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CONVERGENCE OF ISHIKAWA ITERATES FOR A MULTI-VALUED MAPPING WITH A FIXED POINT

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Abstract. Existence of fixed points of multivalued mappings that satisfy a certain contractive condition was proved by N. Mizoguchi and W. Takahashi. An alternative proof of this theorem was given by Peter Z. Daffer and H. Kaneko. In the present paper, we give a simple proof of that theorem. Also, we define Mann and Ishikawa iterates for a multivalued map T with a fixed point p and prove that these iterates converge to a fixed point q of T under certain conditions. This fixed point q may be different from p. To illustrate this phenomenon, an example is given.

Keywords: multi-valued map, Mann iterates, Ishikawa iterates, fixed points

MSC 2000: 47H10, 54H25.

1. Introduction

Let X be a Hilbert space, C a compact, convex subset of X and T a selfmap on C. For $x_0 \in C$, the sequence of Ishikawa iterates of T are defined by (Ishikawa [3]),

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n], \quad n \geqslant 0 \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the conditions

(i) $0 \leqslant \alpha_n, \beta_n < 1$ for all n,

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$ and

(iii) $\sum \alpha_n\beta_n = \infty$.

The Mann iterates of T are defined by $x_0 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geqslant 0 \tag{2}$$

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where $\{\alpha_n\}$ is a real sequence satisfying $\alpha_0 = 1$, $0 \leqslant \alpha_n < 1$ for all $n \geqslant 1$ and $\sum \alpha_n = \infty$.

For more details and literature on the convergence of Ishikawa and Mann iterates, we refer to Franks and Marzek [2], Ishikawa [3], Kalinde and Rhoades [4], Liu Qihou [8], [9], Mann [6] and Rhoades [10].

The main purpose of this paper is to extend the convergence results from single valued maps to multi-valued maps by defining Ishikawa and Mann iterates for multi-valued maps with a fixed point.

Existence of fixed points for multi-valued maps that satisfy a certain contractive condition was proved by N. Mizoguchi and W. Takahashi ([7], Theorem 5). An altenative proof of this theorem was given by Daffer and Kaneko ([1], Theorem 2.1). In Section 2, we give a simplified version of the proof of that theorem. In Section 2, we also present necessary lemmas to prove our main theorems in Section 3.

2. Preliminaries

Let (X, d) be a complete metric space. A subset K of X is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = d(x, K) = \inf\{d(x, y)\colon\ y \in K\}.$$

If X is a Hilbert space, clearly, every closed convex subset of X is proximinal.

We denote the family of all bounded proximinal subsets of a set K in X by $P(K)$ and the family of all nonempty bounded closed subsets of X by $CB(X)$.

Let A, B be two bounded subsets of X. The Hausdorff distance between A and B is defined by

$$H(A, B) = max\Big\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\Big\}.$$

Theorem 1 was proved by Mizoguchi and Takahashi ([7], Theorem 5) and an alternative proof of this theorem was given by Peter Z. Daffer and Hideaki Kaneko ([1], Theorem 2.1). The proof given here is comparatively simpler than the other two proofs.

Theorem 1 (Theorem 2.1 of [1]). *Let (X, d) be a complete metric space and $T\colon X \to CB(X)$. If α is a function from $(0, \infty)$ to $[0, 1)$ such that $\limsup_{r\to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$ and if*

$$H(Tx, Ty) \leqslant \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Choose a positive integer n_1 such that

$$\alpha^{n_1}(d(x_0, x_1)) < \{(1 - \alpha(d(x_0, x_1))\}d(x_0, x_1).$$

We now select $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_2, x_1) &\leqslant H(Tx_1, Tx_0) + \alpha^{n_1}(d(x_0, x_1)) \\ &\leqslant \alpha(d(x_0, x_1))d(x_0, x_1) + \alpha^{n_1}(d(x_0, x_1)) \\ &< d(x_1, x_0). \end{aligned}$$

