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w^*-BASIC SEQUENCES AND REFLEXIVITY OF BANACH SPACES

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Abstract. We observe that a separable Banach space X is reflexive iff each of its quotients with Schauder basis is reflexive. Similarly if $\mathscr{L}(X,Y)$ is not reflexive for reflexive X and Y then $\mathscr{L}(X_1,Y)$ is is not reflexive for some $X_1 \subset X$, X_1 having a basis.

Keywords: reflexive Banach space, Schauder basis, quotient space, w*-basic sequence, tensor product

MSC 2000: 46B28

Pełczyński [10] proved that Banach space X is reflexive if each subspace with Schauder basis is reflexive. Actually this result stems from the work of [13] which in turn was inspired by the work of [11]. Here we add simple statements which may be considered as natural complements to the results of [11], [13] and [10]. The first one is a statement similar to that of Pełczyński for separable X and quotients instead of subspaces. Namely we observe that a separable Banach space X is reflexive if each of its quotients with Schauder basis is reflexive. From [7] we know that duals of quotient spaces with basis correspond to subspaces of the dual X^* spanned by w*-basic sequences. Thus our statement reads: A separable Banach space is reflexive if every w*-basic sequence in X^* spans a reflexive subspace. We may proceed similarly as in [10] but we use the tools of w*-basic sequences which were not at hand for the authors of [11], [13] and [10]. Similarly we will consider also reflexivity of spaces of bounded operators or equivalently of π-tensor products of reflexive Banach spaces.

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Following [7] we will denote by $[A]$ the norm closed linear span of a set A and by $\tilde{A}$ its w^* closed linear span if $A \subset X^*$. By $A_\circ$ we denote the polar set in X of a set $A \subset X^*$. By a space having a basis we mean a Banach space with a Schauder basis.

A sequence $\{x_n^*\}$ is called w* basic [7], [8], [2] or [3] provided that there is a sequence $\{x_n\} \subset X$ so that $\{x_n, x_n^*\}$ is biorthogonal and for each $x^* \in \widetilde{[x_n^*]}$ we have $\sum_{i=1}^{n} x^*(x_i)x_i^* \xrightarrow{w^*} x^*$.

From [7] we shall use the following two facts:

A) If $\{x_n^*\}$ is w^* basic sequence then the factor space $X/[x_n^*]_\circ$ has a basis and $\widetilde{[x_n^*]}$ can be identified with $(X/[x_n^*]_\circ)^*$.

B) If X is separable then every w^* null sequence $\{x_n^*\} \subset X^*$ which is not norm null has a w^* basic subsequence $\{x_{n_k}\}$.

Finally we recall two results of Holub and Heinrich [4], [5] (and slightly more restrictive [12]) on the reflexivity of the space $\mathscr{L}(X, Y)$:

C) The space of bounded linear operators $\mathscr{L}(X, Y)$ is reflexive if $\mathscr{L}(X, Y) = \mathscr{K}(X, Y)$ and if X and Y are reflexive Banach spaces.

Conversely,

D) If $\mathscr{L}(X, Y)$ is reflexive and if X or Y has the approximation property then $\mathscr{L}(X, Y) = \mathscr{K}(X, Y)$. Of course X and Y are then reflexive spaces.

The statement C) was proved under more restrictive assumptions by Ruckle [12] and in the approximation property free form by [4], [5]. This approximation property free form seems not to be generally known as e.g. the recent paper [9] shows.

