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A NOTE ON THE OSCILLATION OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We give a sufficient condition for the oscillation of linear homogeneous second order differential equation $y'' + p(x)y' + q(x)y = 0$, where $p(x), q(x) \in C[\alpha, \infty)$ and α is positive real number.

Keywords: oscillatory, second order differential equations

MSC 2000: 34A30, 34C10

1. Introduction

In this paper a sufficient condition is given for the oscillation of linear second order differential equation with variable coefficients

$$y'' + p(x)y' + q(x)y = 0, \tag{1.1}$$

where $p(x)$ and $q(x)$ are continuous functions on the interval $[\alpha, \infty)$, where α is real number. Here we are concerned with sufficient conditions on the coefficient $p(x)$ so that the equation (1.1) is oscillatory.

The study of oscillation of second order differential equations is of great interest. Many criteria have been found which involve the behavior of the integral of a combination of the coefficients. This approach has been motivated by some authors (for example see [1], [5], [6], [7], [9]).

Definition 1. A solution $y(x)$ of the differential equation (1.1) is said to be nontrivial if $y(x) \neq 0$ for at least one $x \in [\alpha, \infty)$.

Definition 2. A nontrivial solution $y(x)$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros on $[\alpha, \infty)$, otherwise it said to be "nonoscillatory".

2. Main Results

We prove he following theorems.

Theorem 1. *If* $p(x) < 0$ *on* $[\alpha, \infty)$ *is such that*

$$\lim_{x\to\infty}\left[\frac{1}{4}\int_\alpha^x (4q(s) - (p(s))^2)\,\mathrm{d}s\right] = \infty, \tag{2.1}$$

then any solution of the differential equation (1.1) *is oscillatory on* $[\alpha, \infty)$.

Proof. Suppose that the differential equation (1.1) is non oscillatory, then there exists a non trivial solution of (1.1) that has no zeros on $[\beta, \infty)$ for $\beta > \alpha$.

Let $w(x)$ be the function defined by $w(x) = -y^{-1}(x)y'(x)$ for $x \in [\beta, \infty)$.

Then $w(x)$ is well defined and satisfies the Riccati equation

$$w'(x) = w^2(x) - p(x)w(x) + q(x), \quad \text{on } [\beta, \infty).$$

Integrating both sides of this equation from β to x we get

$$\begin{aligned} w(x) &= w(\beta) + \int_\beta^x [(w(s))^2 - p(s)w(s) + q(s)]\,\mathrm{d}s \\ &= w(\beta) + \int_\beta^x \left(w(s) - \frac{p(s)}{2}\right)^2 \mathrm{d}s + \frac{1}{4}\int_\beta^x (4q(s) - (p(s))^2)\,\mathrm{d}s. \end{aligned}$$

Now the hypothesis (2.1) implies that there exists $\gamma > \beta$ such that

$$w(x) > \int_\gamma^x \left(w(s) - \frac{p(s)}{2}\right)^2 \mathrm{d}s \quad \text{on } [\gamma, \infty).$$

Define

$$G(x) = \int_\gamma^x \left(w(s) - \frac{p(s)}{2}\right)^2 \mathrm{d}s, \tag{2.2}$$

then $w(x) > G(x) > 0$ on $[\gamma, \infty)$.

Differentiating (2.2) we get

$$G'(x) = \left(w(x) - \frac{p(x)}{2}\right)^2 > w^2(x),$$

since $p(x) < 0$, therefore $G'(x) > G^2(x)$, thus

$$1 < \frac{G'(x)}{G^2(x)},$$

this inequality holds for $x > \gamma$.

