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GENERALIZED FIRST CLASS SELECTORS FOR UPPER SEMI-CONTINUOUS SET-VALUED MAPS IN BANACH SPACES

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Abstract. In this paper we deal with weakly upper semi-continuous set-valued maps, taking arbitrary non-empty values, from a non-metric domain to a Banach space. We obtain selectors having the point of continuity property relative to the norm topology for a large class of compact spaces as a domain. Exact conditions under which the selector is of the first Borel class are also investigated.

Keywords: measurable selectors, upper semi-continuous maps, point of continuity property

MSC 2000: 46B22

0. Introduction

A map F from a topological space X to the power set of a topological space E is said to be *upper semi-continuous* (usc, for short) if for every $x \in X$ and every open set U in E such that $F(x) \subset U$, there exists an open neighbourhood V of x such that $F(V) \subset U$. When F takes non-empty values, a map $f\colon X \to E$ is said to be a *selector* for F if $f(x) \in F(x)$ for every $x \in X$.

We are primarily interested in the case when E is a Banach space endowed with its weak topology. In this case we shall talk about weakly upper semi-continuous maps to avoid any confusion with the norm topology.

The case of X being a metric space and $F\colon X \to 2^E$ a weak usc map, has been studied by several authors in a series of papers [9], [3], [7], [13]. The definitive result was given by Srivatsa in [13]: If F is as above, then it admits a selector f which is

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a pointwise $\|\cdot\|$-limit of $\|\cdot\|$-continuous functions, i.e., f is in the first Baire class relative to the norm topology. Later, in [8] Srivatsa's result was extended to some duals to Asplund spaces, in the sense that, if $F\colon X \to 2^{(E^*,\mathrm{weak}^*)}$ is upper semi-continuous, then F admits a selector which is of the first Baire class relative to the norm topology.

In [13, Remark 2.7], the author asks whether his main theorem could be extended to non-metric domains, and particularly to the case of Eberlein compact spaces. However, not even the *single-valued* version of the Main Theorem of [13] holds when the domain space is a non-metrizable Eberlein compact. For example, if K is weakly compact, then the identity, $\mathrm{id}\colon (K, \mathrm{weak}) \to (K, \|\cdot\|)$ is Baire 1 only if K is metrizable (compare the footnote on page 622 of [13]). In this paper we give a best possible answer in terms of *generalized* first class selectors by proving the following:

Main Theorem. *Let K be an Eberlein or Gul'ko compact space, E a Banach space and $F\colon K \to 2^E$ a weak upper semi-continuous map taking non-empty arbitrary values. Then F admits a selector f which is, relative to the norm topology on E, $(\mathscr{F} \wedge \mathscr{G})_\sigma$-measurable and has the point of continuity property.*

1. Some definitions and notation

All topological spaces are assumed to be Hausdorff. Let us begin by recalling some definitions. A family $\mathscr{A}$ of subsets of a topological space X is said to be:

- *Scattered* if $\mathscr{A}$ can be well ordered $\mathscr{A} = \{A_\alpha\colon\ \alpha < \lambda\}$ in such a way that there exists a family $\{U_\alpha\colon\ \alpha < \lambda\}$ of open subsets of X such that for all $\alpha < \lambda$

$$A_\alpha \subset U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta.$$

- *Discrete* if for each $x \in X$ there exists $U \subset X$ open with $x \in U$ such that U intersects at most one element from $\mathscr{A}$.
- *Relatively discrete* if $\mathscr{A}$ is discrete in its union with the relative topology.
- *σ-scattered* (resp. *σ-discrete*, *σ-relatively discrete*) if $\mathscr{A} = \bigcup_{n\in\mathbb{N}} \mathscr{N}_n$ where each family $\mathscr{N}_n$ is scattered (resp. discrete, relatively discrete).
- *A network* if each open set in X can be written as a union of members from $\mathscr{A}$.

