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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 3, 681–696

Persistent URL: http://dml.cz/dmlcz/127920

Czechoslovak Mathematical Journal, 54 (129) (2004), 681–696

ON CONNECTED RESOLVING DECOMPOSITIONS IN GRAPHS

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(Received December 3, 2001)

Abstract. For an ordered k-decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G and an edge e of G, the $\mathscr{D}$-code of e is the k-tuple $c_{\mathscr{D}}(e) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ is the distance from e to G_i. A decomposition $\mathscr{D}$ is resolving if every two distinct edges of G have distinct $\mathscr{D}$-codes. The minimum k for which G has a resolving k-decomposition is its decomposition dimension $\dim_d(G)$. A resolving decomposition $\mathscr{D}$ of G is connected if each G_i is connected for $1 \leqslant i \leqslant k$. The minimum k for which G has a connected resolving k-decomposition is its connected decomposition number $\mathrm{cd}(G)$. Thus $2 \leqslant \dim_d(G) \leqslant \mathrm{cd}(G) \leqslant m$ for every connected graph G of size $m \geqslant 2$. All nontrivial connected graphs of size m with connected decomposition number 2 or m have been characterized. We present characterizations for connected graphs of size m with connected decomposition number $m-1$ or $m-2$. It is shown that each pair s, t of rational numbers with $0 < s \leqslant t \leqslant 1$, there is a connected graph G of size m such that $\dim_d(G)/m = s$ and $\mathrm{cd}(G)/m = t$.

Keywords: distance, resolving decomposition, connected resolving decomposition

MSC 2000: 05C12

1. Introduction

For two edges e and f in a connected graph G of positive size, the *distance* $d(e, f)$ *between* e *and* f is the minimum nonnegative integer k for which there exists a sequence $e = e_0, e_1, \ldots, e_k = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. Thus $d(e, f) = 0$ if and only if $e = f$, $d(e, f) = 1$ if and only if e and f are adjacent, and $d(e, f) = 2$ if and only if e and f are nonadjacent edges that are adjacent to a common edge of G. Also, this distance equals the standard distance between vertices e and f in the line graph $L(G)$. For an edge e of G and a subgraph F of positive size in G, we define the *distance between* e *and* F as

$$d(e, F) = \min_{f \in E(F)} d(e, f).$$

A *decomposition* of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition into k subgraphs is a *k-decomposition.* A decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ is *ordered* if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathscr{D}$.

For an ordered k-decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ of G and an edge $e \in E(G)$, the *$\mathscr{D}$-code* of e is the k-vector

$$c_{\mathscr{D}}(e) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k)).$$

Hence exactly one coordinate of $c_{\mathscr{D}}(e)$ is 0, namely the ith coordinate if $e \in E(G_i)$. The decomposition $\mathscr{D}$ is said to be a *resolving decomposition* for G if every two distinct edges of G have distinct $\mathscr{D}$-codes. The minimum k for which G has a resolving k-decomposition is its *decomposition dimension* $\dim_d(G)$. A resolving decomposition of G with $\dim_d(G)$ elements is a *minimum resolving decomposition* for G. These concepts were first introduced and studied in [1]. A resolving decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ of G is defined to be *connected* in [13] if each subgraph G_i $(1 \leqslant i \leqslant k)$ is a connected subgraph in G. The minimum k for which G has a connected resolving k-decomposition is its *connected decomposition number* $\mathrm{cd}(G)$. A connected resolving decomposition of G with $\mathrm{cd}(G)$ elements is a *minimum connected resolving decomposition* for G. Since every connected resolving k-decomposition is a resolving k-decomposition, it follows that

$$2 \leqslant \dim_d(G) \leqslant \mathrm{cd}(G) \leqslant m$$

for every connected graph G of size $m \geqslant 2$.

To illustrate these concepts, consider the graph G of Fig. 1. Let $\mathscr{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_5, f_1, f_5, f_4\}$, $E(G_2) = \{e_2, e_3, f_2\}$, and $E(G_3) = \{e_4, e_6, f_3, f_6, f_7\}$. The $\mathscr{D}$-codes of the vertices of G are:

$$\begin{aligned}
&c_{\mathscr{D}}(e_1) = (0,1,2), \ c_{\mathscr{D}}(e_2) = (1,0,2), \ c_{\mathscr{D}}(e_3) = (2,0,1), \ c_{\mathscr{D}}(e_4) = (2,1,0),\\
&c_{\mathscr{D}}(e_5) = (0,4,1), \ c_{\mathscr{D}}(e_6) = (1,4,0), \ c_{\mathscr{D}}(f_1) = (0,1,1) \ \ c_{\mathscr{D}}(f_2) = (1,0,1),\\
&c_{\mathscr{D}}(f_3) = (1,1,0), \ c_{\mathscr{D}}(f_4) = (0,2,1), \ c_{\mathscr{D}}(f_5) = (0,3,1) \ \ c_{\mathscr{D}}(f_6) = (1,3,0),\\
&c_{\mathscr{D}}(f_7) = (1,2,0).
\end{aligned}$$

Thus, $\mathscr{D}$ is a resolving decomposition of G. In fact, $\mathscr{D}$ is a minimum resolving decomposition of G and so $\dim_d(G) = |\mathscr{D}| = 3$. However, $\mathscr{D}$ is not connected since G_1 and G_2 are not connected subgraphs in G. On the other hand, let $\mathscr{D}^* = \{G_1^*, G_2^*, G_3^*, G_4^*, G_5^*\}$, where $E(G_1^*) = \{e_1, f_1\}$, $E(G_2^*) = \{e_5, f_4, f_5\}$, $E(G_3^*) = \{e_2, e_3, f_2\}$, $E(G_4^*) = \{e_4, f_3\}$, and $E(G_5^*) = \{e_6, f_6, f_7\}$. Then $\mathscr{D}^*$ is a

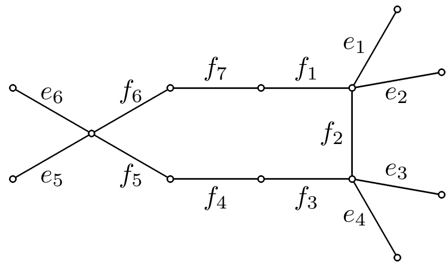

Figure 1. A graph G with $\dim_d(G) = 3$ and $\mathrm{cd}(G) = 4$.

connected resolving decomposition of G. But $\mathscr{D}^*$ is not minimum since the decomposition $\mathscr{D}' = \{G_1', G_2', G_3', G_4'\}$, where $E(G_1') = \{e_1\}$, $E(G_2') = \{e_3\}$, $E(G_3') = \{e_5\}$, and $E(G_4') = E(G) - \{e_1, e_3, e_5\}$, is a connected resolving decomposition of G with fewer elements. Indeed, it can be verified that $\mathscr{D}'$ is a minimum connected resolving decomposition of G and so $\mathrm{cd}(G) = |\mathscr{D}'| = 4$.

