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SPECTRA OF EXTENDED DOUBLE COVER GRAPHS

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Abstract. The construction of the extended double cover was introduced by N. Alon [1] in 1986. For a simple graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, the extended double cover of G, denoted G^*, is the bipartite graph with bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which x_i and y_j are adjacent iff $i = j$ or v_i and v_j are adjacent in G.

In this paper we obtain formulas for the characteristic polynomial and the spectrum of G^* in terms of the corresponding information of G. Three formulas are derived for the number of spanning trees in G^* for a connected regular graph G. We show that while the extended double covers of cospectral graphs are cospectral, the converse does not hold. Some results on the spectra of the nth iterared double cover are also presented.

Keywords: charateristic polynomial of graph, graph spectra, extended double cover of graph

MSC 2000: 05C50, 05C30

1. INTRODUCTION

The spectra of graphs have long been studied and the study in this field has found applications in a variety of problems in theoretical chemistry, quantum mechanics, statistical physics, computer and information sciences, as well as some areas of mathematics including spectral Riemannian geometry (see [2], [4]–[7], [9]–[11] and the cited references there).

For studying networks N. Alon [1] introduced, in 1986, the extended double over of a graph to obtain expanders from magnifiers. This motivated our interest in studying the spectra of the extended double cover graphs.

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Throughout the paper G is always used to denote a simple graph with $n \geqslant 1$ vertices. For a simple graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, the extended double cover of G, denoted as G^*, is the bipartite graph with bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which x_i and y_j are adjacent iff $i = j$ or v_i and v_j are adjacent in G.

For example, the complete bipartite graph $K_{n,n}$ is the extended double cover of the complete graph K_n. It is easy to see that G^* is connected iff G is connected, and G^* is regular of degree $r+1$ iff G is regular of degree r.

For a graph G with adjacency matrix A, the charateristic polynomial of G is $\chi(G, \lambda) = |\lambda I - A|$, where I denotes the identity matrix. The eigenvalues of A (i.e. the zeros of $\chi(G, \lambda)$) and the spectrum of A (which consists of the n eigenvalues) are called the eigenvalues and the spectrum of G, respectively. For other notation and terminology not defined here the reader may refer to the books [2] and [3].

In the next section we shall give formulas for the characteristic polynomial and the spectrum of G^* in terms of the corresponding information of G. Three formulas are derived for the number of spanning trees in G^* for a connected regular graph G.

While the extended double covers of cospectral graphs are cospectral, we show the converse does not hold. Some results on the spectra of the nth iterated double cover are also presented.

2. Results

Theorem 1.
(i) $\chi(G^*, \lambda) = (-1)^n \chi(G, \lambda - 1)\chi(G, -\lambda - 1)$.
(ii) *Let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be the spectrum of* G.
Then the spectrum of G^* *consists of* $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \ldots, \pm(\lambda_n + 1)$.

Proof. Let G have the adjacency matrix A. Then it is not difficult to see that the partitioned matrix $\begin{bmatrix} 0 & A+I \\ A+I & 0 \end{bmatrix}$ is the adjacency matrix of G^*, in which all 0, A and I are $n \times n$ matrices. So,

$$\chi(G^*, \lambda) = \begin{vmatrix} \lambda I & -(A+I) \\ -(A+I) & \lambda I \end{vmatrix} = \lambda^n \begin{vmatrix} I & -\lambda^{-1}(A+I) \\ -(A+I) & \lambda I \end{vmatrix}.$$

It is well known in matrix theory (see, for example, [8, p. 45]) that if M is an invertible matrix then $\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M| \cdot |Q - PM^{-1}N|$. So,

$$\begin{aligned} \chi(G^*, \lambda) &= \lambda^n |\lambda I - \lambda^{-1}(A+I)^2| = |\lambda^2 I - (A+I)^2| \\ &= |\lambda I - (A+I)| \cdot |\lambda I + (A+I)| = (-1)^n |(\lambda - 1)I - A| \cdot |(-\lambda - 1)I - A| \\ &= (-1)^n \chi(G, \lambda - 1)\chi(G, -\lambda - 1). \end{aligned}$$

This completes the proof of (i). Then (ii) follows as a direct consequence of (i). $\square$

Example 1. Since $K_{n,n} = (K_n)^*$, and it is well known that the spectrum of K_n consists of $n-1$ and -1 (with multiplicity $n-1$), then by Theorem 1, the spectrum of $K_{n,n}$ consists of $\pm n$ and 0 (with multiplicity $2n-2$).

It should be pointed out that it is not difficult to determine if a given graph is the extended double cover of some graph. First, it must be bipartite. Secondly, for a connected bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$, G is the extended double cover of some graph iff G has a perfect matching $M = \{e_i \mid e_i = x_i y_i,\ x_i \in X,\ y_i \in Y;\ i = 1, 2, \ldots, n\}$ such that for any distinct e_i, e_j in M, we have either $\{x_i y_j, x_j y_i\} \subset E$ or $\{x_i y_j, x_j y_i\} \cap E = \emptyset$.

