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LINEAR LIFTINGS OF SKEW-SYMMETRIC TENSOR FIELDS TO WEIL BUNDLES

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Abstract. We define equivariant tensors for every non-negative integer p and every Weil algebra A and establish a one-to-one correspondence between the equivariant tensors and linear natural operators lifting skew-symmetric tensor fields of type $(p, 0)$ on an n-dimensional manifold M to tensor fields of type $(p, 0)$ on $T^A M$ if $1 \leqslant p \leqslant n$. Moreover, we determine explicitly the equivariant tensors for the Weil algebras $\mathbb{D}_k^r$, where k and r are non-negative integers.

Keywords: natural operator, product preserving bundle functor, Weil algebra

MSC 2000: 53A55, 58A32

Our aim is to give a classification of all linear natural operators lifting skew-symmetric tensor fields of type $(p, 0)$ to tensor fields of type $(p, 0)$ on a Weil bundle T^A. The result of this paper generalizes that of [5], where linear natural operators lifting skew-symmetric tensor fields of type $(2, 0)$ to skew-symmetric tensor fields of type $(2, 0)$ on T^A are studied under a condition imposed on the Weil algebra A. The condition required in [5] seems to be quite restrictive, as the algebras $\mathbb{D}_k^r$ for $k \geqslant 2$ and $r \geqslant 1$ fail to satisfy it. In this paper we will not make any assumptions on A.

Let p be a non-negative integer. We will denote by $\mathrm{te}(M)$ the vector space of tensor fields of type $(p, 0)$ on a manifold M and by $\mathrm{sk}(M)$ the subspace of $\mathrm{te}(M)$ consisting of skew-symmetric tensor fields. Let A be a Weil algebra and T^A the Weil functor corresponding to A, which is a product preserving bundle functor (see [3], [1]). Fix also a non-negative integer n.

A natural operator lifting skew-symmetric tensor fields of type $(p, 0)$ to tensor fields of type $(p, 0)$ on the Weil bundle T^A is, by definition, a system of maps $L_M\colon \mathrm{sk}(M) \longrightarrow \mathrm{te}(T^A M)$ indexed by n-dimensional manifolds and satisfying for all such manifolds M, N, every embedding $f\colon M \longrightarrow N$ and all $t \in \mathrm{sk}(M)$, $u \in \mathrm{sk}(N)$ the following implication

$$\bigwedge^p Tf \circ t = u \circ f \Longrightarrow \bigotimes^p TT^A f \circ L_M(t) = L_N(u) \circ T^A f. \tag{1}$$

Such a natural operator L is said to be linear if L_M is linear for each n-dimensional manifold M.

Our first goal is to construct some natural operators of this kind. The construction will be divided into two parts. We will use equivariant tensors to obtain equivariant maps first, and then equivariant maps to obtain natural operators.

The first part of our construction will be carried out under the condition that $p \geqslant 1$.

Since A is an $\mathbb{R}$-algebra, A-modules can also be treated as $\mathbb{R}$-vector spaces and A-linear maps as $\mathbb{R}$-linear. The functors $\bigotimes^p$, $\bigwedge^p$ may be applied to both categories. Therefore we will use the symbols $\bigotimes^p_A$, $\bigwedge^p_A$ and $\bigotimes^p_{\mathbb{R}}$, $\bigwedge^p_{\mathbb{R}}$ to avoid ambiguity.

Let us denote by $Z_{i,C}$ the map $\bigotimes^p_{\mathbb{R}} A \longrightarrow \bigotimes^p_{\mathbb{R}} A$ induced by $A \times \ldots \times A \ni (X_1, \ldots, X_p) \longrightarrow X_1 \otimes \ldots \otimes X_{i-1} \otimes CX_i \otimes X_{i+1} \otimes \ldots \otimes X_p \in \bigotimes^p_{\mathbb{R}} A$ for $i \in \{1, \ldots, p\}$ and $C \in A$.

Definition. We call a tensor $G \in \bigotimes^p_{\mathbb{R}} A$ equivariant if $Z_{i,C}(G) = Z_{j,C}(G)$ for all $i, j \in \{1, \ldots, p\}$ and every $C \in A$.

