathrm{fin}}$ and $B^p_{\mu,\mathrm{bdd}}$.

In the next two examples we will look at the differences between strong removability for B^p_μ, $B^p_{\mu,\mathrm{fin}}$, $B^p_{\mu,\mathrm{bdd}}$ and $\mathscr{A}^p_\mu$.

Example 14.4. Let $0 < p < \infty$, $w(z) = \min\{1, |z|^{-p-3}\} \min\{1, |\operatorname{Im} z|^{p-1}\}$ and

$$\mathrm{d}\mu = w\ \mathrm{d}m + \sum_{k=1}^{\infty} a_k \delta_k, \quad \text{where } a_k = \begin{cases} 1/k^p, & k \equiv 1 \pmod 2, \\ 1, & k \equiv 2 \pmod 4, \\ k^p, & k \equiv 0 \pmod 4. \end{cases}$$

Lemma 14.1 shows that $\mathbb{N}$ is weakly removable for B^p_μ, $B^p_{\mu,\mathrm{fin}}$, $B^p_{\mu,\mathrm{bdd}}$ and $\mathscr{A}^p_\mu(\mathbb{C} \setminus \mathbb{N})$ ($\{0\}$ is removable!).

Furthermore, $B^p_{\mu,\mathrm{bdd}}(\mathbb{C} \setminus \mathbb{N}) \subset \operatorname{Hol}(\mathbb{C}) = B^p_{\mu,\mathrm{bdd}}(\mathbb{C})$, since any function in $\operatorname{Hol}(\mathbb{C})$ is bounded on bounded sets. Hence $\mathbb{N}$ is strongly removable for $B^p_{\mu,\mathrm{bdd}}(\mathbb{C} \setminus \mathbb{N})$. On the other hand, $z \in B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus \mathbb{N}) = B^p_\mu(\mathbb{C} \setminus \mathbb{N}) = \mathscr{A}^p_\mu(\mathbb{C} \setminus \mathbb{N})$, but $z \notin B^p_{\mu,\mathrm{fin}}(\mathbb{C})$,

$z \notin B^p_\mu(\mathbb{C})$ and $z \notin \mathscr{A}^p_\mu(\mathbb{C})$. Hence $\mathbb{N}$ is not strongly removable for $B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus \mathbb{N})$, $B^p_\mu(\mathbb{C} \setminus \mathbb{N})$ nor for $\mathscr{A}^p_\mu(\mathbb{C} \setminus \mathbb{N})$.

Let next $f \in B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus \mathbb{N}) = B^p_\mu(\mathbb{C} \setminus \mathbb{N}) = \mathscr{A}^p_\mu(\mathbb{C} \setminus \mathbb{N}) \subset \mathrm{Hol}(\mathbb{C})$, $n \in \mathbb{N} \setminus \{0\}$, $|z_0| = n + \frac{1}{2}$, $D = D\big(z_0, \frac{1}{2}\big)$, $q = \max\{2, 2p\}$ and $1/q + 1/q' = 1$. Using the subharmonicity of $|f|^{p/q}$ we find that

$$
\begin{aligned}
|f(z_0)|^{p/q} &\leqslant \frac{4}{\pi} \int_D |f(z)|^{p/q} \,\mathrm{d}m(z) \\
&= \frac{4}{\pi} \int_D |f(z)|^{p/q} |\mathrm{Im}\, z|^{(p-1)/q} |\mathrm{Im}\, z|^{(1-p)/q} \,\mathrm{d}m(z) \\
&\leqslant \frac{4}{\pi} \bigg(\int_D |f(z)|^p |\mathrm{Im}\, z|^{p-1} \,\mathrm{d}m(z) \bigg)^{1/q} \bigg(\int_D |\mathrm{Im}\, z|^{(1-p)q'/q} \,\mathrm{d}m(z) \bigg)^{1/q'} \\
&\leqslant \frac{4}{\pi} |2z_0|^{(p+3)/q} \|f\|^{1/q}_{L^p_\mu(\mathbb{C}\setminus\mathbb{N})} \bigg(\int_D |\mathrm{Im}\, z|^{(1-p)q'/q} \,\mathrm{d}m(z) \bigg)^{1/q'}.
\end{aligned}
$$