Scattered collections can also be characterized intrinsically. One can show that a collection $\mathscr{A}$ of subsets of a space X is scattered if, and only if, each non-empty $H \subset \bigcup \mathscr{A}$ has a non-empty relatively open subset of the form $H \cap A$ for some $A \in \mathscr{A}$ (for a proof of this and other properties of scattered collections see § 2 of [6]).

Let $f\colon X \to Y$ be a map, and $\mathscr{B}$ a family of subsets of X. Then $\mathscr{B}$ is said to be a *function base* for f if, whenever V is open in Y, $f^{-1}(V)$ can be written as a union of members from $\mathscr{B}$. Also, f is said to be *$\mathscr{B}$-measurable* if $f^{-1}(V) \in \mathscr{B}$ whenever V is open in Y.

As usual $\mathscr{F}$ will denote the family of closed sets and $\mathscr{G}$ the family of open subsets of a fixed topological space X. By an *$\mathscr{F} \wedge \mathscr{G}$-set* of X we mean a set that is the intersection of a closed with an open set (equivalently, the difference of two closed or two open sets) of X. By an *$(\mathscr{F} \wedge \mathscr{G})_\sigma$-set* we mean a set that is a countable union of $\mathscr{F} \wedge \mathscr{G}$-sets.

By an *H-set* in X we mean a set that is the union of a scattered collection of $\mathscr{F} \wedge \mathscr{G}$-sets of X. Our Lemma 2.1 below shows that our definition of an H-set is equivalent to that used in [2]. It also follows from this lemma that the H-sets form an algebra of subsets of the space, a fact that is critical to the proof of our Main Theorem. Let us also note that, if $\{A_\alpha\colon \alpha < \lambda\}$ is a scattered collection of subsets of X with associated open sets U_α, then $\left\{\overline{A}_\alpha^X \cap U_\alpha\colon \alpha < \lambda\right\}$ (where the bar denotes closure in X) is a scattered collection of $\mathscr{F} \wedge \mathscr{G}$-sets of X. In particular, note that the members of a scattered partition of X are necessarily $\mathscr{F} \wedge \mathscr{G}$-sets in X.

Finally, a map $f\colon X \to Y$ is said to be *$\mathscr{F} \wedge \mathscr{G}$-simple* if X has a σ-scattered partition into $\mathscr{F} \wedge \mathscr{G}$-sets such that f is constant on each element of the partition.

2. Preliminaries

Let us continue with some preliminary lemmas which will lead us to the proof of our Main Theorem.

The following lemma can be viewed as an extension of Lemma 2.2 of [2] where the equivalence of (a) and (b) is established. The equivalence of (b) and (c) in effect shows that any scattered collection of $\mathscr{F} \wedge \mathscr{G}$-sets in a space X can be extended to a scattered partition of X, a fact we use in the proof of Lemma 2.2. From the equivalence of (a) and (c) we see that the H-sets of a space form an algebra of subsets of the space (compare also Proposition 2.1 of [2]).

Lemma 2.1. *For any subset H of a topological space X the following are equivalent.*

(a) *H has the property that for any non-empty $A \subset X$, A has a non-empty relatively open subset $U \subset A$ such that either $U \subset H$ or $U \subset X \setminus H$.*

(b) *There is a scattered partition $\mathscr{P}$ of X such that*

$$H = \bigcup\{P \in \mathscr{P}\colon P \cap H \neq \emptyset\}.$$

(c) *H is an H-set of X (that is, H is the union of a scattered collection of $\mathscr{F}\wedge\mathscr{G}$-sets of X).*

Proof. (a) $\Leftrightarrow$ (b). This is Lemma 2.2 of [2].

(b) $\Rightarrow$ (c). This is obvious since a subcollection of a scattered collection is again scattered, and the members of a scattered partition of a space are necessarily $\mathscr{F}\wedge\mathscr{G}$-sets of the space.