Now we choose a positive integer $n_2 > n_1$ so that

$$\alpha^{n_2}(d(x_2, x_1)) < \{1 - \alpha(d(x_2, x_1))\}d(x_2, x_1).$$

Now select $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_3, x_2) &\leqslant H(Tx_2, Tx_1) + \alpha^{n_2}(d(x_2, x_1)) \\ &\leqslant \alpha(d(x_2, x_1))d(x_2, x_1) + \alpha^{n_2}(d(x_2, x_1)) \\ &< d(x_2, x_1). \end{aligned}$$

In general, we can select a positive integer n_k such that

$$\alpha^{n_k}(d(x_k, x_{k-1})) < \{1 - \alpha(d(x_k, x_{k-1}))\}d(x_k, x_{k-1}).$$

We now choose $x_{k+1} \in Tx_k$ so that

$$\begin{aligned} d(x_{k+1}, x_k) &\leqslant H(Tx_k, Tx_{k-1}) + \alpha^{n_k}(d(x_k, x_{k-1})) \\ &\leqslant \alpha(d(x_k, x_{k-1}))d(x_k, x_{k-1}) + \alpha^{n_k}(d(x_k, x_{k-1})) \\ &< d(x_k, x_{k-1}). \end{aligned} \tag{3}$$

Write $t_k = d(x_{k+1}, x_k)$. Then $\{t_k\}$ is a monotonically decreasing sequence of non-negative numbers.

Suppose $\lim\limits_{k\to\infty} t_k = \delta > 0$. From (3) we have

$$t_k \leqslant \alpha^{n_k}(t_{k-1}) + \alpha(t_{k-1})t_{k-1}.$$

Taking limits as $k \to \infty$ on both sides we have

$$\delta \leqslant \limsup_{k\to\infty} \alpha(t_{k-1})\delta < \delta,$$

a contradiction, and hence $\lim\limits_{k\to\infty} t_k = 0$.

Let $a = \limsup_{r\to 0^+} \alpha(r) < 1$. Let b be such that $a < b < 1$. Then for sufficiently large k, $\alpha(t_k) < b$. From (3), we have for sufficiently large k, $t_k < bt_{k-1} + b^{n_k}$. Consequently

$$t_k < b^k t_0 + (b^{k-1}b^{n_1} + b^{k-2}b^{n_2} + \ldots + bb^{n_{k-1}}) + b^{n_k}.$$

Write

$$c_{k-1} = b^{k-1}b^{n_1} + b^{k-2}b^{n_2} + \ldots + bb^{n_{k-1}}.$$

Since c_{k-1} is the $(k-1)$th term in the Cauchy product of the absolutely convergent series $\sum b^k$ and $\sum b^{n_k}$, it follows that $\sum c_k$ is convergent absolutely and hence $\sum t_k$ is convergent so that $\{x_n\}$ forms a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to q (say) in X. Then

$$\begin{aligned} d(q, Tq) &\leqslant d(q, x_k) + d(x_k, Tq) \\ &\leqslant d(q, x_k) + H(Tx_{k-1}, Tq) \\ &\leqslant d(q, x_k) + \alpha(d(x_{k-1}, q))d(x_{k-1}, q) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence $q \in Tq$.

The following lemma can be easily established.

Lemma 2. *Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences such that*

(i) $0 \leqslant \alpha_n$, $\beta_n < 1$,

(ii) $\beta_n \to 0$ *as* $n \to \infty$ *and*

(iii) $\sum \alpha_n\beta_n = \infty$.

Let $\{\gamma_n\}$ be a non-negative real sequence such that $\sum \alpha_n\beta_n(1-\beta_n)\gamma_n$ is bounded. Then γ_n has a subsequence which converges to zero.

