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Witold Seredyński

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SOME CHARACTERIZATION OF LOCALLY NONCONICAL CONVEX SETS

WITOLD SEREDYŃSKI, Wrocław

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Abstract. A closed convex set Q in a local convex topological Hausdorff spaces X is called locally nonconical (LNC) if for every $x, y \in Q$ there exists an open neighbourhood U of x such that $(U \cap Q) + \frac{1}{2}(y - x) \subset Q$. A set Q is local cylindric (LC) if for $x, y \in Q$, $x \neq y$, $z \in (x, y)$ there exists an open neighbourhood U of z such that $U \cap Q$ (equivalently: $\mathrm{bd}(Q) \cap U$) is a union of open segments parallel to $[x, y]$. In this paper we prove that these two notions are equivalent. The properties LNC and LC were investigated in [3], where the implication LNC $\Rightarrow$ LC was proved in general, while the inverse implication was proved in case of Hilbert spaces.

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1. INTRODUCTION

In the sequel, X denotes a locally convex Hausdorff topological space, and Q is a nonempty closed convex subset of X.

Given x, $y \in Q$, by $[x, y]$ the closed segment joining the points x and y is denoted. Similarly (x, y), $[x, y)$, $(x, y]$ we denote the open, left-sided closed, right-sided closed segment, respectively. By $\langle x, y\rangle$ we denote the ordered pair.

Definition 1.1. The ordered pair $\langle x, y\rangle$ is called locally nonconical (LNC) in Q if $x, y \in Q$ and there exists an open neighbourhood U of the point x such that $(U \cap Q) + \frac{1}{2}(y - x) \subset Q$.

Definition 1.2. The set Q is called LNC if every ordered pair $\langle x, y\rangle \in Q \times Q$ is LNC.

Definition 1.3. The ordered pair $\langle x, y\rangle$ is called LC (local cylindric) if $x = y$ or for every $z \in (x, y)$ there exists an open neighbourhood U of z such that $U \cap Q$ is the union of open segments parallel to $[x, y]$.

Definition 1.4. The set Q is called LC if every pair $\langle x, y\rangle \in Q \times Q$ is LC.

In [3] a property of closed convex sets in the local convex topological Hausdorff spaces was considered. This property was called local nonconicality and was denoted LNC. The theorem proved in this paper is a generalization of Theorem 2.2 in [3], which was proved for subsets of Hilbert spaces. In our paper the proof of the general case is presented.

Definition 1.2 is taken from [3], where the notion of LC occured implicitly. We introduce the above definitions 1.1 and 1.3 for technical reasons.

The following useful reformulation of property LNC can be found in [3], which we quote in local version:

Proposition 1.1. *The ordered pair* $\langle x, y\rangle$ *is* LNC *in* Q, *if and only if* $x, y \in Q$ *and for every net* $(x_\alpha)_{\alpha\in\Gamma}$ *contained in* Q, *if* $\lim\limits_{\alpha\in\Gamma} x_\alpha = x$ *then* $x_\alpha + \frac{1}{2}(y - x) \in Q$ *for* $\alpha \in \Gamma$ *large enough.*

Below we list some basic properties of LNC, (see [3]):

(a) If $\dim X \leqslant 2$ then Q is LNC.
(b) If $Q_1, Q_2 \subset X$ are LNC then $Q_1 \cap Q_2$ is LNC, too.
(c) If $Q_1 \subset X_1$, $Q_2 \subset X_2$ are LNC then $Q_1 \times Q_2 \subset X_1 \times X_2$ is LNC, too.
(d) If Q is strictly convex then Q is LNC.
(e) If Q is LNC then Q is stable (i.e. $f\colon Q \times Q \to Q$ defined $f(x, y) := \frac{1}{2}(x + y)$ is an open map; cf. [2]).
(f) The unit ball in c_0 is LNC, but the unit ball l^∞ is not LNC.