Equivariant tensors may be multiplied by elements of A. Indeed, since $p \geqslant 1$, there is $i \in \{1, \ldots, p\}$ and it sufficies to set $CG = Z_{i,C}(G)$ for $C \in A$ and every equivariant tensor G. Since G is equivariant, it is immaterial which i we choose. It is evident that equivariant tensors form an A-module, because $Z_{i,C} \circ Z_{j,D} = Z_{j,D} \circ Z_{i,C}$ for all $i, j \in \{1, \ldots, p\}$ and $C, D \in A$.

We call an $\mathbb{R}$-linear map $H\colon \bigwedge^p_A A^n \longrightarrow \bigotimes^p_{\mathbb{R}} A^n$ equivariant if

$$H \circ \bigwedge^p_A F = \bigotimes^p_{\mathbb{R}} F \circ H \tag{2}$$

for every A-linear $F\colon A^n \longrightarrow A^n$.

Every $X \in \bigotimes^p_A A^n$ can be written as $X^{i_1 \ldots i_p} E_{i_1} \otimes \ldots \otimes E_{i_p}$, where $X^{i_1 \ldots i_p} \in A$ for $i_1, \ldots, i_p \in \{1, \ldots, n\}$ are uniquely determined and $E_1, \ldots, E_n$ stand for the standard basis of the A-module A^n. Of course, $\bigwedge^p_A A^n$ is the subset of $\bigotimes^p_A A^n$ consisting of X with the property that $X^{i_{\sigma(1)} \ldots i_{\sigma(p)}} = \operatorname{sgn} \sigma X^{i_1 \ldots i_p}$ for all $i_1, \ldots, i_p \in \{1, \ldots, n\}$ and every $\sigma \in S_p$, where S_p denotes the set of permutations of $\{1, \ldots, p\}$. If $F\colon A^n \longrightarrow A^n$ is A-linear, then there are $F^i_j \in A$ for $i, j \in \{1, \ldots, n\}$ such that $F(E_j) = F^i_j E_i$ for every $j \in \{1, \ldots, n\}$ and

$$\Big(\bigwedge^p_A F\Big)(X) = F^{i_1}_{j_1} \ldots F^{i_p}_{j_p} X^{j_1 \ldots j_p} E_{i_1} \otimes \ldots \otimes E_{i_p}$$

for every $X \in \bigwedge^p_A A^n$.

The isomorphism $A^n \ni X \longrightarrow X^i \otimes e_i \in A \otimes \mathbb{R}^n$, where $e_1, \ldots, e_n$ stand for the standard basis of the vector space $\mathbb{R}^n$, enables us to identify A^n with $A \otimes \mathbb{R}^n$, and consequently $\bigotimes_{\mathbb{R}}^p A^n$ with $(\bigotimes_{\mathbb{R}}^p A) \otimes (\bigotimes^p \mathbb{R}^n)$. Hence every $X \in \bigotimes_{\mathbb{R}}^p A^n$ can be written as $X^{i_1 \ldots i_p} \otimes e_{i_1} \otimes \ldots \otimes e_{i_p}$, where $X^{i_1 \ldots i_p} \in \bigotimes_{\mathbb{R}}^p A$ for $i_1, \ldots, i_p \in \{1, \ldots, n\}$ are uniquely determined. An easy computation shows that if $F\colon A^n \longrightarrow A^n$ is A-linear, then

$$\Big(\bigotimes_{\mathbb{R}}^p F\Big)(X) = \Big(\Big(Z_{1, F_{j_1}^{i_1}} \circ \ldots \circ Z_{p, F_{j_p}^{i_p}}\Big)(X^{j_1 \ldots j_p})\Big) \otimes e_{i_1} \otimes \ldots \otimes e_{i_p}$$

for every $X \in \bigotimes_{\mathbb{R}}^p A^n$.

Let G be an equivariant tensor. We define $H^G\colon \bigwedge_A^p A^n \longrightarrow \bigotimes_{\mathbb{R}}^p A^n$ by the formula

$$H^G(X) = (X^{i_1 \ldots i_p} G) \otimes e_{i_1} \otimes \ldots \otimes e_{i_p}$$

for $X \in \bigwedge_A^p A^n$. It is easily seen that H^G is an equivariant map. Thus the first part of our construction is complete.

Before we start the second part of our construction we make a few remarks dealing with the symmetry and skew-symmetry of tensors.