Lemma 3 (Liu Qihou, [9]). *Let $\{x_n\}$ be a sequence of reals that satisfy $x_{n+1} \leqslant \alpha x_n + \beta_n$ where $x_n \geqslant 0$, $\beta_n \geqslant 0$ and $\lim_{n\to\infty} \beta_n = 0$, $0 \leqslant \alpha < 1$. Then $\lim_{n\to\infty} x_n = 0$.*

Notation. In the rest of the paper, X is real Hilbert space.

Let $T\colon X \to P(X)$ and let p be a fixed point of T. We define

(A) *the sequence of Mann iterates* by $x_0 \in X$,

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \quad n \geqslant 0$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$ and $\{\alpha_n\}$ is a real sequence such that $0 \leqslant \alpha_n < 1$ and $\sum \alpha_n = \infty$.

(B) *the sequence of Ishikawa iterates* by $x_0 \in X$, $y_n = (1-\beta_n)x_n + \beta_n z_n$ where $z_n \in Tx_n$ is such that $\|z_n - p\| = d(p, Tx_n)$ and $x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n$ where $z'_n \in Ty_n$ is such that $\|z'_n - p\| = d(p, Ty_n)$, $n \geqslant 0$ where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying
  (i) $0 \leqslant \alpha_n, \beta_n < 1$,
  (ii) $\beta_n \to 0$ and
  (iii) $\sum \alpha_n \beta_n = \infty$.

Definitions. We say that the mapping $T\colon X \to P(X)$ is
  (I) *nonexpansive* if $H(Tx, Ty) \leqslant \|x - y\|$ for all $x, y \in X$.
  (II) *quasi-nonexpansive* if it has at least one fixed point p (say) and satisfies

$$d(Tx, p) \leqslant \|x - p\|$$

for all $x \in X$.
  (III) *generalized nonexpansive* if

$$H(Tx, Ty) \leqslant a\|x - y\| + b\{d(x, Tx) + d(y, Ty)\} + c\{d(x, Ty) + d(y, Tx)\}$$

for all $x, y \in X$, where $a + 2b + 2c \leqslant 1$.
  (IV) *quasi-contractive* if for some constant k, $0 \leqslant k < 1$,

$$H(Tx, Ty) \leqslant \max\{\|x - y\|, d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

Lemma 4 (Ishikawa [3]). *Let $\mu \in [0,1]$. Then for any $x, y \in X$,*

$$\|(1-\mu)x + \mu y\|^2 = (1-\mu)\|x\|^2 + \mu\|y\|^2 - \mu(1-\mu)\|x - y\|^2.$$

3. Main results

Theorem 5. *Let K be a compact convex subset of a Hilbert space X. Suppose that a nonexpansive map $T\colon K \to P(K)$ has a fixed point p. Assume that*
(i) $0 \leqslant \alpha_n$, $\beta_n < 1$,
(ii) $\beta_n \to 0$ *and*
(iii) $\sum \alpha_n \beta_n = \infty$. *Then the sequence of Ishikawa iterates defined by* (B) *converges to a fixed point q of T.*

P r o o f. By using Lemma 4, we have

$$
\begin{aligned}
(4)\quad \|x_{n+1}-p\|^2 &= \|(1-\alpha_n)x_n+\alpha_n z'_n-p\|^2\\
&= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|z'_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-z'_n\|^2\\
&\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n H^2(Ty_n,Tp)-\alpha_n(1-\alpha_n)\|x_n-z'_n\|^2\\
&\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-z'_n\|^2,
\end{aligned}
$$

$$
\begin{aligned}
(5)\quad \|y_n-p\|^2 &= \|(1-\beta_n)x_n+\beta_n z_n-p\|^2\\
&\leqslant (1-\beta_n)\|x_n-p\|^2+\beta_n\|z_n-p\|^2-\beta_n(1-\beta_n)\|x_n-z_n\|^2\\
&\leqslant (1-\beta_n)\|x_n-p\|^2+\beta_n H^2(Tx_n,Tp)-\beta_n(1-\beta_n)\|x_n-z_n\|^2\\
&\leqslant (1-\beta_n)\|x_n-p\|^2+\beta_n\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-z_n\|^2\\
&= \|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-z_n\|^2.
\end{aligned}
$$