Integrating both sides of this inequality from γ to x we get

$$\int_\gamma^x \mathrm{d}s < \frac{1}{G(\gamma)} - \frac{1}{G(x)}.$$

Therefore since $G(x) > 0$, we conclude that

$$\lim_{x\to\infty} \int_\gamma^x \mathrm{d}s < \frac{1}{G(\gamma)}.$$

But this is not true. Thus the differential equation (1.1) is oscillatory, and this completes the proof. □

Theorem 2. *If* $p(x) < 0$ *on* $[\alpha, \infty)$ *and there exists a non vanishing function* $g(x) \in C^1[\alpha, \infty)$; $g(x) > 0$, *such that*

$$\lim_{x\to\infty} \int_\alpha^x (g(s))^{-1}\,\mathrm{d}s = \infty \tag{2.3}$$

and

(2.4)

$$\lim_{x\to\infty} \left[-\frac{1}{4} \int_\alpha^x \left\{ p(s)^2 g(s) + \frac{((g(s))')^2}{g(s)} - 2p(s)(g(s))' - 4g(s)q(s) \right\} \mathrm{d}s + \frac{1}{2}(g(x))' \right] = \infty,$$

then any solution of the differential equation (1.1) *is oscillatory.*

Proof. Suppose that there exists a nonoscillatory solution $y(x)$ of the differential equation (1.1), i.e. $y(x)$ has no zeros on $[\alpha, \infty)$.

Define for $x > \alpha$

$$w(x) = -g(x)y'(x)y^{-1}(x), \tag{2.5}$$

where $g(x)$ is a non vanishing function belonging to $C^1[\alpha, \infty)$ and $1/g(x) > 0$ on $[\alpha, \infty)$.

Differentiating (2.5) with respect to x, on the interval $[\alpha, \infty)$, $w(x)$ satisfies the Riccati equation we see that.

$$w'(x) = \frac{1}{g(x)}[w^2(x) - g(x)p(x)w(x) + g'(x)w(x)] + g(x)q(x).$$

Now for $x \in [\alpha, \infty)$, defining

$$H(x) = w(x) + \frac{1}{2}g'(x), \tag{2.6}$$

we will have

$$\begin{aligned} w'(x) &= \frac{1}{g(x)}\bigg[\bigg(H(x) - \frac{p(x)g(x)}{2}\bigg)^2 - \bigg(\frac{p(x)g(x)}{2}\bigg)^2 - \frac{1}{4}(g'(x))^2 \\ &\quad + \frac{1}{2}p(x)g(x)g'(x)\bigg] + g(x)q(x) \\ &= \frac{1}{g(x)}\bigg(H(x) - \frac{p(x)g(x)}{2}\bigg)^2 - \frac{p^2(x)g(x)}{4} - \frac{1}{4}\frac{(g'(x))^2}{g(x)} + \frac{1}{2}p(x)g'(x) \\ &= \frac{1}{g(x)}\bigg(H(x) - \frac{p(x)g(x)}{2}\bigg)^2 \\ &\quad - \frac{1}{4}\bigg[p^2(x)g(x) + \frac{(g'(x))^2}{g(x)} - 2p(x)g'(x) - 4g(x)q(x)\bigg]. \end{aligned}$$

Integrating both sides of the above equation from α to x we get

$$\begin{aligned} w(x) &= w(\alpha) + \int_\alpha^x g^{-1}(s)\bigg(H(s) - \frac{1}{2}p(s)g(s)\bigg)^2\,\mathrm{d}s \\ &\quad - \frac{1}{4}\int_\alpha^x [p^2(s)g(s) + g^{-1}(s)(g'(s))^2 - 2p(s)g'(s) - 4g(s)q(s)]\,\mathrm{d}s. \end{aligned}$$

Then by using (2.6) we have

$$\begin{aligned} H(x) &= w(\alpha) + \int_\alpha^x g^{-1}(s)\bigg(H(s) - \frac{1}{2}p(s)g(s)\bigg)^2\,\mathrm{d}s \\ &\quad - \frac{1}{4}\int_\alpha^x [p^2(s)g(s) + g^{-1}(s)(g'(s))^2 - 2p(s)g'(s) - 4g(s)q(s)]\,\mathrm{d}s + \frac{1}{2}g'(x). \end{aligned}$$