The concept of resolvability in graphs has appeared in the literature. Slater introduced and studied these ideas with different terminology in [11], [12]. Slater described in [8], [9], [10] the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [7] discovered these concepts independently. Recently, these concepts were rediscovered by Johnson [5], [6] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. Resolving decompositions in graphs were first introduced and studied in [1] and further studied in [3], [4]. The connected resolving decompositions in graph have been studied in [13]. We refer to the book [2] for graph theory notation and terminology not described here.

Connected graphs of size $m \geqslant 2$ with decomposition number 2 or m are characterized [13], as we state next.

Theorem 1.1 [13]. *Let G be a connected graph of order $n \geqslant 3$ and of size m. Then*

(a) $\mathrm{cd}(G) = 2$ *if and only if* $G = P_n$,

(b) $\mathrm{cd}(G) = m$ *if and only if* $G = K_3$ *or* $G = K_{1,n-1}$.

2. Characterizing graphs with connected decomposition number $m - 1$

In this section, we establish a characterization of connected graphs of size $m \geqslant 3$ with decomposition number $m-1$. In order to do this, we first present several results established in [13]. The following three results present bounds for the connected decomposition numbers of connected graphs in terms of other graphical parameters.

Theorem 2.1 [13]. *Let G be a connected graph that is not a star. If G contains a vertex that is adjacent to $k \geqslant 1$ end-vertices, then* $\dim_d(G) \geqslant k+1$ *and* $\mathrm{cd}(G) \geqslant k+1$.

Theorem 2.2 [13]. *If G is a connected graph of size $m \geqslant 2$ and diameter d, then*

$$2 \leqslant \mathrm{cd}(G) \leqslant m - d + 2.$$

The *girth* of a graph is the length of its shortest cycle.

Theorem 2.3 [13]. *If G is a connected graph of size $m \geqslant 3$ and girth $l \geqslant 3$, then*

$$3 \leqslant \mathrm{cd}(G) \leqslant m - l + 3.$$

Moreover, $\mathrm{cd}(G) = m - l + 3$ *if and only if G is a cycle of order at least* 3.

Although there is no general formula for the decomposition dimension of a tree that is not a path, a formula has been established in [13] for the connected decomposition number of a tree that is not a path. In order to present this formula, we need some additional definitions. A vertex of degree at least 3 in a connected graph G is called a *major vertex* of G. An end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $\mathrm{ex}(G)$ denote the number of exterior major vertices of G. If G is a tree that is not path, then $\sigma(G)$ is the number of end-vertices of G.

Theorem 2.4 [13]. *If T is a tree that is not a path, then*

$$\mathrm{cd}(T) = \sigma(T) - \mathrm{ex}(T) + 1.$$

We are now prepared to present a characterization of connected graphs of size $m \geqslant 3$ with connected decomposition number $m - 1$. For $n \geqslant 4$, let T_n be the graph of order n obtained from the path P_3 by adding $n-3$ pendant edges at an end-vertex of P_3. The graph T_n is shown in Fig. 2. In particular, $T_4 = P_4$.

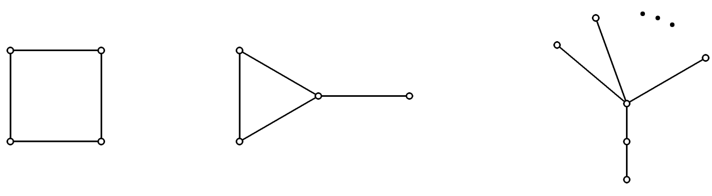

Figure 2. The graphs C_4, $(K_1 \cup K_2) + K_1$, and T_n in Theorem 2.5.

Theorem 2.5. *Let G be a connected graph of size $m \geqslant 3$. Then $\operatorname{cd}(G) = m - 1$ if and only if G is one of the graphs in Fig. 2.*

Proof. It is routine to verify that the graphs mentioned in the theorem have connected decomposition number $m-1$. For the converse, assume that G is a connected graph of size $m \geqslant 3$ and connected decomposition number $m-1$. If $m = 3$, then $G \in \{P_4, K_3, K_{1,3}\}$. Since $\operatorname{cd}(P_4) = 2$ and $\operatorname{cd}(K_3) = \operatorname{cd}(K_{1,3}) = 3$ by Theorem 1.1, it follows that $P_4 = T_4$ is the only graph with the desired property. If $m = 4$, then $G \in \{C_4, (K_1 \cup K_2) + K_1, K_{1,4}, P_5, T_5\}$. Since $\operatorname{cd}(G) = 3 = m - 1$ if $G = C_4, (K_1 \cup K_2) + K_1, T_5$, it follows by Theorem 1.1 that $C_4, (K_1 \cup K_2) + K_1$, and T_5 are the only connected graphs with the desired property for $m = 4$.

We now assume that $m \geqslant 5$. First, suppose that G is not a tree. Let l be the girth of G. If $l \geqslant 5$, then $\operatorname{cd}(G) \leqslant m - 2$ by Theorem 2.3. Thus $l = 4$ or $l = 3$. We consider these two cases.

Case 1: $l = 4$. Let $C\colon v_1, v_2, v_3, v_4, v_1$ be a cycle of length 4 in G. Since G is connected and $m \geqslant 5$, there exists a vertex v not in C such that v is adjacent to a vertex of C, say v is adjacent to v_1. Since $l = 4$, it follows that v is not adjacent to v_i for $i = 2, 4$. Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-2}\}$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_1v_4, v_3v_4\}$, $E(G_3) = \{v_2v_3\}$, and each of G_i $(4 \leqslant i \leqslant m-2)$ contains exactly one edge in $E(G) - (E(C) \cup \{vv_1\})$. Then $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (0, 1, 2, \ldots)$, $c_{\mathscr{D}}(v_1v_2) = (0, 1, 1, \ldots)$, $c_{\mathscr{D}}(v_1v_4) = (1, 0, 2, \ldots)$, and $c_{\mathscr{D}}(v_3v_4) = (2, 0, 1, \ldots)$, it follows that $\mathscr{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathscr{D}| = m - 2$, which is a contradiction.

Case 2: $l = 3$. If the order of G is 4, then $G = K_4 - e$ or $G = K_4$. Since $\operatorname{cd}(K_4 - e) = 3$ and $\operatorname{cd}(K_4) = 4$, it follows that $\operatorname{cd}(G) = m - 2$ for all connected graphs G of order 4 and size $m \geqslant 5$. Thus we may assume that $n \geqslant 5$. Let $C\colon v_1, v_2, v_3, v_1$ be a 3-cycle in G. Then there exists a vertex v not in C such that v is adjacent to a vertex of C, say $vv_1 \in E(G)$. Since $n \geqslant 5$ and G is connected, there exists a vertex $w \in V(G) - \{v, v_1, v_2, v_3\}$ such that w is adjacent to at least one vertex in $\{v, v_1, v_2, v_3\}$. We consider three subcases.