2. Main lemma

Lemma 2.1. *Let* $x, y \in Q$, $x \neq y$ *be fixed, and let* $z := \frac{1}{2}(x + y)$. *Consider* $x' \in (x, z)$, *and* $y' := 2z - x'$. *If the pair* $\langle x', y'\rangle$ *is* LNC *in* Q, *then* $\langle x, y\rangle$ *is* LNC, *too.*

Proof. Let $(x_\alpha)_{\alpha\in\Gamma}$ be a net contained in Q such that $\lim\limits_{\alpha\in\Gamma} x_\alpha = x$. We need to prove that $x_\alpha + \frac{1}{2}(y - x) \in Q$ for $\alpha \in \Gamma$ large enough. Without loss of generality, we can assume that $x_\alpha \neq x$ for every $\alpha \in \Gamma$. Let $0 < \varepsilon < \frac{1}{2}$ be such that $x' = x + \varepsilon(y - x)$, $y' = y - \varepsilon(y - x)$. For $\alpha \in \Gamma$ we put

$$\delta'_\alpha := \sup\{\delta \geqslant 0\colon\ \delta x_\alpha + (1 - \delta)x + \varepsilon(y - x) \in Q\},$$

and

$$\delta_\alpha := \min\{\delta'_\alpha, 2\}.$$

We remark that

$$(1-2\varepsilon)x_\alpha + 2\varepsilon x + \varepsilon(y-x) = (1-2\varepsilon)x_\alpha + 2\varepsilon z \in Q,$$

and hence $\delta \geqslant 1 - 2\varepsilon > 0$.

Because $0 \leqslant \delta_\alpha \leqslant 2$, we have $\lim_{\alpha\in\Gamma} \delta_\alpha(x_\alpha - x) = 0$. For $\alpha \in \Gamma$ put

$$x'_\alpha := \delta_\alpha x_\alpha + (1-\delta_\alpha)x + \varepsilon(y-x).$$

Since Q is closed, it follows from the definition of δ_α that $x'_\alpha \in Q$ for $\alpha \in \Gamma$. Moreover

$$\lim_{\alpha\in\Gamma} x'_\alpha = x + \varepsilon(y-x) = x'.$$

Since the pair $\langle x', y'\rangle$ is LNC, we get

$$x'_\alpha + \frac{1}{2}(y'-x') = x'_\alpha + \Big(\frac{1}{2} - \varepsilon\Big)(y-x) \in Q$$

for $\alpha \in \Gamma$ large enough. Now convexity of Q implies that

$$(1-2\varepsilon)x_\alpha + 2\varepsilon\bigg(x'_\alpha + \Big(\frac{1}{2}-\varepsilon\Big)(y-x)\bigg) \in Q$$

for $\alpha \in \Gamma$ large enough. Observe that

$$\begin{aligned}
(1-2\varepsilon)x_\alpha + 2\varepsilon\bigg(x'_\alpha + \Big(\frac{1}{2}-\varepsilon\Big)(y-x)\bigg)& \\
&= (1-2\varepsilon)x_\alpha + 2\varepsilon\Big(x + \delta_\alpha(x_\alpha - x) + \frac{1}{2}(y-x)\Big)\\
&= (2\varepsilon\delta_\alpha + 1 - 2\varepsilon)x_\alpha + (2\varepsilon - 2\varepsilon\delta_\alpha)x + \varepsilon(y-x).
\end{aligned}$$

Letting $\delta := 2\varepsilon\delta_\alpha + 1 - 2\varepsilon$, we see that $\delta \geqslant 1 - 2\varepsilon > 0$, and

$$\delta x_\alpha + (1-\delta)x + \varepsilon(y-x) \in Q.$$

Therefore $\delta'_\alpha \geqslant \delta$. If $\delta_\alpha = \delta'_\alpha$ then $\delta_\alpha \geqslant 2\varepsilon\delta_\alpha + 1 - 2\varepsilon$; hence $\delta_\alpha \geqslant 1$. If $\delta_\alpha \neq \delta'_\alpha$, then $\delta_\alpha = 2$ and, again, $\delta_\alpha \geqslant 1$.