Proposition 1. *Let X be a separable Banach space. Then X is reflexive iff each of its quotients which has a basis is reflexive.*

Proof. Only the if part of the proposition is to be established. Thus we shall suppose that X^* is not reflexive i.e. that the closed unit ball B_{X^*} is not weakly compact. The Eberlein-Šmulian theorem yields a sequence $\{x_n^*\}$ in the unit ball B_{X^*} no subsequence of which is weakly converging. Due to the separability of X the closed unit ball B_{X^*} is metrizable in the w^* topology and thus the sequence $\{x_n^*\} \subset B_{X^*}$ has a w^* converging subsequence. For simplicity we will denote this subsequence by $\{x_n^*\}$ again. We may suppose that $x_n^* \xrightarrow{w^*} 0$ (otherwise we take $x_n^* - w^* \lim x_n^*$). By our assumptions the sequence $\{x_n^*\}$ is not norm converging (to zero). The above mentioned result B) of [7] yields a w^* basic subsequence which we shall call $\{x_n^*\}$ again. Having in mind the identification mentioned in A) we see that $\{x_n^*\}$ is in the unit ball of $(X/[x_n^*]_\circ)^* = \widetilde{[x_n^*]}$. Because $\{x_n^*\}$ has no weakly convergent subsequence we conclude that the dual unit ball of $X/[x_n^*]_\circ$ is not weakly compact and thus $X/[x_n^*]_\circ$ is not reflexive. From A) we also know that $X/[x_n^*]_\circ$ has a basis. □

Remark 1. Note that actually we have proved slightly more, namely:

Let X be a separable Banach space. Then X is reflexive iff every w^* basic sequence $\{x_n^*\}$ spans a normed closed reflexive subspace $[x_n^*] \subset X^*$.

Remark 2. We do not know if Proposition 1 holds also without the separability assumption. This general statement would then imply (similarly as also the statement mentioned in A) and B) does) a positive answer to the following question which is still not settled: Has every Banach space a separable quotient space?

Similarly we may consider quotients of the space X by subspaces $A \subset X$ such that A has a basis and the quotient space X/A is not reflexive:

Proposition 2. *Let X be a nonreflexive Banach space. Then there is a subspace $A \subset X$ such that A has a basis and the quotient space X/A is not reflexive.*

Proof is contained in the proof of Lemma 2 in [1] and for the sake of completeness we will list it here: Suppose that X is not reflexive. From the results of Singer [13] and from the above cited result of Pełczyński we conclude that there is a basic sequence $\{x_n\} \subset X$ with $\|x_n\| \geqslant 1$ such that $\left\{\sum_1^p x_n\right\}_p$ is bounded. We put $A = [x_{2n-1}]$ and let P be the quotient map of X onto X/A. Then evidently $\{x_{2n-1}\}$ and $\{P(x_{2n})\}$ are basic sequences, $\{P(x_{2n})\}$ is not a norm null sequence and $\left\{\sum_1^p P(x_{2n})\right\}_p = \left\{\sum_1^{2p} P(x_n)\right\}_p$ is bounded (in p). We conclude [13] that the sequence $\{P(x_{2n})\}$ spans a non reflexive subspace of X/A. □

Next we will consider the reflexivity of the space of bounded operators $\mathscr{L}(X,Y)$:

Proposition 3. *Let X, Y be reflexive Banach spaces and suppose that $\mathscr{L}(X,Y)$ is not reflexive. Then there is a subspace $X_1 \subset X$ such that X_1 has Schauder basis and such that $\mathscr{L}(X_1,Y)$ is not reflexive.*

Proof. Suppose that $\mathscr{L}(X,Y)$ is not reflexive. The result C) mentioned in the introduction yields a noncompact operator $f \in \mathscr{L}(X,Y)$. Let $\{x_n\}$ be a bounded sequence in $\mathscr{L}(X,Y)$ such that $\{f(x_n)\}$ has no norm convergent subsequence. Then $\{x_n\}$ also has no norm convergent subsequence. The reflexivity of the space X implies that there is a subsequence of the sequence $\{x_n\}$ weakly converging to $x \in X$. Let us denote this subsequence again by $\{x_n\}$ and put $z_n = x_n - x$. Then $z_n \xrightarrow{w} 0$. The classical theorem of Pełczyński mentioned in the introduction yields a basic subsequence of the sequence $\{z_n\}$. As above we call this subsequence again $\{z_n\}$ and put $X_1 = [\{\{z_n\} \cup \{x\}\}]$. Then X_1 has a basis. Indeed, if $x \in [z_n]$ then $[x_n] = [z_n]$ and $\{z_n\}$ is a basis of X_1. If $x \notin [z_n]$ then X_1 is the direct sum of $[z_n]$ and the one dimensional subspace spanned by x and thus X_1 again has a basis. In any case $\{x_n\} \subset X_1$. This last inclusion evidently implies that the restriction