Example 2. Let M_n denotes the Möbius ladder (see [6]) that is the graph with $2n$ vertices $1, 2, \ldots, 2n$ in which the following pairs of vertices are adjacent:

$$\begin{aligned} &(i, i+1), \quad i = 1, 2, \ldots, 2n-1, \\ &(1, 2n), \\ &(i, i+n), \quad i = 1, 2, \ldots, n. \end{aligned}$$

Clearly, M_n is bipartite iff n is odd. It is not difficult to see that when $n \geqslant 3$ is odd $M_n = (C_n)^*$ where C_n is the cycle with n vertices. It is well known that the spectrum of C_n consists of $2\cos(2\pi i/n)$, $i = 1, 2, \ldots, n$. Then by Theorem 1, the spectrum of M_n (n is odd) consists of $\pm(2\cos(2\pi i/n) + 1)$, $i = 1, 2, \ldots, n$.

Example 3. Let $K_{n,n} \underline{\bowtie} K_{m,m}$ denote the graph obtained by idetifying one edge of $K_{n,n}$ with one edge of $K_{m,m}$. Then $K_{n,n} \underline{\bowtie} K_{m,m} = (K_n \bowtie K_m)^*$, where $K_n \bowtie K_m$ denotes the graph obtained by idetifying one vertex of K_n with one vertex of K_m. Applying a formula of Schwenk [10] (also see [9, p. 210]), it is easy to calculate $\chi(K_n \bowtie K_m, \lambda)$. Then $\chi(K_{n,n} \underline{\bowtie} K_{m,m}, \lambda)$ can be obtained easily from Theorem 1. The detail is left to the reader.

Let $\tau(G^*)$ ($\tau(G)$, resp.) denote the number of spanning trees of G^* (G, resp.). To obtain formulas for $\tau(G^*)$ we need the following well known result (see, for example, Corollary 6.5 in [2]).

Lemma 1. *Let G be a connected regular graph of degree r with the spectrum $r = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. Then*

$$\tau(G) = \frac{1}{n} \prod_{i=2}^{n} (r - \lambda_i) = \frac{1}{n} \chi'(G, r)$$

where χ' denotes the derivative of the characteristicc polynomial χ of G.

Now we are ready to give formulas for $\tau(G^*)$.

Corollary 1. *Let G be a connected regular graph of degree r with the spectrum $r = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. Then*
(i) $\tau(G^*) = n^{-1}(r+1) \prod_{i=2}^{n} [(r-\lambda_i)(r+\lambda_i+2)]$,
(ii) $\tau(G^*) = \tau(G) \cdot (r+1) \prod_{i=2}^{n} (r+\lambda_i+2)$.

Proof. (i) From the given condition, we can see that G^* is a connected regular graph of degree $r+1$ with $2n$ vertices. By Theorem 1 we can get the spectrum of G^*. Then by Lemma 1 we have

$$\begin{aligned}\tau(G^*) &= \frac{1}{2n}[(r+1)-(-\lambda_1+1))] \\ &\quad \times \prod_{i=2}^{n} \big[(r+1)-(\lambda_i+1)\big]\big[(r+1)-(-\lambda_i+1))\big] \\ &= \frac{r+1}{n} \prod_{i=2}^{n} [(r-\lambda_i)(r+\lambda_i+2)].\end{aligned}$$

(ii) directly follows from (i) and Lemma 1. □

Using Corollary 1 we can get the number of spanning trees of G^* from the spectrum of graph G. However, it often happens that the accurate spectrum of G is not easy, even impossible to determine although we can find the characteristic polynomial of G. The following Corollary 2 allows us to obtain the number $\tau(G^*)$ directly from the characteristic polynomial of graph G.

Corollary 2. *Let G be a connected regular graph of degree r with n vertices. Then*

$$\tau(G^*) = \frac{(-1)^n}{2n}\chi'(G,r)\chi(G,-r-2),$$

where χ' denotes the derivative of the characteristic polynomial χ.

Proof. By Lemma 1, we have

$$\tau(G^*) = \frac{1}{2n}\chi'(G^*, r+1),$$

where χ' denotes the derivative of the characteristic polynomial χ.