(c) $\Rightarrow$ (a). Let $\mathscr{H} = \{H_\alpha\colon\ \alpha < \lambda\}$ be a scattered collection of $\mathscr{F}\wedge\mathscr{G}$-sets in X, with associated open sets $\{U_\alpha\colon\ \alpha < \lambda\}$, such that $H = \bigcup \mathscr{H}$. Since each H_α is an $\mathscr{F}\wedge\mathscr{G}$-set in X, choose open sets G_α of X such that

$$H_\alpha = \overline{H}_\alpha^X \cap G_\alpha \quad (\alpha < \lambda),$$

where we may (and do) assume that $G_\alpha \subset U_\alpha$. To see that H satisfies (a), let $\emptyset \neq A \subset X$ be given. If $A \subset X \setminus H$, then (a) holds with $U = A$. Otherwise, there exists a least $\alpha < \lambda$ such that $A \cap H_\alpha \neq \emptyset$. Now, if $A \cap (G_\alpha \setminus H_\alpha) = \emptyset$, then

$$\emptyset \neq A \cap G_\alpha \subset H_\alpha \subset H,$$

hence (a) holds with $U = G_\alpha \cap A$. Otherwise,

$$\emptyset \neq A \cap (G_\alpha \setminus H_\alpha) \subset X \setminus H,$$

hence (a) holds with $U = (G_\alpha \setminus H_\alpha) \cap A$. $\square$

Corollary 2.2. *Every σ-scattered cover by $\mathscr{F}\wedge\mathscr{G}$-sets of a space X can be refined to a σ-scattered partition of $\mathscr{F}\wedge\mathscr{G}$-sets.*

Proof. Let $\bigcup\limits_{n\in\mathbb{N}} \mathscr{H}_n$ be a cover of X where each $\mathscr{H}_n$ is a scattered collection of $\mathscr{F}\wedge\mathscr{G}$-sets of X. By (c) $\Rightarrow$ (b) of Lemma 2.1, for each n

$$X \setminus \bigcup \mathscr{H}_n = \bigcup \mathscr{K}_n$$

where $\mathscr{K}_n$ is a scattered collection $\mathscr{F}\wedge\mathscr{G}$-sets of X.

Now we define $\mathscr{L}_1 = \mathscr{H}_1$ and

$$\mathscr{L}_n = \{H \cap K\colon\ H \in \mathscr{H}_n \text{ and } K \in \mathscr{K}_{n-1}\},$$

for $n \geqslant 2$. It follows that $\bigcup\limits_{n\in\mathbb{N}} \mathscr{L}_n$ is a σ-scattered collection of X by $\mathscr{F}\wedge\mathscr{G}$-sets and is a refinement of $\bigcup\limits_{n\in\mathbb{N}} \mathscr{H}_n$. $\square$

Lemma 2.3. *If X is a regular topological space with a σ-scattered network, then X has a σ-scattered network of $\mathscr{F} \wedge \mathscr{G}$-sets. If X has a σ-scattered network of $\mathscr{F} \wedge \mathscr{G}$-sets, then X has a network $\bigcup\limits_{n \in \mathbb{N}} \mathscr{N}_n$ where each $\mathscr{N}_n$ is a scattered partition of X and $\mathscr{N}_{n+1}$ is a refinement of $\mathscr{N}_n$ for each n.*

Proof. Let $\mathscr{N}$ be any scattered collection in X. Then, as noted above, there is a scattered collection of $\mathscr{F} \wedge \mathscr{G}$-sets $\{H_N \colon N \in \mathscr{N}\}$ such that, for each $N \in \mathscr{N}$,

$$N \subset H_N \subset \overline{N}^X.$$

Using this and the fact that each point in a regular space has a base of closed neighbourhoods, it easily follows that X will have a σ-scattered network of $\mathscr{F} \wedge \mathscr{G}$-sets whenever X has a σ-scattered network.