The latter integral has a bound independent of z_0, which shows that f does not have an essential singularity at ∞. Since, $\mu(\mathbb{C} \setminus 2\mathbb{N}) < \infty$ and $z \notin B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus 2\mathbb{N})$, we have that $B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus 2\mathbb{N}) = B^p_\mu(\mathbb{C} \setminus 2\mathbb{N}) = \mathscr{A}^p_\mu(\mathbb{C} \setminus 2\mathbb{N}) = \{f\colon f \text{ is constant}\}$. Moreover, $1 \in B^p_{\mu,\mathrm{fin}}(\mathbb{C})$, but $1 \notin B^p_\mu(\mathbb{C}) = \mathscr{A}^p_\mu(\mathbb{C})$. Hence $2\mathbb{N}$ is strongly removable for $B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus 2\mathbb{N})$, but not for $B^p_\mu(\mathbb{C} \setminus 2\mathbb{N})$, nor for $\mathscr{A}^p_\mu(\mathbb{C} \setminus 2\mathbb{N})$.

Finally, let $\Omega = \mathbb{C} \setminus \overline{\mathbb{D}}$ and $E = 4\mathbb{N} \setminus \{0\}$. Lemma 14.1 shows that E is weakly removable for $B^p_\mu(\Omega \setminus E) = \mathscr{A}^p_\mu(\Omega \setminus E)$. Let $f \in B^p_\mu(\Omega \setminus E) = \mathscr{A}^p_\mu(\Omega \setminus E)$, then $f(z) = \sum_{j=1}^{\infty} a_j z^{-j}$, $|z| > 1$. Since $z^{-1} \in B^p_\mu(\Omega)$, it follows that $f \in B^p_\mu(\Omega)$, and hence E is strongly removable for $B^p_\mu(\Omega \setminus E)$. On the other hand, $z^{-1} \in \mathscr{A}^p_\mu(\Omega \setminus E)$, but $z^{-1} \notin \mathscr{A}^p_\mu(\Omega)$, so E is not strongly removable for $\mathscr{A}^p_\mu(\Omega \setminus E)$.

By Remarks 4.2 we see that strong removability for $\mathscr{A}^p_\mu(\Omega \setminus E)$ does not imply strong removability for any of the spaces $B^p_\mu(\Omega \setminus E)$, $B^p_{\mu,\mathrm{fin}}(\Omega \setminus E)$ or $B^p_{\mu,\mathrm{bdd}}(\Omega \setminus E)$. In the next example we will show that strong removability for $B^p_\mu(\Omega \setminus E)$ does not imply strong removability for $B^p_{\mu,\mathrm{fin}}(\Omega \setminus E)$.

The author has not been able to determine whether it is true or false that if E is strongly removable for $B^p_\mu(\Omega \setminus E)$ (or $B^p_{\mu,\mathrm{fin}}(\Omega \setminus E)$), then E is strongly removable for $B^p_{\mu,\mathrm{bdd}}(\Omega \setminus E)$. Though, since $B^p_{\mu,\mathrm{bdd}}(\mathbb{C}) = \mathrm{Hol}(\mathbb{C})$, a counterexample would have to be of a little different nature.

Example 14.5. Let $0 < p < \infty$, $w(z) = \min\{1, |z|^{-p-3}\} \min\{1, |\mathrm{Im}\, z|^{p-1}\}$ and

$$
\mathrm{d}\mu = w \,\mathrm{d}m + \sum_{k=1}^{\infty} \delta_{-k} + \frac{1}{k^p} \delta_k.
$$

Let also $E_1 = \mathbb{N} \setminus \{0\}$, $E_2 = -E_1$ and $E = \mathbb{Z} \setminus \{0\}$. Lemma 14.1 shows that E is weakly removable for $B^p_\mu(\mathbb{C} \setminus E)$ and $B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus E)$.

Since $z \in \mathscr{A}^p_w(\mathbb{C})$ and any subset of E_2 with $\mu(E_2) < \infty$ is bounded, $z \in B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus E_1)$. On the other hand, $\mu(\mathbb{C} \setminus E_2) < \infty$ and $z \notin \mathscr{A}^p_\mu(\mathbb{C} \setminus E_2)$, so $z \notin B^p_{\mu,\mathrm{fin}}(\mathbb{C})$, i.e. E_1 is not strongly removable for $B^p_{\mu,\mathrm{fin}}(\mathbb{C} \setminus E_1)$.

