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THE s-PERRON, sap-PERRON AND ap-MCSHANE INTEGRALS

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Abstract. In this paper, we study the s-Perron, sap-Perron and ap-McShane integrals. In particular, we show that the s-Perron integral is equivalent to the McShane integral and that the sap-Perron integral is equivalent to the ap-McShane integral.

Keywords: s-Perron integral, sap-Perron integral, ap-McShane integral

MSC 2000: 26A39, 28B05

1. Introduction

The major and minor functions are first defined using the upper and lower derivates, and then the Perron integral is defined using the major and minor functions. Similarly, the ap-major and ap-minor functions are first defined using the upper and lower approximate derivates, and then the ap-Perron integral is defined using the ap-major and ap-minor functions.

It is well-known [4] that the Perron integral is equivalent to the Henstock integral and that the ap-Perron integral is equivalent to the ap-Henstock integral.

In this paper, we change the definitions of major and minor functions by strong derivates and strong approximate derivates rather than ordinary derivates and approximate derivates, and then define the s-Perron and sap-Perron integrals using such major and minor functions. We also define the ap-McShane integral, and then show that the s-Perron integral is equivalent to the McShane integral and that the sap-Perron integral is equivalent to the ap-McShane integral.

Let $F\colon [a,b] \to \mathbb{R}$ be a function. The *upper* and *lower derivates* of F at c are defined by

$$\overline{D}F(c) = \lim_{\delta\to 0^+} \sup\left\{\frac{F(x)-F(c)}{x-c}\colon\ 0<|x-c|<\delta\right\};$$
$$\underline{D}F(c) = \lim_{\delta\to 0^+} \inf\left\{\frac{F(x)-F(c)}{x-c}\colon\ 0<|x-c|<\delta\right\}.$$

The function F is said to be *differentiable* at $c \in [a,b]$ if $\overline{D}F(c)$ and $\underline{D}F(c)$ are finite and equal. This common value is called the *derivative* of F at c and is denoted by $F'(c)$.

We first define the strong derivates of a function.

Definition 2.1. Let $F\colon [a,b] \to \mathbb{R}$ be a function and let $c \in [a,b]$. The *upper* and *lower strong derivates* of F at c are defined by

$$\overline{SD}F(c) = \lim_{\delta\to 0^+} \sup\left\{\frac{F(y)-F(x)}{y-x}\colon\ [x,y] \subseteq (c-\delta, c+\delta)\cap[a,b]\right\};$$
$$\underline{SD}F(c) = \lim_{\delta\to 0^+} \inf\left\{\frac{F(y)-F(x)}{y-x}\colon\ [x,y] \subseteq (c-\delta, c+\delta)\cap[a,b]\right\}.$$

The function F is *strongly differentiable at* c if $\overline{SD}F(c)$ and $\underline{SD}F(c)$ are finite and equal. This common value is called the *strong derivative* of F at c and is denoted by $F'_s(c)$.

Note that the interval $[x,y]$ need not contain the point c in the above definition. From definition, it is clear that

$$\underline{SD}F \leqslant \underline{D}F \leqslant \overline{D}F \leqslant \overline{SD}F.$$

From this relation, it is obvious that if F is strongly differentiable at c, then it is differentiable at c and $F'_s(c) = F'(c)$.

The derivative F' of a differentiable function $F\colon [a,b] \to \mathbb{R}$ need not be continuous on $[a,b]$. Nonetheless, the following theorem shows that the strong derivative F'_s of a strongly differentiable function F is in fact continuous on $[a,b]$.

Theorem 2.2. *Let $F\colon [a,b]\to\mathbb{R}$ be a function. If F is strongly differentiable on $[a,b]$, then F'_s is continuous on $[a,b]$.*

Proof. Let $c\in[a,b]$ and let $\varepsilon>0$. Since F is strongly differentiable at c, there exists $\delta>0$ such that

$$\left|\frac{F(y)-F(x)}{y-x}-F'_s(c)\right|<\varepsilon$$

for every interval $[x,y]\subseteq(c-\delta,c+\delta)\cap[a,b]$. If $|z-c|<\delta$ and $z\in[a,b]$, then there exists $\delta_1>0$ such that $(z-\delta_1,z+\delta_1)\cap[a,b]\subseteq(c-\delta,c+\delta)\cap[a,b]$ and $\left|\frac{F(q)-F(p)}{q-p}-F'_s(z)\right|<\varepsilon$ for every interval $[p,q]\subseteq(z-\delta_1,z+\delta_1)\cap[a,b]$, since F is strongly differentiable at z. Choose an interval $[p_0,q_0]$ such that $[p_0,q_0]\subseteq(z-\delta_1,z+\delta_1)\cap[a,b]$. Then we have

$$\begin{aligned}|F'_s(z)-F'_s(c)|&\leqslant\left|F'_s(z)-\frac{F(q_0)-F(p_0)}{q_0-p_0}\right|+\left|\frac{F(q_0)-F(p_0)}{q_0-p_0}-F'_s(c)\right|\\&<\varepsilon+\varepsilon=2\varepsilon.\end{aligned}$$