Subcase 2.1: *w is adjacent to v.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-2}\}$, where $E(G_1) = \{v_1v, vw\}$, $E(G_2) = \{v_1v_2, v_2v_3\}$, $E(G_3) = \{v_1v_3\}$, and each of G_i $(4 \leqslant i \leqslant m-2)$ contains exactly one edge in $E(G) - (E(C) \cup \{v_1v, vw\})$. Since $c_{\mathscr{D}}(v_1v_2) = (1, 0 \ldots)$, $c_{\mathscr{D}}(v_2v_3) = (2, 0, \ldots)$, $c_{\mathscr{D}}(v_1v) = (0, 1, \ldots)$, and $c_{\mathscr{D}}(v_2v_3) = (0, 2, \ldots)$, it follows that $\mathscr{D}$ is a connected resolving decomposition of G and so $\mathrm{cd}(G) \leqslant |\mathscr{D}| = m - 2$, which is a contradiction.

Subcase 2.2: *w is adjacent to v_1.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-2}\}$, where $E(G_1) = \{v_1v, v_1v_2\}$, $E(G_2) = \{v_1v_3, v_1w\}$, $E(G_3) = \{v_2v_3\}$, and each of G_i $(4 \leqslant i \leqslant m-2)$ contains exactly one edge in $E(G) - (E(C) \cup \{v_1v, v_1w\})$. Since $c_{\mathscr{D}}(v_1v) = (0, 1, 2 \ldots)$, $c_{\mathscr{D}}(v_1v_2) = (0, 1, 1, \ldots)$, $c_{\mathscr{D}}(v_1v_3) = (1, 0, 1, \ldots)$, and $c_{\mathscr{D}}(v_1w) = (1, 0, 2, \ldots)$, it follows that $\mathscr{D}$ is a connected resolving decomposition of G and so $\mathrm{cd}(G) \leqslant |\mathscr{D}| = m-2$, a contradiction.

Subcase 2: *w is adjacent to v_2 or to v_3, say w is adjacent to v_2.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-2}\}$, where $E(G_1) = \{v_1v_2, v_2w\}$, $E(G_2) = \{v_1v_3, v_1v\}$, $E(G_3) = \{v_2v_3\}$, and each of G_i $(4 \leqslant i \leqslant m-2)$ contains exactly one edge in $E(G) - (E(C) \cup \{v_1v, v_2w\})$. Since $c_{\mathscr{D}}(v_1v_2) = (0, 1, 1 \ldots)$, $c_{\mathscr{D}}(v_2w) = (0, 2, 1, \ldots)$, $c_{\mathscr{D}}(v_1v_3) = (1, 0, 1, \ldots)$, and $c_{\mathscr{D}}(v_1v) = (1, 0, 2, \ldots)$, it follows that $\mathscr{D}$ is a connected resolving decomposition of G and so $\mathrm{cd}(G) \leqslant |\mathscr{D}| = m - 2$, again, a contradiction.

Thus, G is a tree of size $m \geqslant 5$. Since $\mathrm{cd}(P_n) = 2$ for $n \geqslant 3$, it follows that G is not a path. Furthermore, by Theorem 2.2, the diameter d of G is at most 3. If $d = 2$, then G is a star and so $\mathrm{cd}(G) = m$. Thus $d = 3$ and G is a double star. Let u and v be the two central vertices of G; that is, u and v are not end-vertices of G. If $\deg u \geqslant 3$ and $\deg v \geqslant 3$, then u and v are exterior major vertices of G and so $\mathrm{ex}(G) = 2$. Since $\sigma(G) = m-1$, it follows by Theorem 2.4 that $\mathrm{cd}(G) = (m-1) - 2 + 1 = m-2$, which is a contradiction. Thus exactly one of u and v has degree 3 or more. Therefore, $G = T_n$, as desired. □

3. Characterizing graphs with connected decomposition number $m - 2$

In this section we present a characterization of connected graphs of size $m \geqslant 4$ with connected decomposition number $m-2$. For $n \geqslant 5$, let $H_n = (K_2 \cup (n-3)K_1) + K_1$. For $n \geqslant 6$, let X_n be a double star with two central vertices of degree at least 3. For $n \geqslant 5$, let Y_n be the graph obtained from P_4 by adding $n - 4$ pendant edges at an end-vertex of P_4, and let Z_n be the graph obtained from P_5: v_1, v_2, v_3, v_4, v_5 by adding $n-5$ pendant edges at v_3. In particular, $Y_5 = Z_5 = P_5$. The graphs H_n, X_n, Y_n, and Z_n are shown in Fig. 3.

Theorem 3.1. *Let G be a connected graph of size $m \geqslant 4$. Then $\operatorname{cd}(G) = m - 2$ if and only if G is one of the graphs in Fig.* 3.

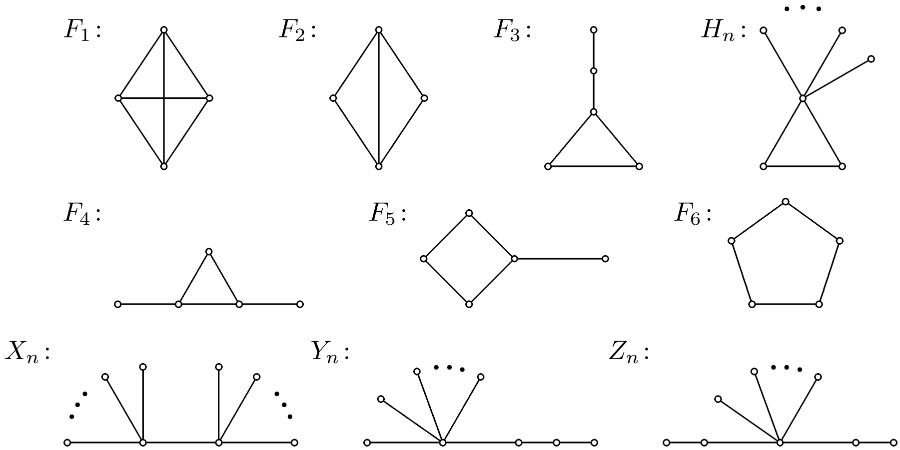

Figure 3. The graphs F_i $(1 \leqslant i \leqslant 6)$, H_n, X_n, Y_n, and Z_n in Theorem 3.1.

P r o o f. It is routine to verify that each graph G in Fig. 3 has connected decomposition number $m - 2$, where m is the size of G. For the converse, assume that G is a connected graph of order $n \geqslant 4$, size $m \geqslant 4$, and connected decomposition number $m - 2$.