As in the previous example a function in $B^p_\mu(\mathbb{C} \setminus E_1) \subset \mathrm{Hol}(\mathbb{C})$ does not have an essential singularity at ∞. Since $\mathbb{C} \setminus E_1$ satisfies condition (2.1), it is easy to see that $1 \notin B^p_\mu(\mathbb{C} \setminus E_1)$. Hence $B^p_\mu(\mathbb{C} \setminus E_1) = \{0\}$, and E_1 is strongly removable for $B^p_\mu(\mathbb{C} \setminus E_1)$.

In the next example we show that Proposition 6.3 does not hold with $B^p_{\mu,\mathrm{fin}}$ replaced by B^p_μ.

Example 14.6. Let $0 < p < q < \infty$ be such that $q/p \in \mathbb{N}$. Let further $w(z) = \min\{1, |z|^{-p-3}\} \min\{1, |\mathrm{Im}\, z|^{p-1}\}$,

$$d\mu = w \; dm + \sum_{j=1}^{\infty} j^p \delta_j,$$

$E = \mathbb{N} \setminus \{0\}$ and $\Omega = \mathbb{C}$. Lemma 14.1 shows that E is weakly removable for B^p_μ.

We will first show that E is strongly removable for $B^p_\mu(\mathbb{C} \setminus E)$. Let $f \in B^p_\mu(\mathbb{C} \setminus E) \subset \mathrm{Hol}(\mathbb{C})$. Assume that $\Omega' \subset \mathbb{C}$ is a domain that satisfies (2.1) for p. Then,

$$\infty > \sum_{j \in \Omega' \cap E} j^p \frac{1}{j^p} = \sum_{j \in \Omega' \cap E} 1 = \mathrm{card}(\Omega' \cap E).$$

It follows that $\Omega' \cap E$ is compact and hence strongly removable for $B^p_\mu(\Omega' \setminus E)$. Hence $f \in B^p_\mu(\mathbb{C} \setminus E) \subset B^p_\mu(\Omega' \setminus E) = B^p_\mu(\Omega')$. It follows that $f \in B^p_\mu(\mathbb{C})$, which shows that E is strongly removable for $B^p_\mu(\mathbb{C} \setminus E)$.

Furthermore $\mathbb{C}$ satisfies condition (2.1) for q. Now $\|1\|_{L^q_\mu(\mathbb{C} \setminus E)} < \infty = \|1\|_{L^q_\mu(\mathbb{C})}$. Hence $1 \in B^q_\mu(\mathbb{C} \setminus E)$, but $1 \notin B^q_\mu(\mathbb{C})$, from which it follows that E is not strongly removable for $B^q_\mu(\mathbb{C} \setminus E)$.

In the next example we show that Proposition 6.3 does not hold when $q/p \notin \mathbb{N}$.

Example 14.7. Let $0 < p < \infty$, $0 < q < \infty$ and let $q/p = N + \varepsilon$, where $0 < \varepsilon < 1$. Let also

$$d\mu = w \; dm + \sum_{n=0}^{\infty} \gamma_n \delta_{2^{-n}}, \quad \text{where } w(z) = \begin{cases} |\mathrm{Im}\, z|^{p-1} |\mathrm{Re}\, z|^{Np-1}, & \text{if } |z| < 1, \\ |z|^{-3}, & \text{if } |z| \geqslant 1, \end{cases}$$

and where γ_n will be specified below. Let further $E = \{2^{-n} \colon n \in \mathbb{N}\}$ and $K = E \cup \{0\}$. Note that $\mathscr{A}^p_\mu(\mathbb{S} \setminus K) = B^p_\mu(\mathbb{S} \setminus K) = B^p_{\mu,\mathrm{fin}}(\mathbb{S} \setminus K) = B^p_{\mu,\mathrm{bdd}}(\mathbb{S} \setminus K)$ and similarly with p replaced by q or $\mathbb{S} \setminus K$ replaced by any domain containing ∞.