Fix $\sigma \in S_p$. We will denote by σ_A the map $\bigotimes_{\mathbb{R}}^p A \longrightarrow \bigotimes_{\mathbb{R}}^p A$ induced by $A \times \ldots \times A \ni (X_1, \ldots, X_p) \longrightarrow X_{\sigma^{-1}(1)} \otimes \ldots \otimes X_{\sigma^{-1}(p)} \in \bigotimes_{\mathbb{R}}^p A$ and by σ_{A^n} the map $\bigotimes_{\mathbb{R}}^p A^n \longrightarrow \bigotimes_{\mathbb{R}}^p A^n$ induced by $A^n \times \ldots \times A^n \ni (X_1, \ldots, X_p) \longrightarrow X_{\sigma^{-1}(1)} \otimes \ldots \otimes X_{\sigma^{-1}(p)} \in \bigotimes_{\mathbb{R}}^p A^n$. Clearly, $\sigma_A \circ Z_{i,C} = Z_{\sigma(i),C} \circ \sigma_A$ for every $i \in \{1, \ldots, p\}$ and every $C \in A$. It follows that for every equivariant tensor G the tensor $\sigma_A(G)$ is also equivariant and the restriction of σ_A to the A-module of equivariant tensors is A-linear. Moreover, it is easily seen that $\sigma_{A^n}(X) = \sigma_A(X^{i_1 \ldots i_p}) \otimes e_{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{i_{\sigma^{-1}(p)}}$ for every $X \in \bigotimes_{\mathbb{R}}^p A^n$. Combining these we get $\sigma_{A^n}(H^G(X)) = \operatorname{sgn} \sigma H^{\sigma_A(G)}(X)$ for every $X \in \bigwedge_A^p A^n$, because $X^{i_{\sigma(1)} \ldots i_{\sigma(p)}} = \operatorname{sgn} \sigma X^{i_1 \ldots i_p}$ for all $i_1, \ldots, i_p \in \{1, \ldots, n\}$. This forces

$$\sigma_{A^n} \circ H^G = \operatorname{sgn} \sigma H^G \Longleftrightarrow \sigma_A(G) = G, \tag{3}$$

$$\sigma_{A^n} \circ H^G = H^G \Longleftrightarrow \sigma_A(G) = \operatorname{sgn} \sigma G, \tag{4}$$

provided $p \leqslant n$.

We now return to our construction and proceed to the second part. Fix an equivariant map H. The task is to construct a natural operator which we will denote by $\widetilde{H}$.

We recall that $A = T^A \mathbb{R}$ and the addition and multiplication in A are obtained by applying T^A to the addition and multiplication in $\mathbb{R}$. Similarly, applying T^A to the addition and multiplication by elements of $\mathbb{R}$ in $\bigwedge^p \mathbb{R}^n$ we obtain an addition and multiplication by elements of A in $T^A \bigwedge^p \mathbb{R}^n$, so it is an A-module. Applying T^A to the canonical map $\mathbb{R}^n \times \ldots \times \mathbb{R}^n \longrightarrow \bigwedge^p \mathbb{R}^n$ we get a skew-symmetric A-p-linar map $A^n \times \ldots \times A^n \longrightarrow T^A \bigwedge^p \mathbb{R}^n$ which induces an A-linear isomorphism $\bigwedge^p_A A^n \longrightarrow T^A \bigwedge^p \mathbb{R}^n$. Therefore we can identify $T^A \bigwedge^p \mathbb{R}^n$ with $\bigwedge^p_A A^n$. Let W be an open subset of $\mathbb{R}^n$ and $f\colon W \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a smooth map such that $f_x\colon \mathbb{R}^n \ni y \longrightarrow f(x,y) \in \mathbb{R}^n$ is linear for every $x \in W$. Then we have $\lambda_f\colon W \times \bigwedge^p \mathbb{R}^n \ni (x,y) \longrightarrow (\bigwedge^p f_x)(y) \in \bigwedge^p \mathbb{R}^n$, and so $T^A\lambda_f\colon T^A W \times \bigwedge^p_A A^n \longrightarrow \bigwedge^p_A A^n$ according to our identification. On the other hand we have $T^A f_X\colon A^n \ni Y \longrightarrow T^A f(X,Y) \in A^n$ which is A-linear for every $X \in T^A W$, as is easy to check, and so we have $\Lambda_{T^A f}\colon T^A W \times \bigwedge^p_A A^n \ni (X,Y) \longrightarrow (\bigwedge^p_A T^A f_X)(Y) \in \bigwedge^p_A A^n$. It is a simple matter to prove that