If $n = 4$ and $m \geqslant 4$, then $G \in \{C_4, (K_2 \cup K_1) + K_1, K_4, K_4 - e\}$. Since $\operatorname{cd}(K_4) = 4$, and $\operatorname{cd}(K_4 - e) = 3$, it follows by Theorem 2.5 that $K_4 = F_1$ and $K_4 - e = F_2$ are the only graphs with the desired property for $n = 4$. If $n = 5$ and $m = 4$, then $G \in \{K_{1,4}, P_5, T_5\}$, where T_5 is the graph of Fig. 2 for $n = 5$. By Theorems 1.1 and 2.5, P_5 is the only graph with the desired property. If $n = 5$ and $m = 5$, then $G \in \{F_i\colon 3 \leqslant i \leqslant 6\} \cup \{H_5\}$. Since $\operatorname{cd}(F_i) = \operatorname{cd}(H_5) = 3 = m - 2$ for $3 \leqslant i \leqslant 6$, the graphs F_i, $3 \leqslant i \leqslant 6$, and H_5 are the only graphs with the desired property for $n = 5$ and $m = 5$.

We now assume that $n \geqslant 5$ and $m \geqslant 6$. We may assume that G is not one of the graphs of Fig. 3. First, suppose that G is not a tree. Let l be the girth of G. Since $\operatorname{cd}(G) = m - 2$, it follows by Theorem 2.3 that $3 \leqslant l \leqslant 5$. We consider three cases, according to whether $l = 5$, $l = 4$, or $l = 3$.

Case 1: $l = 5$. Let $C_5\colon v_1, v_2, v_3, v_4, v_5, v_1$ be a 5-cycle in G. Since $m \geqslant 6$ and C_5 is a smallest cycle in G, there exists a vertex $v \in V(G) - V(C_5)$ such that v is adjacent exactly one vertex of C_5, say v is adjacent to v_1. Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4\}$, $E(G_3) = \{v_4v_5, v_5v_1\}$, and each of G_i $(4 \leqslant i \leqslant m - 3)$ contains exactly one edge in $E(G) - (E(C_5) \cup \{vv_1\})$. Thus $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (0, 2, 1, \ldots)$, $c_{\mathscr{D}}(v_1v_2) =$

$(0,1,1,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (1,0,2,\ldots)$, $c_{\mathscr{D}}(v_3v_4) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_4v_5) = (2,1,0,\ldots)$, and $c_{\mathscr{D}}(v_5v_1) = (1,2,0,\ldots)$, it follows that $\mathscr{D}$ is a connected resolving decomposition of G. Thus $\mathrm{cd}(G) \leqslant |\mathscr{D}| = m-3$, which is a contradiction.

Case 2: $l = 4$. Let $C_4\colon v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G. Since $n \geqslant 5$, there exists a vertex $v \in V(G) - V(C_4)$ such that v is adjacent one vertex of C_4, say, v is adjacent to v_1. Since $m \geqslant 6$, it follows that G contains an edge f such that $f \notin E(C_4) \cup \{vv_1\}$ and f is adjacent to some edge in $E(C_4) \cup \{vv_1\}$. Thus G must contain a subgraph that is isomorphic to one of the graphs A_i $(1 \leqslant i \leqslant 5)$ in Fig. 4.

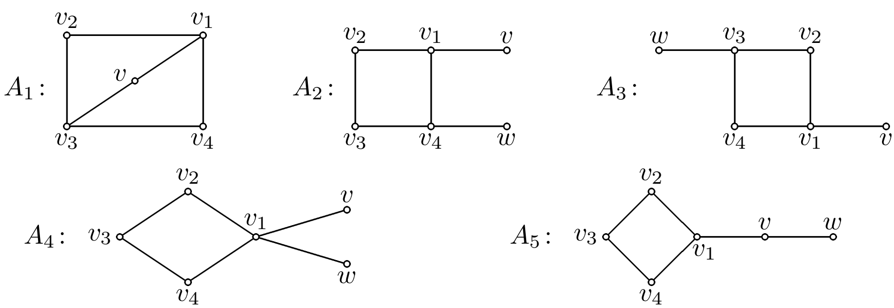

Figure 4. The graphs A_i $(1 \leqslant i \leqslant 5)$.

Subcase 2.1: *G contains a subgraph that is isomorphic to A_1.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vv_3\}$, $E(G_2) = \{vv_1, v_1v_2\}$, $E(G_3) = \{v_1v_4, v_2v_3, v_3v_4\}$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(C_4) \cup \{vv_1, vv_3\})$. Thus $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_4) = (2,1,0,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (1,1,0,\ldots)$, and $c_{\mathscr{D}}(v_3v_4) = (1,2,0,\ldots)$, it follows that $\mathscr{D}$ is a resolving decomposition of G.

Subcase 2.2: *G contains a subgraph that is isomorphic to A_2.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vv_1, v_1v_4\}$, $E(G_2) = \{v_4w, v_3v_4\}$, $E(G_3) = \{v_1v_2, v_2v_3\}$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(C_4) \cup \{vv_1, v_4w\})$. Thus $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (0,2,1,\ldots)$, $c_{\mathscr{D}}(v_1v_4) = (0,1,1,\ldots)$, $c_{\mathscr{D}}(v_4w) = (1,0,2,\ldots)$, $c_{\mathscr{D}}(v_3v_4) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (1,2,0,\ldots)$, and $c_{\mathscr{D}}(v_2v_3) = (2,1,0,\ldots)$, it follows that $\mathscr{D}$ is a resolving decomposition of G.

Subcase 2.3: *G contains a subgraph that is isomorphic to A_3.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vv_1, v_1v_4\}$, $E(G_2) = \{v_1v_2, v_2v_3\}$, $E(G_3) = \{v_3w, v_3v_4\}$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(C_4) \cup \{vv_1, v_3w\})$. Thus $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (0,1,2,\ldots)$, $c_{\mathscr{D}}(v_1v_4) = (0,1,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (1,0,2,\ldots)$,

$c_{\mathscr{D}}(v_2v_3) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_3w) = (2,1,0,\ldots)$, and $c_{\mathscr{D}}(v_3v_4) = (1,1,0,\ldots)$, it follows that $\mathscr{D}$ is a resolving decomposition of G.

Subcase 2.4: *G contains a subgraph that is isomorphic to A_4.* Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4\}$, $E(G_3) = \{v_1v_4, v_1w\}$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(C_4) \cup \{vv_1, v_1w\})$. Thus $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (0,2,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (0,1,1,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (1,0,2,\ldots)$, $c_{\mathscr{D}}(v_3v_4) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_4) = (1,1,0,\ldots)$, and $c_{\mathscr{D}}(v_1w) = (1,2,0,\ldots)$, it follows that $\mathscr{D}$ is a resolving decomposition of G.