Let $\mathscr{N} = \bigcup\limits_{n \in \mathbb{N}} \mathscr{N}_n$ be a network for X where each $\mathscr{N}_n$ is a scattered collection of $\mathscr{F} \wedge \mathscr{G}$-sets in X. By (c) $\Rightarrow$ (b) of Lemma 2.1 we may assume that each $\mathscr{N}_n$ is a partition of X. Replacing $\mathscr{N}_n$ by the scattered partition

$$\{N_1 \cap N_2 \cap \ldots \cap N_n \colon N_i \in \mathscr{N}_i,\ \ i = 1, 2, \ldots, n\}$$

yields the desired σ-scattered network of X. □

Lemma 2.4. *Let f be a map from a topological space X to a metric space (Y, d). Then the following are equivalent.*

(a) *f has a σ-scattered function base of $\mathscr{F} \wedge \mathscr{G}$-sets of X.*
(b) *There exists a sequence of $\mathscr{F} \wedge \mathscr{G}$-simple maps $f_n \colon X \to (Y, d)$ which converges uniformly to f.*

Proof. (a) $\Rightarrow$ (b). Let $\mathscr{B}$ be a σ-scattered function base for f consisting of $\mathscr{F} \wedge \mathscr{G}$-sets in X. Fix $n \in \mathbb{N}$. By the property of a function base there exists a σ-scattered collection $\mathscr{B}_n \subset \mathscr{B}$ which is a refinement of the family

$$\bigl\{f^{-1}(B(y, \tfrac{1}{2n})) \colon y \in Y\bigr\},$$

where $B(y, \varepsilon)$ denotes the open ball about y of radius ε in (Y, d).

By Corollary 2.2, $\mathscr{B}_n$ can be refined by a σ-scattered partition $\mathscr{K}_n$ of $\mathscr{F} \wedge \mathscr{G}$-sets. For each non-empty $K \in \mathscr{K}_n$ fix a point $x_K \in K$ and define $f_n \colon X \to Y$ by

$$f_n(x) = f(x_K) \quad \text{for each } x \in K.$$

It follows that f_n is $\mathscr{F} \wedge \mathscr{G}$-simple and, for each $x \in K$, $d(f(x), f_n(x)) < 1/n$. Thus, the maps f_n converge uniformly to f as required.

(b) $\Rightarrow$ (a). For each $n \in \mathbb{N}$, let $\mathscr{H}_n$ be a σ-scattered partition of X and let $f_n\colon X \to Y$ be a map that is constant on each member of $\mathscr{H}_n$. Suppose the sequence $\langle f_n \rangle$ converges uniformly to the map f. It suffices to show that $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is a function base for f. Thus let V be an open subset of Y and consider any $x_0 \in X$ such that $f(x_0) \in V$. Let $\varepsilon > 0$ be such that the ball $B(f(x_0); \varepsilon) \subset V$. We need to show that for some $H \in \mathscr{H}$ we have $x_0 \in H$ and $f(H) \subset V$.

For the given ε there exists $m \in \mathbb{N}$ such that $d(f_m(x), f(x)) < \frac{1}{2}\varepsilon$ for any $x \in X$. Now choose $H \in \mathscr{H}_m$ such that $x_0 \in H$. To see that $H \subset f^{-1}(V)$, let $x \in H$ and note that, since f_m is constant on H, we have

$$\begin{aligned} d(f(x_0), f(x)) &\leqslant d(f(x_0), f_m(x_0)) + d(f_m(x_0), f_m(x)) + d(f_m(x), f(x)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $f(x) \in B(f(x_0); \varepsilon) \subset V$ as required. $\square$

The following lemma is similar to Lemma 2.1 of [13] and Theorem 5 of [8] and makes crucial use of the fact that the H-sets form an algebra of the space.