$$T^A \lambda_f = \Lambda_{T^A f}. \tag{5}$$

Of course, if W is an open subset of $\mathbb{R}^n$, then TW may be interpreted as $W \times \mathbb{R}^n$. Similarly, since $T^A W$ is an open subset of A^n, $TT^A W$ may be interpreted as $T^A W \times A^n = T^A(W \times \mathbb{R}^n)$. Consequently if $f\colon W \longrightarrow \mathbb{R}^n$ is a smooth map, then both $T^A T f$ and $TT^A f$ are maps $T^A W \times A^n \longrightarrow A^n \times A^n$. It is a simple matter to prove that

$$T^A T f = TT^A f. \tag{6}$$

Let M be an n-dimensional manifold and $t \in \mathrm{sk}(M)$. Taking a chart $\varphi\colon U \longrightarrow \mathbb{R}^n$ on M and interpreting $\bigwedge^p T\mathbb{R}^n$ as $\mathbb{R}^n \times \bigwedge^p \mathbb{R}^n$ we have the map $T^A(\bigwedge^p T\varphi \circ t \circ \varphi^{-1})\colon T^A(\varphi(U)) \longrightarrow A^n \times T^A \bigwedge^p \mathbb{R}^n = A^n \times \bigwedge^p_A A^n$ according to our identification. We can also interpret $\bigotimes^p TT^A\mathbb{R}^n$ as $A^n \times \bigotimes^p_{\mathbb{R}} A^n$. Of course, $T^A\varphi\colon T^A U \longrightarrow A^n$ is a chart on $T^A M$. This enables us to define $\widetilde{H}_M(t)$ by the requirement that

$$\bigotimes^p TT^A\varphi \circ \widetilde{H}_M(t) \circ T^A\varphi^{-1} = (\mathrm{id}_{A^n} \times H) \circ T^A\Big(\bigwedge^p T\varphi \circ t \circ \varphi^{-1}\Big)$$

for every chart $\varphi\colon U \longrightarrow \mathbb{R}^n$ on M. A trivial verification shows that taking another chart $\psi\colon V \longrightarrow \mathbb{R}^n$ on M yields the same $\widetilde{H}_M(t)$ on $T^A U \cap T^A V$, which is due to (2), (5) for $f = P \circ T(\psi \circ \varphi^{-1})$, where P stands for the projection $\psi(U \cap V) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and (6) for $f = \psi \circ \varphi^{-1}$. This means that $\widetilde{H}_M(t)$ is well defined and it is easy to show that $\widetilde{H}$ is a linear natural operator. Thus the second part of our construction is complete.

Finally, we have the natural operator $\widetilde{H^G}$ for every equivariant tensor G and we can now formulate our main result.

Theorem. *If $1 \leqslant p \leqslant n$, then for every linear natural operator L lifting skew-symmetric tensor fields of type $(p,0)$ to tensor fields of type $(p,0)$ on the Weil bundle T^A there is a uniquely determined equivariant tensor G such that $L = \widetilde{H^G}$.*

$\widetilde{H^G}$ lifts all skew-symmetric tensor fields to skew-symmetric tensor fields if and only if G is symmetric.

$\widetilde{H^G}$ lifts all skew-symmetric tensor fields to symmetric tensor fields if and only if G is skew-symmetric.

Remark. For $p = 0$ the theorem and the lemma below are not true, but the second part of our construction still works and we have a one-to-one correspondence between the natural operators and the equivariant functions (cf. [4]).

Let $e \in \operatorname{sk}(\mathbb{R}^n)$ be given by $e(x) = (x, e_1 \wedge \ldots \wedge e_p)$ for $x \in \mathbb{R}^n$.

Lemma. *If $1 \leqslant p \leqslant n$ and J, K are two linear natural operators such that $J_{\mathbb{R}^n}(e) = K_{\mathbb{R}^n}(e)$, then $J = K$.*

Proof of Lemma. Since linear natural operators form a vector space, it suffices to prove that if L is a linear natural operator such that $L_{\mathbb{R}^n}(e) = 0$ then $L = 0$. We will write ${T^A}_0\mathbb{R}^n$ for the fibre of $T^A\mathbb{R}^n$ over 0.