The hypothesis of Theorem 2 implies that there exists $\beta > \alpha$ such that

$$H(x) > \int_\beta^x g^{-1}(s)\bigg(H(s) - \frac{1}{2}p(s)g(s)\bigg)^2\,\mathrm{d}s$$

holds for $x > \beta$. Define a function $Q(x)$ for $x > \beta$ by

$$Q(x) = \int_\beta^x g^{-1}(s)\bigg(H(s) - \frac{1}{2}p(s)g(s)\bigg)^2\,\mathrm{d}s. \tag{2.7}$$

Since $p(x) < 0$, $H(x) > Q(x) > 0$. Differentiating (2.7) we have

$$\begin{aligned} Q'(x) &= g^{-1}(x)\left(H(x) - \frac{1}{2}p(x)g(x)\right)^2 \\ &\geqslant g^{-1}(x)\left(Q(x) - \frac{1}{2}p(x)g(x)\right)^2 \\ &> g^{-1}(x)Q^2(x), \end{aligned}$$

therefore

$$g^{-1}(x) < \frac{Q'(x)}{Q^2(x)}.$$

Integrating both sides of this inequality with respect to x (with x replaced by s) from β to x we get for $x > \beta$

$$\int_\beta^x g^{-1}(s)\,\mathrm{d}s < \frac{1}{Q(\beta)} - \frac{1}{Q(x)};$$

since $Q(x) > 0$, therefore

$$\int_\beta^x g^{-1}(s)\,\mathrm{d}s < \frac{1}{Q(\beta)}$$

which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory. □

3. Example

Following is an illustrative example showing the applicability of both theorems. Consider the second order differential equation

$$y''(x) - \frac{2}{x}y'(x) + \left(1 + \frac{2}{x^2}\right)y(x) = 0, \quad \text{for } x > 0.$$

For this differential equation we have

$$p(x) = \frac{-2}{x} < 0, \quad \text{and} \quad q(x) = \left(1 + \frac{2}{x^2}\right).$$

To show the applicability of Theorem 1, the hypothesis is satisfied as follows

$$\begin{aligned} \lim_{x\to\infty} \int_\alpha^x \left(q(s) - \left(\frac{p(s)}{2}\right)^2\right)\mathrm{d}s &= \lim_{x\to\infty} \int_\alpha^x \left(1 + \frac{2}{s^2} - \frac{1}{4}\frac{4}{s^2}\right)\mathrm{d}s \\ &= \lim_{x\to\infty} \int_\alpha^x \left(1 + \frac{1}{s^2}\right)\mathrm{d}s = \lim_{x\to\infty}\left[s - \frac{1}{s}\right]_\alpha^x = \infty. \end{aligned}$$

Therefore the theorem implies that the differential equation is oscillatory.

This fact is directly verified by noting that the solution of the differential equation is given by

$$y(x) = x(k_1 \cos x + k_2 \sin x).$$

To show the applicability of Theorem 2, choose $g(x) = x$. It is clear that the hypothesis (2.4) is satisfied hence

$$\begin{aligned}
&\lim_{x\to\infty}\left[-\frac{1}{4}\int_\alpha^x \left\{(p(s))^2 g(s) + \frac{((g(s))')^2}{g(s)} - 2p(s)(g(s))' - 4g(s)q(s)\right\} \mathrm{d}s + \frac{1}{2}(g(x))'\right] \\
&\quad = \lim_{x\to\infty}\left\{-\frac{1}{4}\left[\int_\alpha^x \frac{4}{s} + \frac{4}{s} + \frac{1}{s} - 4s - \frac{8}{s}\right] \mathrm{d}s + \frac{1}{2}\right\} \\
&\quad = \lim_{x\to\infty}\left\{\left[\frac{s^2}{2} - \frac{1}{4}\ln s\right]_\alpha^x + \frac{1}{2}\right\} = \infty.
\end{aligned}$$

Hence Theorem 2 is applicable.

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