Subcase 2.5: *G contains a subgraph that is isomorphic to A_5.* Since $l = 4$, it follows that $v_1w, vv_2, vv_4 \notin E(G)$. If $vv_3 \in E(G)$, then G contains a subgraph that is isomorphic to A_1, and so the result follows by Subcase 2.1. If $v_2w \in E(G)$ or $v_4w \in E(G)$, then G contains a subgraph that is isomorphic to A_2, and so the result follows by Subcase 2.2. If $v_3w \in E(G)$, then G contains a subgraph that is isomorphic to A_3, and so the result follows by Subcase 2.3. Thus we may assume that none of vv_2, vv_3, vv_4, v_1w, v_2w, v_3w, v_4w is an edge of G. Let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vw\}$, $E(G_2) = \{vv_1, v_1v_2\}$, $E(G_3) = \{v_1v_4, v_2v_3, v_3v_4\}$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(C_4) \cup \{vv_1, vw\})$. Thus $\mathscr{D}$ is connected. Since $c_{\mathscr{D}}(vv_1) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (3,1,0,\ldots)$, $c_{\mathscr{D}}(v_1v_4) = (2,1,0,\ldots)$, and $c_{\mathscr{D}}(v_3v_4) = (3,2,0,\ldots)$, it follows that $\mathscr{D}$ is a resolving decomposition of G.

Thus, in each case, G has a connected resolving decomposition with $m-3$ elements, and so $\operatorname{cd}(G) \leqslant m-3$, which is a contradiction.

Case 3: $l = 3$. Let $C_3\colon v_1, v_2, v_3, v_1$ be a 3-cycle in G. Since G is not one of the graphs in Fig. 3, it follows that $G \neq H_n$. Since $n \geqslant 5$, there exist $v, w \in V(G) - V(C_3)$ such that the subgraph $\langle\{v_1, v_2, v_3, v, w\}\rangle$ induced by $\{v_1, v_2, v_3, v, w\}$ is a connected subgraph of G. This fact together with $m \geqslant 6$ implies that G must contain a subgraph that is isomorphic to one of the graphs B_i $(1 \leqslant i \leqslant 10)$ in Fig. 5. We proceed by cases. In each of the following subcases, we construct a connected resolving decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ of G by choosing G_1, G_2, G_3 such that $|E(G_1)| + |E(G_2)| + |E(G_3)| = 6$ and each G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge from $E(G) - (E(G_1) \cup E(G_2) \cup E(G_3))$.

Subcase 3.1: *G contains B_1.* Let $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, and $E(G_3) = \{v_1v_3, v_1v, vw, wv_3\}$. Then $c_{\mathscr{D}}(v_1v_3) = (1,1,0,\ldots)$, $c_{\mathscr{D}}(v_1v) = (1,2,0,\ldots)$, $c_{\mathscr{D}}(vw) = (2,2,0,\ldots)$, and $c_{\mathscr{D}}(wv_3) = (2,1,0,\ldots)$.

Subcase 3.2: *G contains B_2.* Let $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_1v_3, vv_1\}$, and $E(G_3) = \{v_2v_3, v_2w, wv_3\}$. Then $c_{\mathscr{D}}(v_1v_3) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v) = (1,0,2,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (1,1,0,\ldots)$, $c_{\mathscr{D}}(v_2w) = (1,2,0,\ldots)$, and $c_{\mathscr{D}}(wv_3) = (2,1,0,\ldots)$.

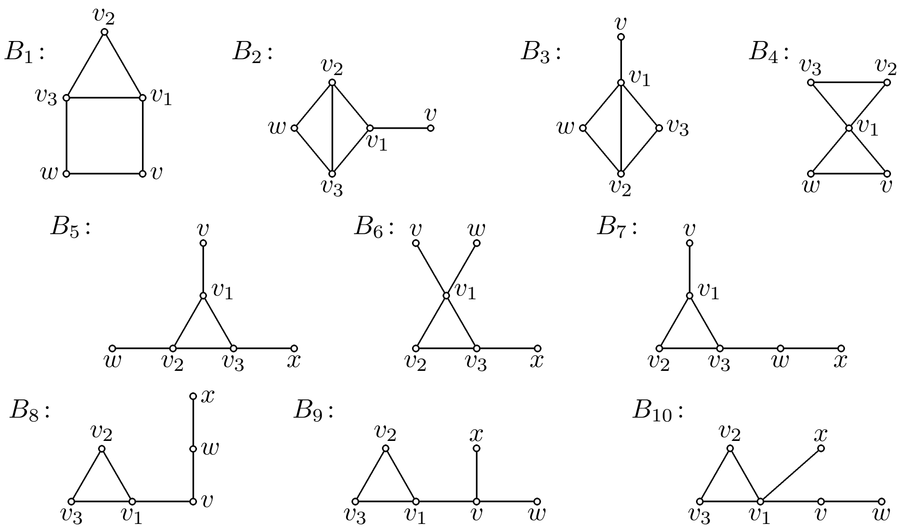

Figure 5. The graphs B_i $(1 \leqslant i \leqslant 10)$.

Subcase 3.3: *G contains* B_3. Let $E(G_1) = \{vv_1, v_1v_3\}$, $E(G_2) = \{v_2v_3\}$, and $E(G_3) = \{v_1v_2, v_2w, wv_1\}$. Then $c_{\mathscr{D}}(vv_1) = (0,2,1,\ldots)$, $c_{\mathscr{D}}(v_1v_3) = (0,1,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (1,1,0,\ldots)$, $c_{\mathscr{D}}(v_2w) = (2,1,0,\ldots)$, and $c_{\mathscr{D}}(wv_1) = (1,2,0,\ldots)$.

Subcase 3.4: *G contains* B_4. If $vv_2 \in E(G)$ or $v_3w \in E(G)$, then G contains B_3 and so the result follows by Subcase 3.3. Thus we may assume that $vv_2 \notin E(G)$ and $v_3w \notin E(G)$. Similarly, we may assume that $wv_2 \notin E(G)$ and $vv_3 \notin E(G)$. Let $E(G_1) = \{vw\}$, $E(G_2) = \{vv_1, v_1v_2\}$, and $E(G_3) = \{v_2v_3, v_3v_1, wv_1\}$. Then $c_{\mathscr{D}}(vv_1) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (3,1,0,\ldots)$, $c_{\mathscr{D}}(v_3v_1) = (2,1,0,\ldots)$, and $c_{\mathscr{D}}(wv_1) = (1,1,0,\ldots)$.

Subcase 3.5: *G contains* B_5. Let $E(G_1) = \{vv_1\}$, $E(G_2) = \{v_2w, v_1v_2\}$, and $E(G_3) = \{v_1v_3, v_2v_3, v_3x\}$. Then $c_{\mathscr{D}}(v_2w) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_3) = (1,1,0,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (2,1,0,\ldots)$, and $c_{\mathscr{D}}(v_3x) = (2,2,0,\ldots)$.

Subcase 3.6: *G contains* B_6. Let $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3x\}$, and $E(G_3) = \{v_1v_3, v_1w\}$. Then $c_{\mathscr{D}}(vv_1) = (0,2,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (0,1,1,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(v_3x) = (2,0,1,\ldots)$, $c_{\mathscr{D}}(v_1v_3) = (1,1,0,\ldots)$, and $c_{\mathscr{D}}(v_1w) = (1,2,0,\ldots)$.