Let $\alpha \in (\mathbb{N} \cup \{0\})^n$.

We prove that $L_{\mathbb{R}^n}(e_{\alpha,i})|_{{T^A}_0\mathbb{R}^n} = 0$ for every $i \in \{0, \ldots, p-1\}$, where $e_{\alpha,i} \in \operatorname{sk}(\mathbb{R}^n)$ is given by $e_{\alpha,i}(x) = (x, (x^1)^{\alpha_1} \ldots (x^i)^{\alpha_i} e_1 \wedge \ldots \wedge e_p)$ for $x \in \mathbb{R}^n$. This holds for $i = 0$, because $e_{\alpha,0} = e$. Assume that $i \geqslant 1$ and the formula holds for $i-1$. It is well known that there exist a neighbourhood I of 0 in $\mathbb{R}$ and an embedding $g\colon I \longrightarrow \mathbb{R}$ such that $g(0) = 0$ and $Tg(x,1) = (g(x), 1 + g(x)^{\alpha_i})$ for every $x \in I$. Then (1) for $f = \mathrm{id}_{\mathbb{R}^{i-1}} \times g \times \mathrm{id}_{\mathbb{R}^{n-i}}$, $t = e_{\alpha,i-1}$ and $u = e_{\alpha,i-1} + e_{\alpha,i}$ yields the desired formula.

Let $e_\alpha \in \operatorname{sk}(\mathbb{R}^n)$ be given by $e_\alpha(x) = (x, x^\alpha e_1 \wedge \ldots \wedge e_p)$ for $x \in \mathbb{R}^n$. It is well known that there are a neighbourhood I of 0 in $\mathbb{R}^{n-p+1}$ and an embedding $g\colon I \longrightarrow \mathbb{R}^{n-p+1}$ such that $g(0) = 0$ and $Tg(x, (1, 0, \ldots, 0)) = (g(x), (1 + (g^1(x))^{\alpha_p} \ldots (g^{n-p+1}(x))^{\alpha_n}, 0, \ldots, 0))$ for every $x \in I$. Then (1) for $f = \mathrm{id}_{\mathbb{R}^{p-1}} \times g$, $t = e_{\alpha,p-1}$ and $u = e_{\alpha,p-1} + e_\alpha$ yields $L_{\mathbb{R}^n}(e_\alpha)|_{{T^A}_0\mathbb{R}^n} = 0$.

Let $\beta \in (\mathbb{N} \cup \{0\})^n$, $i_1, \ldots, i_p \in \{1, \ldots, n\}$ be such that $i_1 < \ldots < i_p$ and let $e_{\beta,i_1\ldots i_p} \in \operatorname{sk}(\mathbb{R}^n)$ be given by $e_{\beta,i_1\ldots i_p}(x) = (x, x^\beta e_{i_1} \wedge \ldots \wedge e_{i_p})$ for $x \in \mathbb{R}^n$. Clearly, there are $\tau \in S_n$ such that $\tau(1) = i_1, \ldots, \tau(p) = i_p$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$ such that $\alpha_1 = \beta_{\tau(1)}, \ldots, \alpha_n = \beta_{\tau(n)}$. Then (1) for $f\colon \mathbb{R}^n \ni x \longrightarrow (x^{\tau^{-1}(1)}, \ldots, x^{\tau^{-1}(n)}) \in \mathbb{R}^n$, $t = e_\alpha$ and $u = e_{\beta,i_1\ldots i_p}$ yields $L_{\mathbb{R}^n}(e_{\beta,i_1\ldots i_p})|_{{T^A}_0\mathbb{R}^n} = 0$.

Obviously, for every $t \in \operatorname{sk}(\mathbb{R}^n)$ and every $r \in \mathbb{N}$ there is a polynomial $q \in \operatorname{sk}(\mathbb{R}^n)$ such that $j_0^r t = j_0^r q$. But we have proved that $L_{\mathbb{R}^n}(q)|_{{T^A}_0\mathbb{R}^n} = 0$. Hence the base-extending Peetre theorem (see [3]) gives $L_{\mathbb{R}^n}(t)|_{{T^A}_0\mathbb{R}^n} = 0$. This forces $L = 0$, as is easy to show, and the lemma is proved. $\square$

P r o o f of Theorem. It is seen at once that

$$\widetilde{H^G}_{\mathbb{R}^n}(e)(X) = \Bigl(X, \frac{1}{p!} \sum_{\sigma \in S_p} \operatorname{sgn} \sigma G \otimes e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(p)}\Bigr) \tag{7}$$

for every $X \in A^n$.