From (4) and (5),

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|x_n-p\|^2-\alpha_n\beta_n(1-\beta_n)\|x_n-z_n\|^2\\
-\alpha_n(1-\alpha_n)\|x_n-z'_n\|^2.
\end{aligned}
$$

Therefore

$$\alpha_n\beta_n(1-\beta_n)\|x_n-z_n\|^2+\alpha_n(1-\alpha_n)\|x_n-z'_n\|^2\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

This implies

$$\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\|x_n-z_n\|^2\leqslant\|x_1-p\|^2<\infty.$$

Hence by Lemma 2, there exists a subsequence $\{x_{n_l}-z_{n_l}\}$ of $\{x_n-z_n\}$ such that $\|x_{n_l}-z_{n_l}\|\to 0$ as $l\to\infty$. Since $z_{n_l}\in Tx_{n_l}$, $d(Tx_{n_l},x_{n_l})\leqslant\|x_{n_l}-z_{n_l}\|\to 0$ as $l\to\infty$ and $\{x_{n_l}\}\subset K$, K being compact, without loss of generality, we may assume that $x_{n_l}\to q$ as $l\to\infty$. Now $d(Tx_{n_l},q)\leqslant d(Tx_{n_l},x_{n_l})+\|x_{n_l}-q\|\to 0$ as $l\to\infty$. Also $H(d(Tx_{n_l},Tq))\to 0$ as $l\to\infty$. Hence

$$d(q,Tq)\leqslant d(q,Tx_{n_l})+H(Tx_{n_l},Tq)\to 0\ \text{ as }\ l\to\infty.$$

This shows that $q\in Tq$. Hence the theorem follows. □

Theorem 6. *Let K be a compact convex subset of a Hilbert space X. Suppose $T\colon K\to P(K)$ is nonexpansive and has a fixed point p. Assume that*

(i) $0\leqslant\alpha_n<1$ *and*

(ii) $\sum\alpha_n=\infty$.

Then the sequence of Mann iterates defined by (A) *converges to a fixed point q of T.*

P r o o f is similar to that of Theorem 5. □

Theorem 7. *Let K be a compact convex subset of a Hilbert space X. Suppose $T\colon K \to P(K)$ is a generalized nonexpansive map with a fixed point p. Assume that*

(i) $0 \leqslant \alpha_n, \beta_n < 1$ *for all* n,

(ii) $\beta_n \to 0$ *and*

(iii) $\sum \alpha_n \beta_n = \infty$.

Then the sequence of Ishikawa iterates defined by (B) *converges to a fixed point q of T.*

Proof. We have

$$(6)\qquad \|x_{n+1} - p\|^2 = (1-\alpha_n)\|x_n - p\|^2 + \alpha_n H(Tp, Ty_n) - \alpha_n(1-\alpha_n)\|x_n - z'_n\|^2.$$

By the generalized nonexpansive property of T, we have

$$\begin{aligned} H(Tp, Ty_n) &\leqslant a\|y_n - p\| + bd(y_n, Ty_n) + c\{d(p, Ty_n) + d(y_n, Tp)\} \\ &\leqslant a\|y_n - p\| + b\{\|y_n - p\| + d(p, Ty_n)\} + c\{d(p, Ty_n) + d(y_n, Tp)\} \\ &\leqslant (a+b+c)\|y_n - p\| + (b+c)d(p, Ty_n) \\ &\leqslant (a+b+c)\|y_n - p\| + (b+c)H(Tp, Ty_n). \end{aligned}$$

Hence

$$H(Ty_n, Tp) \leqslant \left(\frac{a+b+c}{1-(b+c)}\right)\|y_n - p\|.$$

Since $(a+b+c)/(1-(b+c)) \leqslant 1$, it follows that

$$(7)\qquad H(Ty_n, Tp) \leqslant \|y_n - p\|.$$

From (6) and (7), we have

$$\|x_{n+1} - p\|^2 \leqslant (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - z'_n\|^2$$

which is the inequality (4).

