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REMARKS ON DENSE SUBSPACES

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Abstract. Some constructions of spaces with/without dense subspaces satisfying stronger separation axioms are presented.

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0. Introduction

The problem of finding dense subspaces satisfying stronger separation axioms than their originating spaces has been treated recently in [1]. In the present paper we develop the topic further in order to answer some questions raised in the quoted article and in [2]. We reformulate other questions from these articles for the current situation.

I. Terminology and notation

A space X is called *Urysohn* if its points can be pairwise separated by neighbourhoods with disjoint closures. Other separation axioms we deal with are standard.

Denote by τ the topology on X. A system $\pi \subset \tau \setminus \{\emptyset\}$ is a *π-base* of X if every nonempty $O \in \tau$ is a superset of some $P \in \pi$. The cardinal $\pi w(X) = \min\{|\pi|;\ \pi$ is a π-base of $X\}$ is called the *π-weight* of X. Interiors (resp. boundaries) of sets are denoted by int (bd, respectively).

As usual, we identify an ordinal number with the set of all its predecessors. Let $[\omega]^\omega$ be the set of all infinite subsets of ω and ω^ω the set of all functions $f\colon \omega \to \omega$.

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Such functions will be identified with their graphs. We shall use two cardinal characteristics of natural numbers:

$$\begin{aligned}\mathfrak{p} = \min\bigl\{|\mathscr{P}|;\ &\mathscr{P} \subset [\omega]^\omega \ \&\ \mathscr{P} \text{ is closed under finite intersections}\\ &\qquad\qquad \&\ \neg(\exists X \in [\omega]^\omega)\,(\forall P \in \mathscr{P})\ |X \setminus P| < \aleph_0\bigr\},\\ \mathfrak{d} = \min\bigl\{|\mathscr{D}|;\ &\mathscr{D} \subset \omega^\omega \ \&\ (\forall f \in \omega^\omega)\,(\exists g \in \mathscr{D})\,(\forall n \in \omega)\ f(n) \leqslant g(n)\bigr\}.\end{aligned}$$

It is easy to see that $\aleph_1 \leqslant \mathfrak{p} \leqslant \mathfrak{d} \leqslant 2^{\aleph_0}$. These cardinals can consistently differ ([5], [6]).

For ordinal numbers κ and λ put $\kappa^{<\lambda} = \{f\colon \alpha \to \kappa;\ \alpha < \lambda\}$. We consider sets of this type with the usual tree ordering $\subseteq$ ($\subset$, respectively), i.e. with (strict) inclusion of sequences as sets of ordered pairs.

II. Countable spaces

It has been proved [1, Theorem 2.1] that every countable Hausdorff space of weight less than $\mathfrak{p}$ has a dense regular subspace. It turned out that this estimate is exact, i.e. the following holds true.

Theorem 1.

$$\mathfrak{p} = \min\{w(X);\ |X| = \aleph_0 \ \&\ X \textit{ is a } T_2\textit{-space without dense } T_3\textit{-subspace}\}.$$

Proof. It remains to find a countable Hausdorff space T of weight $w(T) \leqslant \mathfrak{p}$, which has no dense regular subspace. The following construction completes the proof and was suggested to the author by M. Hrušák [4].

Example 1. Let $T = \omega^{<\omega}$ and let $\mathscr{P} \subset [\omega]^\omega$ be a system witnessing $\mathfrak{p}$. We can assume that $\omega \setminus F \in \mathscr{P}$ for every F finite.

Consider $s = \langle s(0), \ldots, s(|s|-1)\rangle \in T$. Basic open neighbourhoods of s will be all the sets of the form

$$O_{n,P}(s) = \{s\} \cup \{t \in T;\ s \subset t \ \&\ t(|s|) \geqslant n \ \&\ |t| \in P\},$$

where $n \in \omega$ and $P \in \mathscr{P}$.

It is easy to check that T is a Hausdorff space. We shall prove that no dense subset of T is regular at any point.