Hence F'_s is continuous at c. This completes the proof. □

Example 2.3. Let $f\colon[a,b]\to\mathbb{R}$ be a continuous function and let $F(x)=\int_a^x f$ be the indefinite Lebesgue integral of f for each $x\in[a,b]$. Then F is strongly differentiable on $[a,b]$ and $F'_s=f$ on $[a,b]$.

To show this, let $c\in[a,b]$ and let $\varepsilon>0$. Since f is continuous at c, there exists $\delta>0$ such that $|f(x)-f(c)|<\varepsilon$ if $|x-c|<\delta$ and $x\in[a,b]$. Let $[s,t]\subseteq(c-\delta,c+\delta)\cap[a,b]$. Then we have

$$\int_s^t\{f(c)-\varepsilon\}<\int_s^t f<\int_s^t\{f(c)+\varepsilon\};$$

$$f(c)-\varepsilon<\frac{F(t)-F(s)}{t-s}<f(c)+\varepsilon;$$

and it follows that $\left|\frac{F(t)-F(s)}{t-s}-f(c)\right|<\varepsilon$. Hence F is strongly differentiable at c and $F'_s(c)=f(c)$.

Let $f\colon[a,b]\to\mathbb{R}_e$ be a function, where $\mathbb{R}_e=\mathbb{R}\cup\{\pm\infty\}$. A measurable function $U\colon[a,b]\to\mathbb{R}$ is called a *major function* of f on $[a,b]$ if $\underline{D}U(x)>-\infty$ and $\underline{D}U(x)\geqslant f(x)$ for all $x\in[a,b]$. A measurable function $V\colon[a,b]\to\mathbb{R}$ is called a *minor function* of f on $[a,b]$ if $\overline{D}V(x)<\infty$ and $\overline{D}V(x)\leqslant f(x)$ for all $x\in[a,b]$.

Recall that a function $f\colon[a,b]\to\mathbb{R}_e$ is *Perron integrable* on $[a,b]$ if f has at least one major function and one minor function on $[a,b]$ and the numbers

$$\inf\{U_a^b\colon U\ \text{is a major function of}\ f\ \text{on}\ [a,b]\};$$
$$\sup\{V_a^b\colon V\ \text{is a minor function of}\ f\ \text{on}\ [a,b]\}$$

are equal, where $U_a^b=U(b)-U(a)$ and $V_a^b=V(b)-V(a)$.

Using upper and lower strong derivates, we define the strong major and strong minor functions.

Definition 2.4. Let $f\colon [a,b] \to \mathbb{R}_e$ be a function.

(1) A measurable function $U\colon [a,b] \to \mathbb{R}_e$ is an *s-major function* of f on $[a,b]$ if $\underline{SD}U(x) > -\infty$ and $\underline{SD}U(x) \geqslant f(x)$ for all $x \in [a,b]$.

(2) A measurable function $V\colon [a,b] \to \mathbb{R}$ is an *s-minor function* of f on $[a,b]$ if $\overline{SD}V(x) < \infty$ and $\overline{SD}V(x) \leqslant f(x)$ for all $x \in [a,b]$.

Now we define the s-Perron integral.

Definition 2.5. A function $f\colon [a,b] \to \mathbb{R}_e$ is *s-Perron integrable* on $[a,b]$ if f has at least one s-major function and one s-minor function on $[a,b]$ and the numbers

$$\inf\{U_a^b\colon\ U \text{ is an s-major function of } f \text{ on } [a,b]\};$$
$$\sup\{V_a^b\colon\ V \text{ is an s-minor function of } f \text{ on } [a,b]\}$$

are equal. This common value is called the *s-Perron integral* of f on $[a,b]$ and is denoted by $(\mathrm{SP})\int_a^b f$. The function f is s-Perron integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is s-Perron integrable on $[a,b]$.

It follows easily from the definition that every s-Perron integrable function is Perron integrable.

The following theorem is an immediate consequence of the definition.