Subcase 3.7: *G contains* B_7. Let $E(G_1) = \{vv_1, v_1v_3\}$, $E(G_2) = \{v_1v_2, v_2v_3\}$, and $E(G_3) = \{wx, v_3w\}$. Then $c_{\mathscr{D}}(vv_1) = (0,1,2,\ldots)$, $c_{\mathscr{D}}(v_1v_3) = (0,1,1,\ldots)$, $c_{\mathscr{D}}(v_1v_2) = (1,0,2,\ldots)$, $c_{\mathscr{D}}(v_2v_3) = (1,0,1,\ldots)$, $c_{\mathscr{D}}(wx) = (2,2,0,\ldots)$, and $c_{\mathscr{D}}(v_3w) = (1,1,0,\ldots)$.

Subcase 3.8: *G contains* B_8. If there is an edge joining one vertex in $\{v_2, v_3\}$ and one vertex in $\{v, w, x\}$, then G contains at least one of B_1, B_2, and B_7 and so the result follows by Subcases 3.1, 3.2, or 3.7. Assume that v_1 is not adjacent to x and w; for otherwise, G contains B_4 or B_{10}. If G contains B_4, then the result follows by Subcase 3.4. If G contains B_{10}, then this will be verified in Subcase 3.10. Thus we may assume that there is no edge between $\{v_2, v_3\}$ and $\{v, w, x\}$. Let $E(G_1) = \{xw, wv, vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, and $E(G_3) = \{v_1v_3\}$. Then $c_{\mathscr{D}}(xw) = (0, 4, 3, \ldots)$, $c_{\mathscr{D}}(wv) = (0, 3, 2, \ldots)$, $c_{\mathscr{D}}(vv_1) = (0, 2, 1, \ldots)$, and $c_{\mathscr{D}}(v_1v_2) = (0, 1, 1, \ldots)$.

Subcase 3.9: *G contains* B_9. If there is an edge joining one vertex in $\{v_2, v_3\}$ and one vertex in $\{v, x\}$, then G contains B_1 or B_2 and so the result follows by Subcases 3.1 or 3.2. Thus we may assume that there is no edge between $\{v_2, v_3\}$ and $\{v, x\}$. Let $E(G_1) = \{vw, v_1v, v_1v_2\}$, $E(G_2) = \{vx\}$, and $E(G_3) = \{v_1v_3, v_2v_3\}$. Then $c_{\mathscr{D}}(vw) = (0, 1, 2, \ldots)$, $c_{\mathscr{D}}(vv_1) = (0, 1, 1, \ldots)$, $c_{\mathscr{D}}(v_1v_2) = (0, 2, 1, \ldots)$, $c_{\mathscr{D}}(v_1v_3) = (1, 2, 0, \ldots)$, and $c_{\mathscr{D}}(v_2v_3) = (1, 3, 0, \ldots)$.

Subcase 3.10: *G contains* B_{10}. If there is an edge joining one vertex in $\{v_2, v_3\}$ and one vertex in $\{v, w\}$, then G contains B_1 or B_3 and so the result follows by Subcases 3.1 or 3.3. Thus we may assume that there is no edge between $\{v_2, v_3\}$ and $\{v, w\}$. Let $E(G_1) = \{vw, vv_1, v_1v_3\}$, $E(G_2) = \{xv_1, v_1v_2\}$, and $E(G_3) = \{v_2v_3\}$. Then $c_{\mathscr{D}}(vw) = (0, 2, 3, \ldots)$, $c_{\mathscr{D}}(vv_1) = (0, 1, 2, \ldots)$, $c_{\mathscr{D}}(v_1v_3) = (0, 1, 1, \ldots)$, $c_{\mathscr{D}}(xv_1) = (1, 0, 2, \ldots)$, and $c_{\mathscr{D}}(v_1v_2) = (1, 0, 1, \ldots)$.

Thus, in each subcase above, G has a connected resolving decomposition with $m-3$ elements, and so so $\mathrm{cd}(G) \leqslant m-3$, which is a contradiction.

Therefore, G is a tree of order $n \geqslant 5$ and size $m \geqslant 6$. Let d be the diameter of G. By Theorem 1.1, G is neither a path nor a star and so $d \geqslant 3$. On the other hand, by Theorem 2.2, $d \leqslant 4$. Thus, $d = 3$ or $d = 4$. We consider these two cases.

Case 1: $d = 3$. Then G is a double star. Let u and v be the two central vertices of G. Since G is not a star, at least one of u and v has degree 3 or more. On the other hand, if exactly one of u and v has degree 3 or more, then $\mathrm{cd}(G) = m-1$ by Theorem 2.5. Therefore, $G = X_n$ as shown in Fig. 3.

Case 2: $d = 4$. Let $P_5\colon v_1, v_2, v_3, v_4, v_5$ be a path of order 5 in G. Since $G \neq P_5$, at least one of the vertices v_2, v_3, v_4 has degree 3 or more. We claim that $G = Y_n$ or $G = Z_n$ in Fig. 3 in this case. Assume, to the contrary, that this is not true. Then G contains a subgraph that is isomorphic to one of the graphs T_1, T_2, and T_3 in Fig. 6.

If G contains the subgraph that is isomorphic to T_1, then let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{v_2v\}$, $E(G_2) = \{v_3w\}$, $E(G_3) = V(P_5)$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(P_5) \cup \{v_2v, v_3w\})$. If G contains the subgraph that is isomorphic to T_2, then let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{vv_2\}$, $E(G_2) = \{v_4w\}$, $E(G_3) = E(P_5)$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in

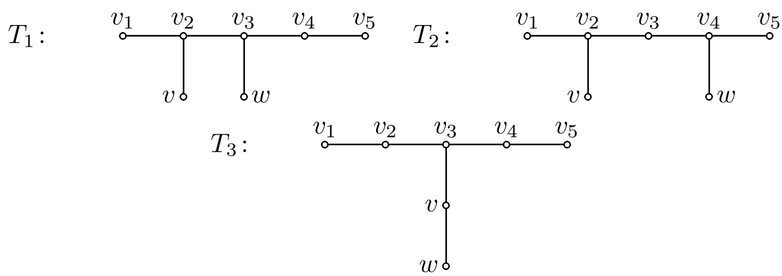

Figure 6. The graphs T_i $(1 \leqslant i \leqslant 3)$.

$E(G)-(E(P_5)\cup\{v_2v, v_4w\})$. If G contain the subgraph that is isomorphic to T_3, then let $\mathscr{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of G, where $E(G_1) = \{v_1v_2, v_2v_3\}$, $E(G_2) = \{v_3v_4, v_4v_5\}$, $E(G_3) = \{v_3v, vw\}$, and each of G_i $(4 \leqslant i \leqslant m-3)$ contains exactly one edge in $E(G) - (E(P_5) \cup \{v_3v, vw\})$. In each case, it can be verified that $\mathscr{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathscr{D}| = m-3$, which is a contradiction. Therefore, $G = Y_n$ or $G = Z_n$, as claimed. □

4. Realizable ratios

We have seen in Theorem 1.1 that a path of size $m \geqslant 2$ is the only connected graph of size $m \geqslant 2$ with connected decomposition number 2. Furthermore, it was shown in [1] that a path of size $m \geqslant 2$ is also the only connected graph of size $m \geqslant 2$ with decomposition dimension 2. Thus, there is no connected graph of size $m \geqslant 2$ with decomposition dimension 2 and connected decomposition number 3 or more. On the other hand, it was shown in [13] that every pair a, b of integers with $3 \leqslant a \leqslant b$ is realizable as decomposition dimension and connected decomposition number of some connected graph, as we state below.