Choose D dense in T and $s \in D$. For $n \in \omega$ define

$$L(s,n) = \bigl\{|t|;\ t \in D \ \&\ s \subset t \ \&\ t(|s|) \geqslant n\bigr\}.$$

Density of D implies that each $L(s,n)$ is infinite. Hence we can find an increasing sequence $\langle k_n;\ n \in \omega\rangle$ with $k_n \in L(s,n)$ $(\forall n)$. Put $L = \{k_n;\ n \in \omega\}$. There is a $P \in \mathscr{P}$ such that $|L \setminus P| = \aleph_0$. Then

$$O = O_{0,P}(s) \cap D$$

is an open neigbourhood of s in D, but

$$(\forall n \in \omega)\,(\forall Q \in \mathscr{P})\ \overline{O_{n,Q}(s) \cap D}^{D} \setminus O = \overline{O_{n,Q}(s)} \cap D \setminus O \neq \emptyset.$$

Indeed, fix $n \in \omega$ and $Q \in \mathscr{P}$, without loss of generality, $Q \subset P$. There is $m \geqslant n$ with $k_m \notin P$ and $t \in D$ such that $s \subset t$, $t(|s|) \geqslant m$, $|t| = k_m$. Hence $t \notin O$. It remains to prove that $t \in \overline{O_{n,Q}(s)}$. But

$$(\forall k \in \omega)\,(\forall R \in \mathscr{P})\ O_{n,Q}(s) \cap O_{k,R}(t) \supset O_{k,Q\cap R}(t) \setminus \{t\} \neq \emptyset.$$

□

On the other hand, the π-weight of a countable Hausdorff space having no dense regular subspaces can be countable [1, Theorem 2.3]. But if "regular" is weakened to "Urysohn" then a π-base has to be uncountable. We strenghten a result from [1] (the Urysohn case of Theorem 2.9) proving

Theorem 2. *Every countable Hausdorff space X with $\pi w(X) < \mathfrak{d}$ has a dense Urysohn subspace.*

P r o o f. Fix a countable family $\{O_n;\ n \in \omega\}$ containing open subsets and separating (in the sense of Hausdorff) points of X. Put $R_n = \operatorname{bd}(O_n) \setminus \bigcup_{k<n} \operatorname{bd}(O_k)$ for every n and $R = \bigcup_{n\in\omega} R_n$. If $\operatorname{int} R = \emptyset$, we are done ($X \setminus R$ is dense and Urysohn).

Suppose $B = \operatorname{int} R \neq \emptyset$. It suffices to find D a dense subset of B such that $(\forall n)$ $|D \cap R_n| < \aleph_0$. Then $D \cup (X \setminus R)$ will be clearly dense in X and Urysohn.

Identify B with a subset of $\omega \times \omega$ putting $R_n \cap B \simeq \{n\} \times |R_n \cap B| \subset \{n\} \times \omega$. We shall use the following equality ([5, Theorem 3.6], attributed to J. Roitman):

$$\mathfrak{d} = \min\big\{|\mathscr{D}| + |\mathscr{A}|;\ \mathscr{A} \subset [\omega]^{\omega} \& \mathscr{D} \subset \omega^{\omega} \\ \&(\forall f \in \omega^{\omega})\,(\exists A \in \mathscr{A})\,(\exists g \in \mathscr{D})\ f \leqslant g \text{ on } A\big\}.$$

Let $\mathscr{O}$ be a π-base of B, $|\mathscr{O}| < \mathfrak{d}$. Each $R_n \cap B$ is nowhere dense, hence $(\forall O \in \mathscr{O})$ $\{n;\ R_n \cap O \neq \emptyset\}$ is infinite. For every $O \in \mathscr{O}$ choose a partial function $g_O \subset O$