Lemma 2.5. *Let (X, τ) be a topological space with a σ-scattered network of $\mathscr{F} \wedge \mathscr{G}$-sets, let $(E, \|\cdot\|)$ be a Banach space, and let $F\colon X \to 2^E$ be a weak upper semi-continuous set-valued map with non-empty values. Then, for every $\varepsilon > 0$, there exists a $\mathscr{F} \wedge \mathscr{G}$-simple map $f_\varepsilon\colon X \to E$ such that*

$$\|\cdot\|\text{-dist}(f_\varepsilon(x), F(x)) < \varepsilon \quad \textit{for all } x \in X.$$

P r o o f. Let $\mathscr{N} = \bigcup_{n \in \mathbb{N}} \mathscr{N}_n$ be a network for X of the type given in Lemma 2.3 so that each $\mathscr{N}_n$ is a partition of X. For each $n \in \mathbb{N}$ and $N \in \mathscr{N}_n$, fix $x_N \in N$ and $y_N \in F(x_N)$ arbitrarily and define $\mathscr{F} \wedge \mathscr{G}$-simple maps $f_n\colon X \to E$ by defining

$$f_n(x) = y_N \quad \text{for all } x \in N \in \mathscr{N}_n.$$

Let $\langle h_n \rangle$ denote the sequence of all finite rational linear combinations of the sequence $\langle f_n \rangle$. One can easily see that each of the maps h_n is also $\mathscr{F} \wedge \mathscr{G}$-simple. For each $n \in \mathbb{N}$ let

$$A_n = \{x \in X\colon \overline{B}(h_n(x); \varepsilon) \cap F(x) \neq \emptyset\},$$

where $\overline{B}(y; \varepsilon) = \{z \in E\colon \|z - y\| \leqslant \varepsilon\}$.

Our first objective is to show that $X = \bigcup_{n \in \mathbb{N}} A_n$. Fix $x \in X$. The point x belongs to exactly one $N_n \in \mathscr{N}_n$ ($n \in \mathbb{N}$), thus $\{N_n\colon n \in \mathbb{N}\}$ forms a local network at x, and it follows that $\langle x_{N_n}\rangle$ τ-converges to x. Now, if $F(x)$ were contained in the complement of the weak closure of $\{f_n(x)\colon\ n\in\mathbb{N}\}$ then, by the upper semi-continuity of F, for some m we would have $F(N_m)$ contained in the complement as well. But this would contradict the fact that $f_m(x)=y_{N_m}\in F(x_{N_m})$. Thus the sequence $\langle f_n(x)\rangle$ has a weak cluster point in $F(x)$, and therefore there is a subsequence of $\langle h_n(x)\rangle$ converging in the norm to a point in $F(x)$. It follows that for some $n\in\mathbb{N}$ we have $\overline{B}(h_n(x);\varepsilon)\cap F(x)\neq\emptyset$, thus proving that $X=\bigcup_{n\in\mathbb{N}} A_n$.

As h_n is $\mathscr{F}\wedge\mathscr{G}$-simple, X has a σ-scattered partition $\mathscr{M}_n$ into $\mathscr{F}\wedge\mathscr{G}$-sets such that h_n takes a constant value, say y_M, on each $M\in\mathscr{M}_n$. Thus

$$A_n=\bigcup_{M\in\mathscr{M}_n} M\cap\bigl\{x\in X\colon\ \overline{B}(y_M;\varepsilon)\cap F(x)\neq\emptyset\bigr\}.$$

As F is weakly usc, $\bigl\{x\in X\colon\ \overline{B}(y_M;\varepsilon)\cap F(x)\neq\emptyset\bigr\}$ is closed in X, and it follows that each A_n is an H-set in X. Since by Lemma 2.1 the H-sets form an algebra we can find disjoint H-sets $B_n\subset A_n$ so that $X=\bigcup_{n\in\mathbb{N}} B_n$. Defining

$$f_\varepsilon(x)=h_n(x)\quad\text{whenever }\ x\in B_n,$$

it follows that f_ε is a $\mathscr{F}\wedge\mathscr{G}$-simple map and $\|\cdot\|$-dist$(f_\varepsilon(x),F(x))\leqslant\varepsilon$, for each $x\in X$. □

For convenience, let us call a partition of a space X *amenable* if it is σ-scattered and consists of $\mathscr{F}\wedge\mathscr{G}$-sets in X. The following lemma isolates the technical part of the proof of Theorem 3.1 below.