The above considerations show that

$$x_\alpha + \varepsilon(y-x) \in [x + \varepsilon(y-x), \delta_\alpha x_\alpha + (1-\delta_\alpha)x + \varepsilon(y-x)] \subset Q,$$

for $\alpha \in \Gamma$ large enough. Let

$$x''_\alpha := x_\alpha + \varepsilon(y - x).$$

We have $x''_\alpha \in Q$ for $\alpha \in \Gamma$ large enough and $\lim\limits_{\alpha\in\Gamma} x''_\alpha = x'$. Because the pair $\langle x', y'\rangle$ is LNC, we infer that for $\alpha \in \Gamma$ large enough $x''_\alpha + \frac{1}{2}(y' - x') \in Q$. But

$$x''_\alpha + \frac{1}{2}(y' - x') = x_\alpha + \varepsilon(y - x) + \Big(\frac{1}{2} - \varepsilon\Big)(y - x) = x_\alpha + \frac{1}{2}(y - x).$$

Therefore, for $\alpha \in \Gamma$ large enough $x_\alpha + \frac{1}{2}(y - x) \in Q$; hence $\langle x, y\rangle$ is LNC, and the proof is complete. $\square$

3. Main theorem

Theorem 3.1. *Properties* LNC *and* LC *are equivalent.*

Proof. In view of [3], Theorem 2.1, we only need to prove that LC implies LNC. Let $x, y \in Q$, $x \neq y$ and let the pair $\langle x, y\rangle$ is LC. Put $z := \frac{1}{2}(x + y)$. We need to show that there is $x' \in (x, z)$ such that for $y' := x + y - x'$ the pair $\langle x', y'\rangle$ is LNC. In view of Lemma 2.1 this will complete the proof.

Let U be a convex neighbourhood of z such that $U \cap Q$ is the union of open segments parallel to $[x, y]$. Let $x', y' \in U \cap [x, y]$, $x' \in [x, z)$, $y' := 2z - x'$. Let $(x_\alpha)_{\alpha\in\Gamma}$ be a net such that $\lim\limits_{\alpha\in\Gamma} x_\alpha = x'$ and $x_\alpha \in U \cap Q$ for $\alpha \in \Gamma$. Let

$$\varepsilon_\alpha := \sup\{\varepsilon\colon\ x_\alpha + \varepsilon(y' - x') \in U \cap Q\}$$

for $\alpha \in \Gamma$.

We have to show that $x_\alpha + \frac{1}{2}(y' - x') \in Q$ for $\alpha \in \Gamma$ large enough. To get a contradiction assume that $x_\alpha + \frac{1}{2}(y' - x') \notin Q$ for all $\alpha \in \Gamma$. But then $0 \leqslant \varepsilon_\alpha \leqslant \frac{1}{2}$ for every $\alpha \in \Gamma$. Upon replacing $(x_\alpha)_\alpha \in \Gamma$ by some subnet of $(x_\alpha)_\alpha \in \Gamma$ we can assume that $\lim\limits_{\alpha\in\Gamma} \varepsilon_\alpha = \varepsilon \geqslant 0$, where $0 \leqslant \varepsilon \leqslant \frac{1}{2}$. Observe that

$$W_\alpha := x_\alpha + \varepsilon_\alpha(y' - x') \in \mathrm{cl}(U) \cap Q = (\mathrm{bd}(U) \cap Q) \cup (U \cap Q).$$

If W_α belongs to $U \cap Q$ then either $x_\alpha + \varepsilon(y' - x') \in U \cap Q$ for some $\varepsilon > \varepsilon_\alpha$ which contradicts the definition ε_α, or $x_\alpha + \varepsilon_\alpha(y' - x')$ belongs to no open segment contained in $U \cap Q$, parallel to $[x, y]$ in contrary to the assumption. Therefore $x_\alpha + \varepsilon_\alpha(y' - x') \in \mathrm{bd}(U) \cap Q$ for $\alpha \in \Gamma$. Hence

$$\lim_{\alpha\in\Gamma}(x_\alpha + \varepsilon_\alpha(y' - x')) = x' + \varepsilon(y' - x') \in \mathrm{bd}(U) \cap Q.$$

Therefore $x' + \varepsilon(y' - x') \in \mathrm{bd}(U) \cap [x', y']$. But $[x', y'] \subset U$, $U \cap \mathrm{bd}\, U = \emptyset$, hence $\mathrm{bd}(U) \cap [x', y'] = \emptyset$, and we get a contradiction. $\square$

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Author's address: Institute of Mathematics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland, e-mail: `seredyn@im.pwr.wroc.pl`.