Theorem 4.1. *For every pair a, b of integers with $3 \leqslant a \leqslant b$, there exists a connected graph G such that $\dim_d(G) = a$ and $\operatorname{cd}(G) = b$.*

However, there is no restriction on the size of such a graph in Theorem 4.1. On the other hand, it is routine to verify that every graph described in Theorem 2.5 has size m and decomposition dimension $m-1$. Thus, if a, m are integers with $2 \leqslant a \leqslant m-2$, then there is no connected graph G of size m such that $\dim_d(G) = a$ and $\operatorname{cd}(G) = m-1$. Furthermore, it can be verified, for the graphs described in Theorem 3.1, that (1) $\dim_d(F_i) = m-2$ for $1 \leqslant i \leqslant 6$, (2) $\dim_d(H_n) = m-2$ for $n \geqslant 5$, (3) $\dim_d(X_n) = \max\{\deg u, \deg v\} + 1$, where u and v are the central vertices of X_n for $n \geqslant 6$, (4) $\dim_d(Y_n) = m-2$ for $n \geqslant 5$, (5) $\dim_d(Z_n) = m-2$ if $n = 5, 6$ and $\dim_d(Z_n) = m-3$ if $n \geqslant 7$. In each case (1)–(5), the integer m is the size of the graph under consideration. Hence, if a, m are integers with $2 \leqslant a \leqslant \lceil \frac{1}{2}(m-1) \rceil$ and $m \geqslant 7$, then there is no connected graph G of size m such that $\dim_d(G) = a$ and $\mathrm{cd}(G) = m-2$. Therefore, there exist infinitely many triples a, b, m of integers, where $2 \leqslant a \leqslant b \leqslant m$, for which there is no connected graph of size m having decomposition dimension a and connected decomposition number b. This suggests the following definitions.

For a connected graph G of size $m \geqslant 2$, the *decomposition dimension ratio* $r_{\dim}(G)$ of G and the *connected decomposition number ratio* $r_{\mathrm{cd}}(G)$ of G are defined as

$$r_{\dim}(G) = \frac{\dim_d(G)}{m} \qquad \text{and} \qquad r_{\mathrm{cd}}(G) = \frac{\mathrm{cd}(G)}{m}.$$

Since $2 \leqslant \dim_d(G) \leqslant \mathrm{cd}(G) \leqslant m$ for every connected graphs G of size $m \geqslant 2$, it follows that

$$0 < r_{\dim}(G) \leqslant r_{\mathrm{cd}}(G) \leqslant 1.$$

It is shown in [13] that if G be a connected graph of order $n \geqslant 3$ and of size m, then $\dim_d(G) = m$ if and only if $G = K_3$ or $G = K_{1,n-1}$. Thus, by Theorem 1.1, we have the following.

Proposition 4.2. *Let G be a connected graph of size $m \geqslant 2$. Then $r_{\dim}(G) = 1$ if and only if $r_{\mathrm{cd}}(G) = 1$.*

Next, we show that every pair s, t of rational numbers with $0 < s \leqslant t < 1$ is realizable as the decomposition dimension ratio and connected decomposition number ratio for some connected graph. Recall that, for a graph G and a major vertex v of G, $\mathrm{ter}(v)$ is its terminal degree, $\sigma(G)$ is the sum of the terminal degrees of the major vertices of G, and $\mathrm{ex}(G)$ is the number of exterior major vertices of G. These concepts were defined in Section 2.

Theorem 4.3. *For each pair s, t of rational numbers with $0 < s \leqslant t < 1$, there is a connected graph G such that $r_{\dim}(G) = s$ and $r_{\mathrm{cd}}(G) = t$.*

Proof. Let $s = s_1/s_2$ and $t = t_1/t_2$, where s_1, s_2, t_1, t_2 be positive integers. Let $a, b \geqslant 20$ be integers such that $as_2 = bt_2$. Since $0 < s \leqslant t$, it follows that $0 < as_1/as_2 \leqslant bt_1/bt_2$. Because $as_2 = bt_2$, we obtain $0 < as_1 \leqslant bt_1$. Let $bt_1 = kas_1 + k_0$, where $k \geqslant 1$ and $0 \leqslant k_0 \leqslant as_1$. Since $b \geqslant 20$ and $k \geqslant 1$, it follows that $kbt_2 \geqslant 20$ and $kas_1 \leqslant kbt_1$. We construct a connected graph G of size kbt_2 such that $\dim_d(G) = kas_1$ and $\mathrm{cd}(G) = kbt_1$. There are two cases.

Case 1: $0 \leqslant k_0 \leqslant 5$. Let $N = k(k_0+4) - 2 \geqslant 2$ and $L = k[b(t_2 - t_1) - 2k_0 - 8] - 6 \geqslant 2$. Furthermore, let P: $v_1, v_2, \ldots, v_N$ be a copy of a path of order N and Q: $w_1, w_2, \ldots, w_L$ be a copy of a path of order L. Then the graph G is obtained from P and Q by (1) adding kas_1-1 new vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,kas_1-1}$ and joining each of these vertices to v_1, (2) for each i with $2 \leqslant i \leqslant k$, adding $kas_1 - 2$ new vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,kas_1-2}$ and joining each of these vertices to v_i, (3) for each i with $k+1 \leqslant i \leqslant N$, adding two new vertices $u_{i,1}$, $u_{i,2}$ and joining these two vertices to v_i, and (4) adding the edge $u_{N,1}w_1$. Then the size of G is

$$\begin{aligned} m &= |E(P)| + |E(Q)| + (kas_1 - 1) + (k-1)(kas_1 - 2) + 2(N-k) + 1 \\ &= (N-1) + (L-1) + (kas_1 - 1) + (k-1)(kas_1 - 2) + 2(N-k) + 1 \\ &= kbt_2 - kbt_1 + k^2as_1 + kk_0 \\ &= kbt_2 - kbt_1 + k(bt_1 - k_0) + kk_0 = kbt_2 = kas_2. \end{aligned}$$