Lemma 2.6. *Let X be a topological space with a σ-scattered network of $\mathscr{F}\wedge\mathscr{G}$-sets, and let $(E,\|\cdot\|)$ be a Banach space. Suppose given an amenable partition $\mathscr{H}$ of X, weakly closed sets $\{B_H\colon\ H\in\mathscr{H}\}$, and a set-valued map $F\colon\ X\to 2^E$ such that, for every $H\in\mathscr{H}$, the restriction $F\big|H$ is weakly usc and $F(x)\cap B_H\neq\emptyset$ for all $x\in H$. Then, for any $\varepsilon>0$, there exists an amenable partition $\mathscr{M}$ of X which is a refinement of $\mathscr{H}$, a set-valued map $G\colon\ X\to 2^E$, and a $\mathscr{F}\wedge\mathscr{G}$-simple map $g\colon\ X\to E$ such that, for each $M\in\mathscr{M}$ with $M\subset H\in\mathscr{H}$, the following hold:*

(i) *for each $x\in M$*

$$G(x)=\begin{cases}\{g(x)\} & \textit{if }\ g(x)\in F(x),\\ F(x)\cap B_H\cap\overline{B}(g(x);\varepsilon) & \textit{otherwise};\end{cases}$$

(ii) *there exists at least one $x_M\in M$ such that $G(x_M)=\{g(x_M)\}$;*

(iii) *$\|\cdot\|$-dist$(g(x),G(x))\leqslant\varepsilon$ for all $x\in M$;*

(iv) *the restriction $G\big|M$ is weakly usc.*

P r o o f. For each $H \in \mathscr{H}$ the assumptions of Lemma 2.5 apply to the set-valued map $F_H\colon H \to 2^E$ where $F_H(x) = F(x) \cap B_H$ for each $x \in H$, hence there is an $\mathscr{F} \wedge \mathscr{G}$-simple map $f_H\colon H \to E$ such that $\|\cdot\|$-dist$(f_H(x), F_H(x)) < \frac{1}{2}\varepsilon$ for each $x \in H$. Let $\mathscr{M}_H$ be an amenable partition of H associated with f_H and note that each member of $\mathscr{M}_H$ is an $\mathscr{F} \wedge \mathscr{G}$-set in X since H has this property. For each $M \in \mathscr{M}_H$ fix

$$x_M \in M \quad \text{and} \quad y_M \in F(x_M) \cap B_H \cap \overline{B}(f_H(x_M); \tfrac{1}{2}\varepsilon),$$

and define

$$M^+ = \{x \in M\colon y_M \in F(x)\} \quad \text{and} \quad M^- = M \setminus M^+.$$

Since $F\big|H$ is weakly usc, M^+ is closed in M and so both M^+ and M^- are $\mathscr{F} \wedge \mathscr{G}$-sets in X. Now let

$$\mathscr{M} = \bigcup_{H \in \mathscr{H}} \bigcup_{M \in \mathscr{M}_H} \{M^+, M^-\}$$

and define $g\colon X \to E$ and $G\colon X \to 2^E$ as follows: If $x \in M \in \mathscr{M}_H$, then

$$g(x) = y_M \quad \text{and} \quad G(x) = \begin{cases} \{y_M\} & x \in M^+ \\ F(x) \cap B_H \cap \overline{B}(y_M; \varepsilon) & x \in M^-. \end{cases}$$

Note that, if f_H takes the fixed value w_M on $M \in \mathscr{M}_H$, then

$$y_M \in \overline{B}(w_M; \tfrac{1}{2}\varepsilon) \quad \text{and} \quad \|\cdot\|\text{-dist}(w_M, F(x) \cap B_H) < \tfrac{1}{2}\varepsilon$$

for all $x \in M$, so

$$F(x) \cap B_H \cap \overline{B}(y_M; \varepsilon) \neq \emptyset \quad \text{for all } x \in M.$$

The remainder of the property (i) follows from the above definitions and the fact that each member of $\mathscr{M}$ has either the form M^+ or M^- for some $M \in \mathscr{M}_H$.