Theorem 2.6. *A function $f\colon [a,b] \to \mathbb{R}_e$ is s-Perron integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there exist an s-major function U and an s-minor function V of f on $[a,b]$ such that $U_a^b - V_a^b < \varepsilon$.*

Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [c,d]$, and a free tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [a,b]$. The (free) tagged interval $(x,[c,d])$ is said to be *subordinate* to δ if

$$[c,d] \subseteq (x-\delta(x), x+\delta(x)).$$

Let $\mathscr{P} = \{(x_i,[c_i,d_i])\colon\ 1 \leqslant i \leqslant n\}$ be a finite collection of non-overlapping (free) tagged intervals in $[a,b]$. If $(x_i,[c_i,d_i])$ is subordinate to δ for each i, then we say that $\mathscr{P}$ is subordinate to δ. If $\mathscr{P}$ is subordinate to δ and $[a,b] = \bigcup\limits_{i=1}^{n} [c_i,d_i]$, then we say that $\mathscr{P}$ is a (free) tagged partition of $[a,b]$ that is subordinate to δ.

Recall that a function $f\colon [a,b] \to \mathbb{R}$ is *McShane integrable* on $[a,b]$ if there exists a real number A with the following property: for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f(\mathscr{P})-A|<\varepsilon$ whenever $\mathscr{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ, where $f(\mathscr{P})=\sum\limits_{i=1}^{n} f(x_i)(d_i-c_i)$ if $\mathscr{P}=\{(x_i,[c_i,d_i])\colon 1\leqslant i\leqslant n\}$ is a free tagged partition of $[a,b]$. The real number A is called the *McShane integral* of f on $[a,b]$ and is denoted by $(M)\int_a^b f$. The function $f\colon [a,b]\to\mathbb{R}$ is said to be *Henstock integrable* on $[a,b]$ if we replace 'free tagged partition' by 'tagged partition' in the definition of the McShane integral.

The following two theorems show that the s-Perron integral is equivalent to the McShane integral.

Theorem 2.7. *If $f\colon [a,b]\to\mathbb{R}$ is s-Perron integrable on $[a,b]$, then f is McShane integrable on $[a,b]$ and the integrals are equal.*

Proof. Let $\varepsilon>0$. By the definition, there exist an s-major function U and an s-minor function V of f on $[a,b]$ such that

$$-\varepsilon<V_a^b-(\mathrm{SP})\int_a^b f\leqslant 0\leqslant U_a^b-(\mathrm{SP})\int_a^b f<\varepsilon.$$

Since $\overline{SD}V\leqslant f\leqslant \underline{SD}U$ on $[a,b]$, for each $c\in[a,b]$ there exists $\delta(c)>0$ such that

$$\frac{U(y)-U(x)}{y-x}\geqslant f(c)-\varepsilon \quad\text{and}\quad \frac{V(y)-V(x)}{y-x}\leqslant f(c)+\varepsilon$$

whenever $[x,y]\subseteq(c-\delta(c),c+\delta(c))\cap[a,b]$. Now let

$$\mathscr{P}=\{(x_i,[c_i,d_i])\colon 1\leqslant i\leqslant n\}$$

be a free tagged partition of $[a,b]$ that is subordinate to δ. Then we have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)(d_i-c_i)-(\mathrm{SP})\int_a^b f&=\sum_{i=1}^{n}(f(x_i)(d_i-c_i)-U_{c_i}^{d_i})+U_a^b-(\mathrm{SP})\int_a^b f\\&<\sum_{i=1}^{n}\varepsilon(d_i-c_i)+\varepsilon=\varepsilon(b-a+1).\end{aligned}$$

Similarly, using the s-minor function V, we arrive at

$$\sum_{i=1}^{n} f(x_i)(d_i-c_i)-(\mathrm{SP})\int_a^b f>-\varepsilon(b-a+1).$$

Since $|f(\mathscr{P})-(\mathrm{SP})\int_a^b f|<\varepsilon(b-a+1)$, f is McShane integrable on $[a,b]$ and $(\mathrm{M})\int_a^b f=(\mathrm{SP})\int_a^b f$. $\square$

Theorem 2.8. *If $f\colon [a,b]\to\mathbb{R}$ is McShane integrable on $[a,b]$, then f is s-Perron integrable on $[a,b]$.*

Proof. Let $\varepsilon>0$. By the definition, there exists a positive function δ on $[a,b]$ such that $\bigl|f(\mathscr{P})-(M)\int_a^b f\bigr|<\varepsilon$ whenever $\mathscr{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ. For each $x\in(a,b]$, let

$$U(x)=\sup\{f(\mathscr{P})\colon\ \mathscr{P}\ \text{is a free tagged partition of}\ [a,x]\ \text{that is subordinate to}\ \delta\};$$
$$V(x)=\inf\{f(\mathscr{P})\colon\ \mathscr{P}\ \text{is a free tagged partition of}\ [a,x]\ \text{that is subordinate to}\ \delta\};$$

and let $U(a)=0=V(a)$. By the Saks-Henstock Lemma, the functions U and V are finite-valued on $[a,b]$. We prove that U is an s-major function of f on $[a,b]$; the proof that V is an s-minor function of f on $[a,b]$ is quite similar.