($\subset \omega \times \omega$) with an infinite domain. Put $\mathscr{A} = \{\operatorname{dom} g_O ;\ O \in \mathscr{O}\}$, $\mathscr{D} = \{g_O \cup \{\langle n, 0\rangle ;$ $n \notin \operatorname{dom} g_O\} ;\ O \in \mathscr{O}\}$. As $|\mathscr{D}| + |\mathscr{A}| < \mathfrak{d}$, there is $f\colon\ \omega \to \omega$ such that

$$(\forall O \in \mathscr{O})\ (\exists n \in \operatorname{dom} g_O)\ f(n) > g_O(n).$$

Define

$$D = B \cap \{\langle n, k\rangle ;\ n \in \omega\ \&\ k \leqslant f(n)\}.$$

The set D intersects every graph of g_O, thus it is dense in B. Moreover, for each n, $|D \cap R_n| \leqslant f(n)$. □

So far, all known examples of countable Hausdorff spaces with no dense Urysohn subsets have bases of cardinality $\mathfrak{c}$ [1, Example 2.19].

Question 1. *Is there (in ZFC) a Hausdorff space X, $|X| = \aleph_0$, $\pi w(X) = \mathfrak{d}$ without dense Urysohn subspaces?*

III. First countable spaces

It can be proved in various ways (see, e.g., [7], [1]) that any second countable Hausdorff space contains a dense zero-dimensional subspace. O. T. Alas and R. G. Wilson [2, Example 2] presented a first countable T_2 example of cardinality $\aleph_1$ without dense regular subspaces (comparing with Theorem 1, bigger spaces can have smaller bases so as not to be "nice" on any dense subset). We slightly modify their construction (replacing well ordering by a tree) to obtain a first-countable Hausdorff space without dense Urysohn subspaces.

Example 2. Let $\{A_\alpha ;\ \alpha < \omega_1\}$ be a family of pairwise disjoint countable dense subsets of $\mathbb{R}$. Put

$$X = \bigl\{\langle s, x\rangle ;\ s \in \omega_2^{<\omega_1}\ \&\ x \in A_{\operatorname{dom}(s)}\bigr\}$$

and for every $\langle s, x\rangle \in X$ and every $n \in \omega \setminus \{0\}$ let

$$O_n(s, x) = \bigl\{\langle t, y\rangle \in X ;\ s \subseteq t\ \&\ |x - y| < 1/n\bigr\}.$$

The sets $O_n(s, x)$ form a base of a first countable Hausdorff topology on X (to check the latter, note that if $\langle s, x\rangle \neq \langle t, y\rangle$ then either $x \neq y$ or s, t are incompatible with respect to the tree order of $\omega_2^{<\omega_1}$).

Fix a dense subset $D \subset X$. We shall show that D endowed with the subspace topology is not Urysohn.

It is easy to compute that for each basic open set

$$\overline{O_n(s,x)} = \big\{\langle t,y\rangle \in X\,;\ (s \subseteq t \vee t \subset s)\ \&\ |x-y| \leqslant 1/n\big\}$$

and that

$$(1) \qquad \big(\forall s \in \omega_2^{<\omega_1}\big)\ \big|D \cap \{\langle t,y\rangle;\ s \subset t\}\big| \geqslant \aleph_2.$$

Denote by p the natural projection $X \to \mathbb{R}$, i.e. $p\colon \langle s,x\rangle \mapsto x$.

Claim. *There is $s \in \omega_2^{<\omega_1}$ satisfying*

$$(2) \qquad p\big[D \cap \{\langle t,y\rangle \in X\,;\ t \subset s\}\big] \textit{ is dense in } \mathbb{R}.$$

Proof. Let $\{B_n;\ 1 \leqslant n \in \omega\}$ be a countable base of $\mathbb{R}$. Put $s_0 = \emptyset$.