Taking $f = h \operatorname{id}_{\mathbb{R}^n}$, where $h \in \mathbb{R} \setminus \{0\}$, $t = e$ and $u = h^p e$ in (1) and letting $h \to 0$ we obtain $L_{\mathbb{R}^n}(e)(X) = L_{\mathbb{R}^n}(e)(0)$ for every $X \in A^n$. Therefore there are $G^{i_1 \ldots i_p} \in \bigotimes_{\mathbb{R}}^p A$ for $i_1, \ldots, i_p \in \{1, \ldots, n\}$ such that

$$L_{\mathbb{R}^n}(e)(X) = \Bigl(X, \frac{1}{p!} G^{i_1 \ldots i_p} \otimes e_{i_1} \otimes \ldots \otimes e_{i_p}\Bigr)$$

for every $X \in A^n$.

Taking $f = \operatorname{id}_{\mathbb{R}^p} \times h \operatorname{id}_{\mathbb{R}^{n-p}}$, where $h \in \mathbb{R} \setminus \{0\}$, $t = e$ and $u = e$ in (1) and letting $h \to 0$ we obtain $G^{i_1 \ldots i_p} = 0$ whenever there is $j \in \{1, \ldots, p\}$ such that $i_j > p$.

Taking $f = \operatorname{id}_{\mathbb{R}^{l-1}} \times h \operatorname{id}_{\mathbb{R}} \times \operatorname{id}_{\mathbb{R}^{n-l}}$, where $l \in \{1, \ldots, p\}$ and $h \in \mathbb{R} \setminus \{0\}$, $t = e$ and $u = he$ in (1) and letting $h \to 0$ we obtain $G^{i_1 \ldots i_p} = 0$ whenever there are $j, k \in \{1, \ldots, p\}$ such that $j \neq k$, $i_j = l$ and $i_k = l$.

Thus if $G^{i_1 \ldots i_p} \neq 0$, then there is $\sigma \in S_p$ such that $\sigma(1) = i_1, \ldots, \sigma(p) = i_p$. Taking $f \colon \mathbb{R}^n \ni x \longrightarrow (x^{\tau^{-1}(1)}, \ldots, x^{\tau^{-1}(n)}) \in \mathbb{R}^n$, where $\tau \in S_n$ is such that $\tau|_{\{1,\ldots,p\}} = \sigma$, $t = e$ and $u = \operatorname{sgn} \sigma e$ in (1) we obtain $G^{1 \ldots p} = \operatorname{sgn} \sigma G^{i_1 \ldots i_p}$. Therefore there is $G \in \bigotimes_{\mathbb{R}}^p A$ such that

$$L_{\mathbb{R}^n}(e)(X) = \Bigl(X, \frac{1}{p!} \sum_{\sigma \in S_p} \operatorname{sgn} \sigma G \otimes e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(p)}\Bigr) \tag{8}$$

for every $X \in A^n$.

Let $i, j \in \{1, \ldots, p\}$ be such that $i \neq j$. Then (1) for

$$f \colon \mathbb{R}^n \ni x \longrightarrow \Bigl(x^1, \ldots, x^{i-1}, x^i + \frac{(x^j)^2}{2}, x^{i+1}, \ldots, x^n\Bigr) \in \mathbb{R}^n,$$

$t = e$ and $u = e$ yields $\bigotimes^p TT^A f(L_{\mathbb{R}^n}(e)(X)) = L_{\mathbb{R}^n}(e)(T^A f(X))$ for every $X \in A^n$. An easy computation shows that

$$\Bigl(\bigotimes^p TT^A f(L_{\mathbb{R}^n}(e)(X))\Bigr)^{1,\ldots,j-1,i,j+1,\ldots,p} = \frac{1}{p!}(Z_{j,X^j}(G) - Z_{i,X^j}(G)),$$

whereas $(L_{\mathbb{R}^n}(e)(T^A f(X)))^{1,\ldots,j-1,i,j+1,\ldots,p} = 0$. But for every $C \in A$ there is $X \in A^n$ such that $X^j = C$, and so $Z_{i,C}(G) = Z_{j,C}(G)$. This means that G is equivariant. The lemma now leads to $L = \widetilde{H^G}$, on account of (7) and (8).