By Theorem 1, $\chi(G^*, x) = (-1)^n \chi(G, x-1)\chi(G, -x-1)$. So

$$\begin{aligned}\chi'(G^*, x) &= (-1)^n[\chi(G, x-1)\chi(G, -x-1)]' \\ &= (-1)^n[\chi'(G, x-1)\chi(G, -x-1) - \chi(G, x-1)\chi'(G, -x-1)].\end{aligned}$$

Since G is regular of degree r, r is an eigenvalue of G. So, $\chi(G, x-1) = 0$ when $x = r+1$. Therefore, we have

$$\tau(G^*) = \frac{(-1)^n}{2n}\chi'(G, r)\chi(G, -r-2).$$

□

From Theorem 1, it is clear that if two graphs have the same spectrum, then their extended double covers also have the same spectrum. A natural question is whether the converse is also true. By looking at the spectrum given in Theorem 1, it seems that an affirmative answer is plausible. However, contrary to the intuition, the answer to the above question is negative. This can been seen from the following result, where we use the notation $G_1 \times G_2$ to denote the cartesian product of graphs G_1 and G_2. Recall that the vertex set of $G_1 \times G_2$ is the cartesian product $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. (Note that this graph is denoted differently as $G_1 + G_2$ in [6].)

Theorem 2. *Let G be a connected graph. Then*
(i) *G^* and $G \times K_2$ have the same spectrum iff G is bipartite or $G = K_1$.*
(ii) *$(G^*)^*$ and $(G \times K_2)^*$ have the same spectrum.*

Proof. (i) It is easy to verify the case when G has only one vertex. So we may assume G has $n \geqslant 2$ vertices. Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be the spectrum of G. Then by Theorem 1, the spectrum of G^* consists of $\lambda_1 + 1 \geqslant \lambda_2 + 1 \geqslant \ldots \geqslant \lambda_n + 1$, and $-\lambda_1 - 1 \leqslant -\lambda_2 - 1 \leqslant \ldots \leqslant -\lambda_n - 1$. Recall from [6, p. 70] that if λ_i is an eigenvalue of G_i, $i = 1, 2$, then $\lambda_1 + \lambda_2$ is an eigenvalue of the cartesian product $G_1 \times G_2$, and that K_2 has eigenvalues 1 and -1. Then we see that the spectrum of $G \times K_2$ consists of $\lambda_1 + 1 \geqslant \lambda_2 + 1 \geqslant \ldots \geqslant \lambda_n + 1$, and $\lambda_1 - 1 \geqslant \lambda_2 - 1 \geqslant \ldots \geqslant \lambda_n - 1$. Thus, G^* and $G \times K_2$ have the same spectrum iff $\lambda_i = -\lambda_{n-i+1}$ for $i = 1, 2, \ldots, n$. It is well known [6, Theorem 3.11] that a graph containing at least one edge is bipartite iff its spectrum, considered as the set of points on the real axis, is symmetric with respect to the zero point. Therefore, G^* and $G \times K_2$ have the same spectrum iff G is bipartite or $G = K_1$.

(ii) Since G^* is bipartite, we immediately see from (i) that $(G^*)^*$ and $G^* \times K_2$ have the same spectrum. So we only need show that $G^* \times K_2$ and $(G \times K_2)^*$ have the same spectrum. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the spectrum of a graph G. By Theorem 1, the spectrum of G^* consists of $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \ldots, \pm(\lambda_n + 1)$. Using the known result on the spectrum of cartesian product, we obtain that the spectrum of $G^* \times K_2$ consists of $\pm(\lambda_1 + 1) \pm 1, \pm(\lambda_2 + 1) \pm 1, \ldots, \pm(\lambda_n + 1) \pm 1$, and that the spectrum of $G \times K_2$ consists of $\lambda_1 \pm 1, \lambda_2 \pm 1, \ldots, \lambda_n \pm 1$. Then by Theorem 1, we see that the spectrum of $(G \times K_2)^*$ consists of $\pm(\lambda_1 \pm 1 + 1), \pm(\lambda_2 \pm 1 + 1), \dots, \pm(\lambda_n \pm 1 + 1)$, which obviously is the same as the spectrum of $G^* \times K_2$. □

From Theorem 2, we see that for any connected graph G with an odd cycle, $(G^*)^*$ and $(G \times K_2)^*$ have the same spectrum, but G^* and $G \times K_2$ have different spectra. This gives the negative answer to the question before Theorem 2.

Finally, let's consider the k-th iterated double cover G^{k*} of G, which is defined as follows:

$$G^{1*} = G^* \quad \text{and} \quad G^{k*} = (G^{(k-1)*})^* \ \text{for} \ k > 1.$$

Corollary 3. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the spectrum of the graph G. Then the spectrum of G^{k*} consists of $\pm(\lambda_i + 1)\underbrace{\pm 1 \pm 1 \pm \dots \pm 1}_{k-1}$, $i = 1, 2, \dots, n$.*

P r o o f. By induction and Theorem 1. □

The following two corollaries are directly obtained from Corollary 3.

Corollary 4. *The spectrum of G contains an ineger iff there is a positive integer k such that the spectrum of G^{k*} contains zero.*

Corollary 5. *If the spectrum of G contains zero with multiplicity m, then the spectrum of G^{2k*} contains zero with multiplicity at least $\binom{2k}{k}m$.*

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