Suppose that for some $n \geqslant 1$ we have found

$$s_0 \subset s_1 \subset \ldots \subset s_{n-1},$$
$$y_1,\ \ldots, y_{n-1} \in \mathbb{R},$$

where $(\forall i < n)\ s_i \in \omega_2^{<\omega_1}$, $(\forall 1 \leqslant i < n)\ y_i \in B_i$ and $\langle s_i, y_i\rangle \in D$. Pick $x \in A_{\mathrm{dom}(s_{n-1})}$ and $k \in \omega \setminus \{0\}$ such that $(x - 1/k,\ x + 1/k) \subset B_n$. As D is dense in X, we can fix $\langle s_n, y_n\rangle \in D \cap O_k(s_{n-1}, x)$. Obviously, $s = \bigcup\limits_{n\in\omega} s_n$ is as required. □

Consider an s satisfying (2). It follows from (1) that

$$(\exists \alpha < \omega_1)\ \big|D \cap \{\langle t,y\rangle;\ s \subset t\ \&\ \mathrm{dom}(t) = \alpha\}\big| \geqslant \aleph_2.$$

But $p\big[\{\langle t,y\rangle \in D;\ s \subset t\ \&\ \mathrm{dom}(t) = \alpha\}\big] \subset A_\alpha$, hence it is countable, in particular, there are $y \in A_\alpha$ and $t_1 \neq t_2$ such that

$$s \subset t_1 \cap t_2\ \&\ \mathrm{dom}(t_1) = \mathrm{dom}(t_2) = \alpha\ \&\ \langle t_1, y\rangle, \langle t_2, y\rangle \in D.$$

Let $n \in \omega \setminus \{0\}$ be given. It remains to show that

$$\overline{O_n(t_1,y)} \cap \overline{O_n(t_2,y)} \cap D \neq \emptyset.$$

Using the Claim pick $z \in \mathbb{R}$ and $r \subset s$ such that $|y - z| \leqslant 1/n$ and $\langle r, z\rangle \in D$. Thus $\langle r,z\rangle \in \{\langle t,x\rangle \in D;\ t \subset t_1 \cap t_2\ \&\ |y-x| \leqslant 1/n\} = \overline{O_n(t_1,y)} \cap \overline{O_n(t_2,y)} \cap D$.

The space X of Example 2 has a point-countable base (thus it answers the stronger version of the Question of [2]), but $|X| = \max(\aleph_2, \mathfrak{c})$.

Question 2. *Is there a first countable T_2-space X, $|X| = \aleph_1$, containing no dense Urysohn subspace?*

IV. Products

Problem 3.1 of [1] asks: Suppose $X \times Y$ has a dense regular subspace. Do X and Y have such a subspace? We show that with Y chosen suitably, every space X without dense regular subsets can serve as a counterexample.

Theorem 3. *For every topological space X there is a space Y such that $X \times Y$ contains a dense hereditarily normal zero-dimensional subspace.*

The proof proceeds in three steps.

Proposition 1. *For each topological space X there is a space T and a dense Hausdorff (in fact, Urysohn) $D_1 \subset X \times T$.*

Proof. Since there are κ-resolvable spaces for every cardinal number κ, we can use [1, Theorem 2.14]—note that the construction works for X arbitrary. □

Proposition 2. *For every Hausdorff space X there exists a space W such that $X \times W$ has a dense zero-dimensional subspace.*

Proof. Let X be a Hausdorff space; without loss of generality, the set X' of all nonisolated points of X is nonempty.

Denote by σ the topology on $2^{X'}$ generated by

$$\big\{[\varphi];\ \mathrm{dom}(\varphi) \text{ is nowhere dense in } X \text{ (hence } \mathrm{dom}(\varphi) \subset X') \ \&\ \varphi\colon \mathrm{dom}(\varphi) \to 2\big\}$$

with $[\varphi] = \big\{\Phi \in 2^{X'};\ \varphi \subset \Phi\big\}$. Put $Z = \big\langle 2^{X'}, \sigma\big\rangle$. Then σ is stronger than the product topology on $2^{X'}$, in particular, Z is T_2 and each $[\varphi]$ as above is clopen.