Since the last assertions of the theorem are consequences of (3) and (4), the proof is complete. □

The remainder of this paper will be devoted to an example.

Example. Fix non-negative integers r, k. We recall that the Weil algebra $\mathbb{D}_k^r$ consists of r-jets at 0 of smooth functions $\mathbb{R}^k \longrightarrow \mathbb{R}$ and the addition and multiplication in $\mathbb{D}_k^r$ are induced by the addition and multiplication in the algebra of such functions. Our purpose is to find all equivariant tensors for $\mathbb{D}_k^r$.

Write $|\alpha| = |\alpha_1| + \ldots + |\alpha_k|$ for $\alpha \in \mathbb{Z}^k$ and $I_k^r = \{\alpha \in (\mathbb{N} \cup \{0\})^k \colon |\alpha| \leqslant r\}$. Let J^α be the r-jet at 0 of $\mathbb{R}^k \ni x \longrightarrow x^\alpha \in \mathbb{R}$ for $\alpha \in I_k^r$. Clearly, every $G \in \bigotimes_{\mathbb{R}}^p A$ can be written as

$$\sum_{\alpha \in (I_k^r)^p} G_\alpha J^{\alpha_1} \otimes \ldots \otimes J^{\alpha_p},$$

where $G_\alpha \in \mathbb{R}$ for $\alpha \in (I_k^r)^p$ are uniquely determined. Let $\iota_i \colon I_k^r \longrightarrow (I_k^r)^p$ for $i \in \{1, \ldots, p\}$ be given by $\iota_i(\alpha)_i = \alpha$ and $\iota_i(\alpha)_j = 0$ for $j \in \{1, \ldots, p\}$ such that $i \neq j$ and for $\alpha \in I_k^r$. We claim that if G is an equivariant tensor, then

$$\alpha_i + \beta \in I_k^r \Longrightarrow G_\alpha = \begin{cases} G_{\alpha + \iota_i(\beta) - \iota_j(\beta)}, & \text{if } \alpha_j - \beta \in I_k^r \\ 0, & \text{if } \alpha_j - \beta \notin I_k^r \end{cases} \tag{9}$$

for every $\alpha \in (I_k^r)^p$, every $\beta \in I_k^r$ and all $i, j \in \{1, \ldots, p\}$ such that $i \neq j$. Indeed, (9) is the same as $(Z_{i,J^\beta}(G))_{\alpha + \iota_i(\beta)} = (Z_{j,J^\beta}(G))_{\alpha + \iota_i(\beta)}$.

We first consider the case $k = 1$. Write $|\alpha| = |\alpha_1| + \ldots + |\alpha_p|$ for $\alpha \in (I_1^r)^p$ and

$$D_i = \sum_{\substack{\alpha \in (I_1^r)^p \\ |\alpha| = i}} J^{\alpha_1} \otimes \ldots \otimes J^{\alpha_p}$$

for $i \in \mathbb{Z}$.

If $k = 1$ and $p \geqslant 1$, then D_i for $i \in \{(p-1)r, \ldots, pr\}$ form a basis of the $\mathbb{R}$-vector space of equivariant tensors for $\mathbb{D}_k^r$. Consequently, the dimension of this space equals $r + 1$ and all equivariant tensors are symmetric.

To prove this, observe that if G is an equivariant tensor and $\alpha, \beta \in (I_1^r)^p$ are such that $|\alpha| = |\beta|$, then $G_\alpha = G_\beta$. Indeed, it suffices to use (9) and the induction on $(i, j) \in \{1, \ldots, p\} \times \{0, \ldots, r\}$ with respect to the lexicographic order to show that if $\alpha, \beta \in (I_1^r)^p$ are such that $|\alpha| = |\beta|$, $j = |\beta_i - \alpha_i|$ and $\alpha_l = \beta_l$ for $l \in \{i+1, \ldots, p\}$, then $G_\alpha = G_\beta$. Thus G is a linear combination of D_i for $i \in \{0, \ldots, pr\}$. But if $i \in \{0, \ldots, (p-1)r - 1\}$, then there is $\alpha \in (I_1^r)^p$ such that $|\alpha| = i$, $|\alpha_1| \leqslant r - 1$ and $\alpha_2 = 0$. Then (9) yields $G_\alpha = 0$. Thus G is a linear combination of D_i for $i \in \{(p-1)r, \ldots, pr\}$. On the other hand, D_i for every $i \in \{(p-1)r, \ldots, pr\}$ is an equivariant tensor, because $Z_{j,J^{(1)}}(D_i) = D_{i+1}$ for every $j \in \{1, \ldots, p\}$. This completes the proof. $\square$