Similarly, it is trivial to show that the inequality (5) holds,

$$\|y_n - p\|^2 \leqslant \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2.$$

Now proceeding as in the proof of Theorem 5, the theorem follows. □

Theorem 8. *Let K be a compact convex subset of a Hilbert space X. Suppose $T\colon K \to P(K)$ is a generalized nonexpansive map with a fixed point p. Assume that*

(i) $0 \leqslant \alpha_n < 1$ *for all* n *and*

(ii) $\sum \alpha_n = \infty$.

Then the sequence of Mann iterates defined by (A) *converges to a fixed point q of T.*

Proof is similar to that of Theorem 7 and hence omitted. □

In the following theorem, we show that the sequence of Ishikawa iterates for a quasi-contractive map T with a fixed point p on a closed convex bounded subset of a Hilbert space converges to the same fixed point p, thus removing the compactness restriction on K but imposing quasi-contractive condition on T. The technique adopted in the proof is that of Liu Qihou [9].

Theorem 9. *Let K be a closed convex bounded subset of a Hilbert space X. Suppose $T\colon K \to P(K)$ is a quasi-contractive and has a fixed point p. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences such that $0 \leqslant \alpha_n$, $\beta_n < 1$ for all n and $\beta_n \to 0$ as $n \to \infty$ with $\delta \leqslant \alpha_n \leqslant 1-k^2$ for some $\delta > 0$. Then the sequence of Ishikawa iterates defined by* (B) *converges to p.*

Proof. We have

$$\begin{aligned}(8)\quad \|x_{n+1}-p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n z_n' - p\|^2\\ &= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|z_n'-p\|^2 - \alpha_n(1-\alpha_n)\|x_n - z_n'\|^2,\end{aligned}$$

$$\|z_n' - p\| = d(p, Ty_n) \leqslant \max_{z\in Tp} d(z, Ty_n) \leqslant H(Tp, Ty_n).$$

Therefore

$$\begin{aligned}(9)\qquad \|z_n'-p\|^2 \leqslant H^2(Tp,Ty_n) &\leqslant k^2\max\{\|y_n-p\|^2, d^2(y_n,Ty_n), d^2(p,Ty_n)\}\\ &\qquad (\text{since } d^2(y_n,Tp)\leqslant \|y_n-p\|^2).\end{aligned}$$

If $d(p, Ty_n)$ is the maximum, then

$$H^2(Tp,Ty_n) \leqslant k^2 d^2(p,Ty_n) \leqslant k^2 \max_{z\in Tp} d^2(z,Ty_n) \leqslant k^2 H^2(Tp,Ty_n)$$

so that $0 \leqslant \|z_n'-p\|^2 \leqslant H^2(Tp,Ty_n) = 0$. Hence, from (9) we get, always,

$$\begin{aligned}(10)\qquad \|z_n'-p\|^2 \leqslant H^2(Tp,Ty_n) &\leqslant k^2\max\{\|y_n-p\|^2, d^2(y_n,Ty_n)\}\\ &\leqslant k^2[\|y_n-p\|^2 + d^2(y_n,Ty_n)].\end{aligned}$$

Now consider

$$(11)\qquad \|y_n - p\|^2 = \|(1-\beta_n)x_n + \beta_n z_n - p\|^2$$
$$= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2,$$

$$(12)\quad d^2(y_n, Ty_n) \leqslant \|y_n - z_n'\|^2 = \|(1-\beta_n)x_n + \beta_n z_n - z_n'\|^2$$
$$= (1-\beta_n)\|x_n - z_n'\|^2 + \beta_n\|z_n - z_n'\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2.$$