Consider an analogue of σ-product,

$$W = \big\{\Phi \in Z;\ \Phi^{-1}(1) \text{ is finite}\big\}$$

endowed with the subspace topology and

$$D = \{\langle x, \Phi\rangle \in X' \times W;\ \Phi(x) = 1\} \cup (X^{is} \times W),$$

where $X^{is} = X \setminus X'$.

The set D is dense in $X \times W$: let O be a nonempty open subset of X and let $\varphi\colon S \to 2$ with $S \subset X'$ nowhere dense in X and $\varphi^{-1}(1)$ finite. If there is any $c \in O \cap X^{is}$, then for arbitrary $\Phi \in [\varphi]$: $\langle c, \Phi\rangle \in (O \times [\varphi]) \cap D$. Suppose $O \subset X'$, pick $c \in O \setminus S$ and define $\psi = \varphi \cup \{\langle c, 1\rangle\}$. The domain of ψ is nowhere dense and $|\psi^{-1}(1)| < \aleph_0$, hence there is $\Phi \in W$, $\psi \subset \Phi$ and $\langle c, \Phi\rangle \in (O \times [\varphi]) \cap D$.

It remains to prove that D is zero-dimensional. Fix $\langle x, \Phi\rangle \in D$, choose an open neighbourhood O of x and a function φ such that $\mathrm{dom}(\varphi)$ is nowhere dense and $\Phi \in [\varphi]$. Denote $\widetilde{O} = (O \times [\varphi]) \cap D$. We can assume that x is non-isolated. The space X is Hausdorff, $\Phi(x) = 1$ and $|\Phi^{-1}(1)| < \aleph_0$, so there is an open $U \subset X$ such that $x \in U \subset O$ and $\Phi^{-1}(1) \cap \overline{U} = \{x\}$. In particular, $(\forall y \in \mathrm{bd}(U))\ \Phi(y) = 0$. The set

$$S = \mathrm{dom}(\varphi) \cup \{x\} \cup \mathrm{bd}(U)$$

is nowhere dense in X. Define $\psi\colon S \to 2$,

$$\begin{aligned} &\psi(x) = 1, \\ &\psi(y) = 0, \qquad \text{if } y \in \mathrm{bd}(U), \\ &\psi(y) = \varphi(y), \qquad \text{if } y \in \mathrm{dom}(\varphi). \end{aligned}$$

Put $G = (U \times [\psi]) \cap D$. Obviously, $\langle x, \Phi\rangle \in G$ and

$$\overline{G}^D = (\overline{U \times [\psi]}) \cap D = (\overline{U} \times [\psi]) \cap D.$$

If $\langle y, \Psi\rangle \in (\overline{U} \times [\psi]) \cap D$, then $y \in U$, because either y is isolated or $\Psi(y) = 1$ while $\psi[\mathrm{bd}(U)] \subset \{0\}$ and $\psi \subset \Psi$. Therefore $G = \overline{G}^D \subset \widetilde{O}$. □

Proposition 3. *For every zero-dimensional space X there is a zero-dimensional space $\widetilde{T}$ such that $\widetilde{T} \times X$ contains a dense hereditarily normal subspace.*

Proof. First we associate to a zero-dimensional space X a space T_X which will be later shown to be homeomorphic to the desired dense set. Without loss of generality, X is infinite.

Put $T_X = X^{<\omega} \setminus \{\emptyset\}$ and for every $s = \langle s(0), \ldots, s(|s| - 1)\rangle \in T_X$ let $x(s) = s(|s| - 1)$. Fix such an s, choose a finite set $F \subset X$ containing $x(s)$ and a clopen neigbourhood U of $x(s)$ in X such that $F \cap U = \{x(s)\}$. Put

$$O_{F,U}(s) = \{s\} \cup \{t \supset s;\ t(|s|) \notin F \ \&\ x(t) \in U\}.$$

These sets form a local base in s of a zero-dimensional topology on T_X.