Fix a point $c\in[a,b]$ and let $[x,y]$ be any interval such that $[x,y]\subseteq(c-\delta(c),c+\delta(c))\cap[a,b]$. For each free tagged partition $\mathscr{P}$ of $[a,x]$ that is subordinate to δ, we find that

$$U(y)\geqslant f(\mathscr{P})+f(c)(y-x)$$

and it follows that $U(y)\geqslant U(x)+f(c)(y-x)$. This shows that $\frac{U(y)-U(x)}{y-x}\geqslant f(c)$ and hence $\underline{SD}U(c)\geqslant f(c)>-\infty$. Since $-\infty<\underline{SD}U\leqslant\underline{D}U$ on $[a,b]$, U is BVG_* on $[a,b]$ by [4, Theorem 6.21] and it follows that U is measurable on $[a,b]$ by [4, Corollary 6.9]. Hence, U is an s-major function of f.

Since $|f(\mathscr{P}_1)-f(\mathscr{P}_2)|<2\varepsilon$ for any two free tagged partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $[a,b]$ that are subordinate to δ, it follows that $U_a^b-V_a^b\leqslant 2\varepsilon$. By Theorem 2.6, the function f is s-Perron integrable on $[a,b]$. $\square$

3. The sap-Perron and ap-McShane integrals

Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_cE=\lim_{h\to0^+}\frac{\mu(E\cap(c-h,c+h))}{2h},$$

provided the limit exists. The point c is called *a point of density* of E if $d_cE=1$ and *a point of dispersion* of E if $d_cE=0$. The set E^d represents the set of all points $x\in E$ such that x is a point of density of E. From the definition, it is obvious that a point c is a point of density of E if and only if c is a point of dispersion of the complement E^c of E.

A function $F\colon[a,b]\to\mathbb{R}$ is said to be *approximately differentiable* at $c\in[a,b]$ if there exists a measurable set $E\subseteq[a,b]$ such that $c\in E^d$ and $F|_E$ is differentiable at c. The approximate derivative of F at c is denoted by $F'_{\rm ap}(c)$.

For a measurable function $F\colon [a,b] \to \mathbb{R}$, the *upper* and *lower approximate derivates* of F at $c \in [a,b]$ are defined by

$$\overline{AD}F(c) = \inf\Big\{\alpha \in \mathbb{R}\colon\ c \text{ is a point of dispersion of } \Big\{x \in [a,b]\colon\ \frac{F(x)-F(c)}{x-c} \geqslant \alpha\Big\}\Big\};$$

$$\underline{AD}F(c) = \sup\Big\{\beta \in \mathbb{R}\colon\ c \text{ is a point of dispersion of } \Big\{x \in [a,b]\colon\ \frac{F(x)-F(c)}{x-c} \leqslant \beta\Big\}\Big\}.$$

The measurable function $F\colon [a,b] \to \mathbb{R}$ is approximately differentiable at $c \in [a,b]$ if and only if $\overline{AD}F(c)$ and $\underline{AD}F(c)$ are finite and equal [4, Corollary 16.13].

Then we have the following result.

Theorem 3.1. *Let $F\colon [a,b] \to \mathbb{R}$ be measurable and let $c \in [a,b]$. Then*

$$\overline{AD}F(c) = \inf_E \sup_{x\in E} \frac{F(x)-F(c)}{x-c};$$

$$\underline{AD}F(c) = \sup_E \inf_{x\in E} \frac{F(x)-F(c)}{x-c},$$

where the infimum and supremum are taken over all measurable sets E containing c as a density point.