$f|_{X_1} \in \mathscr{L}(X_1, Y)$ is not a compact operator. We note that X_1 has the approximation property. Again by the result D) of [4] and [5] mentioned in the introduction we conclude that $\mathscr{L}(X_1, Y)$ is not reflexive. □

Remark 3. Note that we have actually observed the following:

Let X, Y be any Banach spaces and suppose that there is noncompact operator $f\colon X \longrightarrow Y$. Then there is a subspace $X_1 \subset X$ such that X_1 has Schauder basis and such that the restriction $f|_{X_1}$ is a noncompact operator.

Remark 4. Dually Proposition 3 can be formulated as follows:

Let X, Y be reflexive Banach spaces and suppose that $\mathscr{L}(X, Y)$ is not reflexive. Then there is a subspace $Y_1 \subset Y$ such that the quotient space Y/Y_1 has Schauder basis and such that $\mathscr{L}(X, Y/Y_1)$ is not reflexive.

Indeed, if there is noncompact operator $f\colon X \longrightarrow Y$ then $f^* \in \mathscr{L}(Y^*, X^*)$ is also noncompact and thus $\mathscr{L}(Y^*, X^*)$ is nonreflexive. Using now Proposition 3 for $\mathscr{L}(Y^*, X^*)$ we get a subspace $Z \subset Y^*$ having a basis such that $\mathscr{L}(Z, X^*)$ is not reflexive. We put now $Y_1 = Z_\circ$. Evidently $Y/Y_1 = Z^*$ has a basis. Proceeding as above and using now the duality of subspaces and quotients we get our claim.

Remark 5. A slightly more general result then stated in the above remark may also be formulated:

Let X, Y be Banach spaces, let Y be separable and suppose that $\mathscr{L}(X, Y) \neq \mathscr{K}(X, Y)$. Then there is a subspace $Y_1 \subset Y$ such that the factor space Y/Y_1 has Schauder basis and such that $\mathscr{L}(X, Y/Y_1) \neq \mathscr{K}(X, Y/Y_1)$.

Indeed, let $f\colon X \longrightarrow Y$ be a noncompact operator. Proceeding as in the proofs of Propositions 3 and 1 we find a w^* basic sequence $\{y_n^*\} \subset Y^*$ such that the restriction $f^*|_{[y_n^*]}$ is noncompact. Let now $Y_1 = [y_n^*]_\circ$ and let P be the projection of Y onto Y/Y_1. Then evidently $Pf\colon X \longrightarrow Y/Y_1$ is a noncompact operator whose dual is $f^*|_{[y_{n^*}]}$.

Remark 6. Having in mind the basic relation of $\mathscr{L}(X, Y)$ to tensor products, namely $(X \widetilde{\otimes}_\pi Y)^* = \mathscr{L}(X, Y)$ we can reformulate Proposition 3:

Let X, Y be reflexive Banach spaces and suppose that $X \widetilde{\otimes}_\pi Y$ is not reflexive. Then there is a subspace $X_1 \subset X$ such that X_1 has Schauder basis and such that $X_1 \widetilde{\otimes}_\pi Y$ is not reflexive.

Question. The statement listed in Remark 5 suggests the following question:

Suppose that there is a noncompact operator $f\colon X \longrightarrow Y$. Does there exist a noncompact operator $g\colon X \longrightarrow Y_1$ and a subspace $Y_1 \subset Y$, Y_1 having a basis?

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