It is worth pointing out that $T^{\mathbb{D}^1_1}$ is the tangent bundle functor. Fix an n-dimensional manifold M and $t \in \mathrm{sk}(M)$. Writing

$$t = t^{i_1 \dots i_p}(q) \frac{\partial}{\partial q^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial q^{i_p}}$$

in local coordinates q on M we easily obtain

$$\begin{aligned} \widetilde{H^{D_{p-1}}}_M(t) = & \frac{\partial t^{i_1 \dots i_p}}{\partial q^j}(q) \dot{q}^j \frac{\partial}{\partial \dot{q}^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \dot{q}^{i_p}} \\ & + \sum_{j=1}^{p} t^{i_1 \dots i_p}(q) \frac{\partial}{\partial \dot{q}^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \dot{q}^{i_{j-1}}} \wedge \frac{\partial}{\partial q^{i_j}} \wedge \frac{\partial}{\partial \dot{q}^{i_{j+1}}} \wedge \dots \wedge \frac{\partial}{\partial \dot{q}^{i_p}} \end{aligned}$$

and

$$\widetilde{H^{D_p}}_M(t) = t^{i_1 \dots i_p}(q) \frac{\partial}{\partial \dot{q}^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \dot{q}^{i_p}}$$

in the local coordinates $(q, \dot{q})$ induced by q on TM (cf. [2]).

We now turn to the case $k \geqslant 2$.

If $k \geqslant 2$ and $p \geqslant 2$, then $J^{\alpha_1} \otimes \dots \otimes J^{\alpha_p}$ for $\alpha \in (I^r_k)^p$ with the property that $|\alpha_1| = r$, …, $|\alpha_p| = r$ form a basis of the $\mathbb{R}$-vector space of equivariant tansors for $\mathbb{D}^r_k$. Consequently, the dimensions of this space and its subspaces consisting of symmetric and skew-symmetric tensors equal

$$\binom{r+k-1}{k-1}^p, \quad \binom{\binom{r+k-1}{k-1} + p - 1}{p}, \quad \binom{\binom{r+k-1}{k-1}}{p}.$$

We only need to show that if G is an equivariant tensor and $\alpha \in (I^r_k)^p$, $i \in \{1, \dots, p\}$, are such that $|\alpha_i| \leqslant r - 1$, then $G_\alpha = 0$. Fix $j \in \{1, \dots, p\}$ such that $i \neq j$ and let $\varepsilon_1, \dots, \varepsilon_k$ be the standard basis of the module $\mathbb{Z}^k$. Then (9) and the induction on $q \in \mathbb{N} \cup \{0\}$ show that either $\alpha_j - q\varepsilon_1, \alpha_i - q\varepsilon_2 \in I^r_k$ and $G_\alpha = G_{\alpha + q(\iota_i(\varepsilon_1) - \iota_j(\varepsilon_1) + \iota_j(\varepsilon_2) - \iota_i(\varepsilon_2))}$ for every $q \in \mathbb{N} \cup \{0\}$, which is impossible, or $G_\alpha = 0$.

References

[1] *J. Gancarzewicz, W. Mikulski and Z. Pogoda*: Lifts of some tensor fields and connections to product preserving functors. Nagoya Math. J. *135* (1994), 1–41.

[2] *J. Grabowski and P. Urbański*: Tangent lifts of Poisson and related structures. J. Phys. A *28* (1995), 6743–6777.

[3] *P. Kolář, P. W. Michor and J. Slovák*: Natural Operations in Differential Geometry. Springer-Verlag, Berlin, 1993.

[4] *M. Mikulski*: Natural transformations transforming functions and vector fields to functions on some natural bundles. Math. Bohem. *117* (1992), 217–223.

[5] *W. M. Mikulski*: The linear natural operators lifting 2-vector fields to some Weil bundles. Note Mat. *19* (1999), 213–217.

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