The property (ii) follows from the fact that $x_M \in M^+$ for each $M \in \mathscr{M}_H$ and $H \in \mathscr{H}$. One easily verifies that $\mathscr{M}$ is an amenable partition of X that is a refinement of $\mathscr{H}$ and that g is an $\mathscr{F} \wedge \mathscr{G}$-simple map associated with $\mathscr{M}$.

Finally, (iii) holds since $G(x) \subset \overline{B}(g(x); \varepsilon)$ for each $x \in X$, and (iv) follows immediately from the definition of G. $\square$

3. Selection theorems for upper semicontinuous set-valued maps

Now we are able to prove our main result.

Theorem 3.1. *Let X be a topological space with a σ-scattered network of $\mathscr{F}\wedge\mathscr{G}$-sets, and let $(E,\|\cdot\|)$ be a Banach space. If $F\colon X\to 2^{(E,\text{weak})}$ is an upper semicontinuous set-valued map with non-empty values, then F has a selector that is a norm uniform limit of $\mathscr{F}\wedge\mathscr{G}$-simple maps defined on X.*

Proof. Let $\mathscr{N}=\bigcup\limits_{n\in\mathbb{N}}\mathscr{N}_n$ be a network for X of the type given in Lemma 2.3. We will call the triple $\langle G,g,\mathscr{M}\rangle$ a partial selector for F of order ε associated with $\{B_H\colon H\in\mathscr{H}\}$ if the components G, g and $\mathscr{M}$ have the properties ascribed to them in the conclusion of Lemma 2.6. Note that if $\mathscr{N}$ is any amenable partition that is a refinement for $\mathscr{M}$, then $\langle G,g,\mathscr{N}\rangle$ is a partial selector for F of order ε associated with $\{B_H\colon H\in\mathscr{H}\}$ whenever $\langle G,g,\mathscr{M}\rangle$ is.

We begin by applying Lemma 2.6 to F with $\mathscr{H}=\{X\}$ and $B_X=X$ to obtain a partial selector $\langle F_1,f_1,\mathscr{M}_1\rangle$ of F of order 2^{-1}, and we may assume that $\mathscr{M}_1$ is a refinement of $\mathscr{N}_1$. Applying Lemma 2.6 again to F, the amenable partition $\mathscr{M}_1$, and the weakly closed sets $B_{M_1}=\overline{B}(y_{M_1};\varepsilon)$, where y_{M_1} is the fixed value taken by f_1 on $M_1\in\mathscr{M}_1$, we obtain a partial selector $\langle F_2,f_2,\mathscr{M}_2\rangle$ of F of order 2^{-2}, and we may assume that $\mathscr{M}_2$ is a refinement of $\mathscr{N}_2$. Repeating this for each $n\in\mathbb{N}$ we obtain a partial selector $\langle F_n,f_n,\mathscr{M}_n\rangle$ for F of order 2^{-n} such that $\mathscr{M}_n$ is a refinement of both $\mathscr{M}_{n-1}$ and $\mathscr{N}_n$. Note that by (i) of Lemma 2.6 we have, for each $x\in X$,

$$F_n(x)=\begin{cases}\{f_n(x)\} & \text{if } f_n(x)\in F(x),\\ F(x)\cap\overline{B}\big(f_1(x);2^{-1}\big)\cap\ldots\cap\overline{B}\big(f_n(x);2^{-n}\big) & \text{otherwise.}\end{cases}$$

Furthermore, by (i) and (iii) of Lemma 2.6,

$$\|\cdot\|\text{-dist}(f_n(x),F_n(x))\leqslant 2^{-n},\quad F_{n+1}(x)\subset F_n(x)$$

for each $n\in\mathbb{N}$, and $\|\cdot\|-\operatorname{diam}(F_n(x))\leqslant 2^{-n+1}$, and thus it follows that

$$\|f_{n+1}(x)-f_n(x)\|\leqslant 3\cdot 2^{-n}$$

for all $x\in X$ and $n\in\mathbb{N}$. Hence the sequence $\langle f_n\rangle$ uniformly converges to some $f\colon X\to E$ relative to the norm topology. It remains to show that f is a selector for F.