Since $\mathrm{ex}(G) = N = k(k_0+4) - 2$ and

$$\sigma(G) = (kas_1 - 1) + (k-1)(kas_1 - 2) + 2(N-k),$$

it then follows by Theorem 2.4 that

$$\mathrm{cd}(G) = k^2as_1 + k_0k = k(bt_1 - k_0) + kk_0 = kbt_1.$$

Thus, it remains to show that $\dim_d(G) = kas_1$. Since v_1 is adjacent to $kas_1 - 1$ end-vertices in G, it follows by Theorem 2.1 that $\dim_d(G) \geqslant kas_1$. On the other hand, let $\mathscr{D} = \{G_1, G_2, \ldots, G_{kas_1}\}$ be a decomposition of G, where $E(G_1) = E(P) \cup E(Q) \cup \{u_{i1}v_i \colon 1 \leqslant i \leqslant N\} \cup \{u_{N,1}w_1\}$, $E(G_2) = \{u_{i2}v_i \colon 1 \leqslant i \leqslant N\}$, $E(G_j) = \{u_{ij}v_i \colon 1 \leqslant i \leqslant k\}$ for $3 \leqslant j \leqslant kas_1 - 2$, $E(G_{kas_1-1}) = \{u_{1,kas_1-1}v_1\}$, and $E(G_{kas_1}) = \{u_{N,2}v_N\}$. Since $d(u_{ij}v_i, G_{kas_1-1}) = i$ for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant kas_1 - 2$, $d(v_iv_{i+1}, G_{kas_1-1}) = i$ for $1 \leqslant i \leqslant N-1$, $d(u_{N,1}w_1, G_{kas_1-1}) = N+1$, $d(w_iw_{i+1}, G_{kas_1-1}) = N+1+i$ for $1 \leqslant i \leqslant L-1$, $d(u_{i1}v_i, G_{kas_1}) = N+1-i$ for $1 \leqslant i \leqslant N$, and $d(v_iv_{i+1}, G_{kas_1}) = N-i$ for $1 \leqslant i \leqslant N-1$, it follows that $\mathscr{D}$ is a resolving decomposition of G, implying that $\dim_d(G) \leqslant |\mathscr{D}| = kas_1$. Therefore, $\dim_d(G) = kas_1$.

Case 2: $5 < k_0 < as_1$. Let $N = 4k+2 \geqslant 6$ and let $L = kb(t_2 - t_1) - 2(4k+1) \geqslant 10$. Furthermore, let $P\colon v_1, v_2, \ldots, v_N$ be a copy of a path of order N and $Q\colon w_1, w_2, \ldots, w_L$ be a copy of a path of order L. Then the graph G is obtained from P and Q by (1) adding $kas_1 - 1$ new vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,kas_1-1}$ and joining these vertices to v_1, (2) for each i with $2 \leqslant i \leqslant k$, adding $kas_1 - 2$ new vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,kas_1-2}$ and joining these vertices to v_i, (3) for each i with $k+1 \leqslant i \leqslant 4k$, adding two new vertices $u_{i,1}$, $u_{i,2}$ and joining these two vertices to v_i, (4) adding $kk_0 - 2$ new vertices $u_{N-1,1}, u_{N-1,2}, \ldots, u_{N-1,kk_0-2}$ and joining these vertices to v_{N-1}, (5) adding two new vertices $u_{N,1}$, $u_{N,2}$ and joining these two vertices to v_N, and (6) adding the edge $u_{N,1}w_1$. Then the size of G is

$$\begin{aligned} m &= |E(P)| + |E(Q)| + (kas_1 - 1) + (k-1)(kas_1 - 2) + 2(3k+1) + (kk_0 - 2) + 1 \\ &= (N-1) + (L-1) + (kas_1 - 1) + (k-1)(kas_1 - 2) + 6k + kk_0 + 1 \\ &= k^2as_1 + kk_0 + kbt_2 - kbt_1 = kbt_2 = kas_2. \end{aligned}$$

Since $\mathrm{ex}(G) = N = 4k+2$ and

$$\sigma(G) = (kas_1 - 1) + (k-1)(kas_1 - 2) + 2(3k+1) + (kk_0 - 2),$$

it follows by Theorem 2.4 that

$$\mathrm{cd}(G) = k^2as_1 + kk_0 = kbt_1.$$

Thus it remains to show that $\dim_d(G) = kas_1$. Since $\mathrm{ter}(v_1) > \mathrm{ter}(v_i) > \mathrm{ter}(v_N)$ for all $2 \leqslant i \leqslant N-1$, it follows by Theorem 2.1 that

$$\dim_d(G) \geqslant \mathrm{ter}(v_1) + 1 = (kas_1 - 1) + 1 = kas_1.$$

On the other hand, let $\mathscr{D} = \{G_1, G_2, \ldots, G_{kas_1}\}$ be a decomposition of G, where $E(G_1) = E(P) \cup E(Q) \cup \{u_{i1}v_i\colon 1 \leqslant i \leqslant N\} \cup \{u_{N,1}w_1\}$, $E(G_2) = \{u_{i2}v_i\colon 1 \leqslant i \leqslant N-1\}$, $E(G_j) = \{u_{ij}v_i\colon 1 \leqslant i \leqslant k,\ i = N-1\}$ for $3 \leqslant j \leqslant kk_0 - 2$, $E(G_j) = \{u_{ij}v_i\colon 1 \leqslant i \leqslant k\}$ for $kk_0 - 1 \leqslant j \leqslant kas_1 - 2$, $E(G_{kas_1-1}) = \{u_{1,kas_1-1}v_1\}$, and $E(G_{kas_1}) = \{u_{N,2}v_N\}$. Since $d(u_{ij}v_i, G_{kas_1-1}) = i$ for $1 \leqslant i \leqslant k$, $i = N_1$, and $1 \leqslant j \leqslant kas_1 - 2$, $d(v_iv_{i+1}, G_{kas_1-1}) = i$ for $1 \leqslant i \leqslant N-1$, $d(u_{N,1}w_1, G_{kas_1-1}) = N+1$, $d(w_iw_{i+1}, G_{kas_1-1}) = N+1+i$ for $1 \leqslant i \leqslant L-1$, $d(u_{i1}v_i, G_{kas_1}) = N+1-i$ for $1 \leqslant i \leqslant N$, and $d(v_iv_{i+1}, G_{kas_1}) = N-i$ for $1 \leqslant i \leqslant N-1$, it follows that $\mathscr{D}$ is a resolving decomposition of G, implying that $\dim_d(G) \leqslant |\mathscr{D}| = kas_1$. Thus $\dim_d(G) = kas_1$.

Hence, in either case, we construct a connected graph G of size kbt_2 such that $\dim_d(G) = kas_1$ and $\mathrm{cd}(G) = kbt_1$. Therefore,

$$r_{\dim}(G) = \frac{kas_1}{kas_2} = \frac{s_1}{s_2} = s \qquad \text{and} \qquad r_{\mathrm{cd}}(G) = \frac{kbt_1}{kbt_2} = \frac{t_1}{t_2} = t,$$

as desired. □

Acknowledgments. We are grateful to Professor Gary Chartrand for suggesting the concept of connected resolving decomposition to us and kindly providing useful information on this topic. Also, we thank Professor Peter Slater for the useful conversation.

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