Claim. *T_X is hereditarily normal.*

P r o o f. We shall use the criterion [3, Theorem 2.16]. Let A, $B \subset T_X$ be separated (i.e. $\overline{A} \cap B = A \cap \overline{B} = \emptyset$). To find open disjoint U, $V \subset T_X$ such that $A \subset U$, $B \subset V$ we proceed by induction on levels in the tree T_X.

For $a \in A$, $|a| = 1$, pick a basic open neigbourhood $O_a = O_{F_a,U_a}(a)$ such that $O_a \cap B = \emptyset$. Similarly for $b \in B$ with $|b| = 1$ choose $O_b = O_{F_b,U_b}(b)$ with $O_b \cap A = \emptyset$.

Now let $|a| = n > 1$, $a \in A$. Denote by $b^0, \ldots, b^k$ all elements of B such that

$$b^i \subset a \ \&\ x(a) \notin U_{b^i}. \tag{3}$$

(In particular, $(\forall i \leqslant k)\ x(b^i) \neq x(a)$.) Then $\widetilde{U} = X \setminus \bigcup_{i \leqslant k} U_{b^i}$ is a clopen neighbourhood of $x(a)$ in X. Take a basic open set $O_{F,U}(a)$ such that $O_{F,U}(a) \cap B = \emptyset$. Put $F_a = F$, $U_a = \widetilde{U} \cap U$, $O_a = O_{F_a,U_a}(a)$. It is easy to see that

$$(\forall i \leqslant k)\ O_a \cap O_{b^i} = \emptyset.$$

Proceed symmetrically for $b \in B$, $|b| = n$.

Finally, put $U = \bigcup_{a \in A} O_a$, $V = \bigcup_{b \in B} O_b$. Hence $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. Indeed, choose $a \in A$, $b \in B$ and let us prove that $O_a \cap O_b = \emptyset$. Without loss of generality, $b \subset a$.

As $a \notin O_{F_b,U_b}(b)$, it is either $a(|b|) \in F_b$ or $x(a) \notin U_b$. In the first case $O_a \cap O_b = \emptyset$, because if $c \in O_b$, $b \subset c$, then $c(|b|) \notin F_b$ and if $a \subset c \in O_a$ then $c(|b|) = a(|b|) \in F_b$. If $x(a) \notin U_b$ then b occurs on the list of b^i for a satisfying (3). Hence $O_a \cap O_b = \emptyset$. $\square$

To prove Proposition 3 let $\widetilde{T} = X^{<\omega} \setminus \{\emptyset\}$ with a local base in $s \in \widetilde{T}$ consisting of all the sets of the type

$$O_F(s) = \{s\} \cup \{t \supset s;\ t(|s|) \notin F\},$$

where $F \subset X$ is finite, $g(s) = s(|s| - 1) \in F$. The space $\widetilde{T}$ is again zero-dimensional.

Denote by G the graph of the mapping $g\colon \widetilde{T} \to X$, i.e.

$$G = \{\langle s, g(s)\rangle;\ s \in \widetilde{T}\}.$$

As for every nonempty open $O \subset \widetilde{T}$ it is $g[O] = X$, G is dense in the product $\widetilde{T} \times X$. And it can be easily verified that G is homeomorphic to T_X via the correspondence $\langle s, g(s)\rangle \mapsto s$. In particular, G is hereditarily normal. $\square$

P r o o f of Theorem 3. Let X be an arbitrary topological space. Pick T and a dense Hausdorff $D_1 \subset X \times T$ from Proposition 1. Then Proposition 2 applied to D_1 gives a space W and a dense zero-dimensional $D \subset D_1 \times W$. Now use Proposition 3 to find a space $\widetilde{T}$ and a dense hereditarily normal $G \subset D \times \widetilde{T} \simeq \widetilde{T} \times D$. Finally put $Y = T \times W \times \widetilde{T}$ and note that G is dense in $X \times Y$. □

Thus, Problem 3.1 we have started with can be reduced to the following

Question 3. *Are there spaces X, Y, both T_2 without dense T_3-subspaces, such that $X \times Y$ has a dense T_3-subspace?*

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