It follows from Lemma 14.1 that E is weakly removable for $B^p_\mu(\mathbb{S}\setminus K)$. It is now straightforward to verify that $z^{-m} \in B^r_\mu(\mathbb{S}\setminus K)$, $m \in \mathbb{N}$, if and only if $m < (N+1)p/r$. Let $m_r = \lceil (N+1)p/r - 1\rceil$. Thus $z^{-m} \in B^r_\mu(\mathbb{S}\setminus K)$ if and only if $m \leqslant m_r$.

Now $pm_p = p\lceil N\rceil = Np$ and

$$qm_q = q\left\lceil \frac{(N+1)p}{q} - 1\right\rceil = (N+\varepsilon)p\left\lceil \frac{N+1}{N+\varepsilon} - 1\right\rceil \geqslant (N+\varepsilon)p > pm_p.$$

Thus we can find γ_n such that

$$\sum_{n=1}^{\infty} \gamma_n 2^{npm_p} < \infty, \tag{14.1}$$

$$\sum_{n=1}^{\infty} \gamma_n 2^{nqm_q} = \infty. \tag{14.2}$$

If follows from (14.1) that E is strongly removable for $B^p_\mu(\mathbb{S}\setminus K)$, whereas from (14.2) it follows that E is not strongly removable for $B^q_\mu(\mathbb{S}\setminus K)$.

15. Isometrically removable sets

We also have a related definition of removability.

Definition 15.1. The set A is *isometrically removable* for $\mathscr{A}^p_\mu(\Omega\setminus A)$ if A is weakly removable for $\mathscr{A}^p_\mu(\Omega\setminus A)$ and

$$\|f\|_{L^p_\mu(\Omega\setminus A)} = \|f\|_{L^p_\mu(\Omega)} \quad \text{for all } f \in \mathscr{A}^p_\mu(\Omega\setminus A). \tag{15.1}$$

Remark. Isometric removability is a stronger requirement than strong removability.

Proposition 15.2. *The set A is isometrically removable for $\mathscr{A}^\infty_\mu(\Omega\setminus A)$ if and only if A is removable for $\mathscr{A}^\infty_\mu(\Omega\setminus A)$.*

P r o o f. This follows directly from the proof of Theorem 5.2. □

Proposition 15.3. *Assume that $0 < p < \infty$. Then a sufficient condition for A to be isometrically removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ is that $\mu(A) = 0$ and A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$. If $\mu(\Omega \setminus A) < \infty$, then the condition is also necessary.*

If Ω satisfies condition (2.1) and $\mu(G) = 0$ for all sets $G \subset \Omega$ with $\dim_H G \leqslant 1$, then A is isometrically removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ if and only if A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.

P r o o f. We start with the sufficiency for the first part. Assume that A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$ and that $\mu(A) = 0$. Let $f \in \mathscr{A}^p_\mu(\Omega \setminus A) \subset \operatorname{Hol}(\Omega)$. Since $\mu(A) = 0$, we have $\|f\|^p_{L^p_\mu(\Omega)} = \|f\|^p_{L^p_\mu(\Omega \setminus A)}$. Hence A is isometrically removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.

For the necessity in the first part we assume that $\mu(\Omega \setminus A) < \infty$ and that A is isometrically removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$. Since $1 \in \mathscr{A}^p_\mu(\Omega \setminus A)$, we have

$$\mu(\Omega \setminus A) = \|1\|^p_{L^p_\mu(\Omega \setminus E)} = \|1\|^p_{L^p_\mu(\Omega)} = \mu(\Omega) = \mu(\Omega \setminus A) + \mu(A).$$

As $\mu(\Omega \setminus A) < \infty$ we have $\mu(A) = 0$. Moreover, A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$.

As for the second part assume that A is weakly removable for $\mathscr{A}^p_\mu(\Omega \setminus A) = B^p_\mu(\Omega \setminus A)$. By Corollary 6.2 we have $\dim_H A \leqslant 1$, and hence $\mu(A) = 0$. From the first part we conclude that A is isometrically removable for $\mathscr{A}^p_\mu(\Omega \setminus A)$. The converse follows directly from Definition 15.1. $\square$

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