Fix $x \in X$ and let M_n be the member of the partition $\mathscr{M}_n$ which contains x. If for some m we have $f_m(x) \in F(x)$, then it follows that

$$f_n(x) = f_m(x) \in F(x) \quad \forall n \geqslant m,$$

and in this case we clearly have $f(x) \in F(x)$. Otherwise, we have $f_n(x) \notin F(x)$ for each n. By (ii) of Lemma 2.6 we can choose $x_n \in M_n$ such that $F_n(x_n) = \{f_n(x_n)\}$, and thus $f_n(x) = f_n(x_n) \in F(x_n)$ for each n. Suppose $f(x) \notin F(x)$. Then $F(x)$ is contained in the weak open set

$$U = E \setminus \{f(x), f_1(x), f_2(x), \ldots\}.$$

By the weak usc of F and the fact that the sets $\{M_n \colon n \in \mathbb{N}\}$ form a local network at x (since $\mathscr{M}_n$ is a refinement of $\mathscr{N}_n$), we must have $F(x') \subset U$ for all $x' \in M_n$ for some n. But this contradicts the fact that $f_n(x) \in F(x_n)$. Thus $f(x) \in F(x)$ for all $x \in X$ showing that f is the desired selector. $\square$

Now if X is a hereditary Baire space (that is, every closed subspace is a Baire space), Y a metric space and $f \colon X \to Y$ has a σ-scattered function base of $\mathscr{F} \wedge \mathscr{G}$-sets then, by Theorem 2.2 in [5], f has the *point of continuity property*, i.e., $f\big|A$ has a point of continuity for each non-empty closed set $A \subset X$ (such maps are called *PC functions* in [5]).

Corollary 3.2. *Let $(E, \|\cdot\|)$ be a Banach space, K a compact space with a σ-scattered network and $F \colon K \to 2^E$ a weak usc map with non-empty values. Then F admits a selector with the point of continuity property.*

The class of compact spaces to which the Corollary above applies is quite large: Every compact space that is *fragmented* (or even *σ-fragmented*) by a metric whose topology is finer than the topology of the space, has a σ-scattered network of $\mathscr{F} \wedge \mathscr{G}$-sets by Theorem 6.4 of [4]; in particular Eberlein, Radon-Nikodým [10] and Gul'ko [12] compact spaces. However, in order to obtain a Borel measurable selector something more is required as the following example shows.

Example 3.3. The identity map $\mathrm{Id} \colon [0, \omega_1] \to ([0, \omega_1]$, discrete topology) is not measurable, since the Borel sets in $[0, \omega_1]$ do not coincide with the Borel sets for the discrete topology [14]. Yet, the identity map has a σ-scattered function base of $\mathscr{F} \wedge \mathscr{G}$-sets [4].

Definition 3.4. A topological space X is said to be hereditary weakly θ-refinable if each open collection in X has a σ-relatively discrete refinement.

A result in [5] shows that if $f\colon X \to (Y,d)$ is a PC map and X is hereditary weakly θ-refinable, then f is $(\mathscr{F} \wedge \mathscr{G})_\sigma$-measurable and has a σ-relatively discrete function base of $(\mathscr{F} \wedge \mathscr{G})$-sets in X. It is known that Eberlein compact spaces [4], [11] and Gul'ko compact spaces [1] are hereditary weakly θ-refinable, and so our Main Theorem in the introduction has been proved.

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