Substituting (11) and (12) into (10), we get

$$(13)\quad \|z_n' - p\|^2 \leqslant k^2[(1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + (1-\beta_n)\|x_n - z_n'\|^2$$
$$+ \beta_n\|z - z_n'\|^2 - 2\beta_n(1-\beta_n)\|x_n - z_n\|^2].$$

Similar to the inequality (10), we get that

$$(14)\qquad \|z_n - p\|^2 = d^2(p, Tx_n) \leqslant H^2(Tp, Tx_n)$$
$$\leqslant k^2[\|x_n - p\|^2 + d^2(x_n, Tx_n)].$$

From (13) and (14) we have

$$(15)\quad \|z_n' - p\|^2 \leqslant k^2[(1-\beta_n)\|x_n - p\|^2 + k^4\beta_n\|x_n - p\|^2 + k^4\beta_n d^2(x_n, Tx_n)$$
$$- 2k^2\beta_n(1-\beta_n)\|z_n - x_n\|^2 + k^2(1-\beta_n)\|x_n - z_n'\|^2$$
$$+ k^2\beta_n\|z_n - z_n'\|^2$$
$$\leqslant k^2(1-\beta_n + k^2\beta_n)\|z_n - p\|^2 - k^2\beta_n(2 - 2\beta_n - k^2)d^2(x_n, Tx_n)$$
$$+ k^2(1-\beta_n)\|x_n - z_n'\|^2 + k^2\beta_n\|z_n - z_n'\|^2$$
$$(\text{since } \|z_n - x_n\|^2 \geqslant d(x_n, Tx_n))$$
$$\leqslant k^2\|x_n - p\|^2 - k^2\beta_n(2 - 2\beta_n - k^2)d^2(x_n, Tx_n)$$
$$+ k^2(1-\beta_n)\|x_n - z_n'\|^2 + k^2\beta_n\|z_n - z_n'\|^2.$$

Substituting (15) in (8), we obtain

$$(16)\quad \|x_{n+1} - p\|^2 \leqslant [1 - \alpha_n(1-k^2)]\|x_n - p\|^2$$
$$- k^2\alpha_n\beta_n(2 - 2\beta_n - k^2)d^2(x_n, Tx_n) + k^2\alpha_n\beta_n\|z_n - z_n'\|^2$$
$$- \alpha_n(1 - \alpha_n - k^2 + k^2\beta_n)\|x_n - z_n'\|^2.$$

As $\delta \leqslant \alpha_n \leqslant 1 - k^2$, we have $1 - \alpha_n(1-k^2) \leqslant 1 - \delta(1-k^2) = \alpha$ (say) and $0 < \alpha < 1$. As $\beta_n \to 0$ as $n \to \infty$, there exists a positive integer N_1 such that $\beta_n \leqslant (2-k^2)/2\ \forall n \geqslant N_1$ so that $2 - 2\beta_n - k^2 \geqslant 0\ \forall n \geqslant N_1$. Also we have

$$1 - \alpha_n - k^2 + k^2\beta_n \geqslant (1-k^2) - (1-k^2) + k^2\beta_n \geqslant 0 \quad \forall n.$$

Consequently from (16), we get that, for sufficiently large n,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant \alpha\|x_n-p\|^2+(1-k^2)k^2\beta_n\|z_n-z_n'\|^2\\ &\leqslant \alpha\|x_n-p\|^2+(1-k^2)k^2\beta_n D\end{aligned}$$

where D is the diameter of K. Now, by using Lemma 3, the sequence $\{x_n\}$ converges to p as $n\to\infty$. Hence the theorem follows. □

We conclude the paper with an example which shows that the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point p and the initial choice of x_0.

Example. Let $X=[0,1]$. Define $T\colon X\to P(X)$ by $Tx=[0,x]$. Here every point of X is a fixed point of T. Let $p,x_0\in[0,1]$. Then
(i) if $p<x_0$, the sequence of Ishikawa iterates converges to p and
(ii) if $x_0\leqslant p$, the sequence of Ishikawa iterates converges to the initial point x_0.

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