P r o o f. Suppose that $\overline{AD}F(c) < \alpha$. By the definition, c is a point of dispersion of the set $\big\{x \in [a,b]\colon \frac{F(x)-F(c)}{x-c} \geqslant \alpha\big\}$. Let $D = \big\{x \in [a,b]\colon \frac{F(x)-F(c)}{x-c} < \alpha\big\}$. Then c is a point of density of the set D. Since $\sup\limits_{x\in D} \frac{F(x)-F(c)}{x-c} \leqslant \alpha$, we have $\inf\limits_E \sup\limits_{x\in E} \frac{F(x)-F(c)}{x-c} \leqslant \alpha$. Hence $\overline{AD}F(c) \geqslant \inf\limits_E \sup\limits_{x\in E} \frac{F(x)-F(c)}{x-c}$.

To prove the reverse inequality, let $\inf\limits_E \sup\limits_{x\in E} \frac{F(x)-F(c)}{x-c} < \beta$. Then there exists a measurable set E with $c \in E^d$ such that $\sup\limits_{x\in E} \frac{F(x)-F(c)}{x-c} < \beta$. Since $E \subseteq \big\{x\colon \frac{F(x)-F(c)}{x-c} < \beta\big\}$, we have $E^c \supseteq \big\{x\colon \frac{F(x)-F(c)}{x-c} \geqslant \beta\big\}$ and c is a point of dispersion of the set $\big\{x\colon \frac{F(x)-F(c)}{x-c} \geqslant \beta\big\}$. This shows that $\overline{AD}F(c) \leqslant \beta$. Hence we have $\overline{AD}F(c) \leqslant \inf\limits_E \sup\limits_{x\in E} \frac{F(x)-F(c)}{x-c}$. The case for the lower approximate derivate is similar. This completes the proof. □

Now we define the upper and lower strong approximate derivates of a measurable function.

Definition 3.2. Let $F\colon [a,b] \to \mathbb{R}$ be measurable and let $c \in [a,b]$. The *upper* and *lower strong approximate derivates* of F at c are defined by

$$\overline{SAD}F(c) = \inf_E \sup \left\{ \frac{F(x)-F(y)}{x-y} \colon x,y \in E, \ x \neq y \right\};$$
$$\underline{SAD}F(c) = \sup_E \inf \left\{ \frac{F(x)-F(y)}{x-y} \colon x,y \in E, \ x \neq y \right\},$$

where the infimum and supremum are taken over all measurable sets E containing c as a density point. The function F is *strongly approximately differentiable* at $c \in [a,b]$ if $\overline{SAD}F(c)$ and $\underline{SAD}F(c)$ are finite and equal. This common value is called the *strong approximate derivative* of F at c and is denoted by $F'_{\mathrm{sap}}(c)$.

For a measurable function $F\colon [a,b] \to \mathbb{R}$, it is easy to see that

$$\underline{SD}F \leqslant \begin{cases} \underline{SAD}F \\ \underline{D}F \end{cases} \leqslant \underline{AD}F \leqslant \overline{AD}F \leqslant \begin{cases} \overline{SAD}F \\ \overline{D}F \end{cases} \leqslant \overline{SD}F.$$

Using strong approximate derivates, it is possible to define the strong approximate major and strong approximate minor functions, and then the sap-Perron integral can be defined.

Definition 3.3. Let $f\colon [a,b] \to \mathbb{R}_e$ be a function.

(1) A measurable function $U\colon [a,b] \to \mathbb{R}$ is an *sap-major function* of f on $[a,b]$ if $\underline{SAD}U(x) > -\infty$ and $\underline{SAD}U(x) \geqslant f(x)$ for all $x \in [a,b]$.

(2) A measurable function $V\colon [a,b] \to \mathbb{R}$ is an *sap-minor function* of f on $[a,b]$ if $\overline{SAD}V(x) < \infty$ and $\overline{SAD}V(x) \leqslant f(x)$ for all $x \in [a,b]$.

Suppose that U is an sap-major function and that V is an sap-minor function of f on $[a,b]$. Since $0 \leqslant \underline{SAD}U - \overline{SAD}V = \underline{SAD}(U-V) \leqslant \underline{AD}(U-V)$, $U-V$ is nondecreasing on $[a,b]$ by [4, Theorem 17.3]. It follows that $V_a^b \leqslant U_a^b$ and that $0 \leqslant U_c^d - V_c^d \leqslant U_a^b - V_a^b$ whenever $[c,d]$ is a subinterval of $[a,b]$.

In particular,

$$\begin{aligned} -\infty &< \sup\{V_a^b \colon V \text{ is an sap-minor function of } f \text{ on } [a,b]\} \\ &\leqslant \inf\{U_a^b \colon U \text{ is an sap-major function of } f \text{ on } [a,b]\} < \infty. \end{aligned}$$

Definition 3.4. A function $f\colon [a,b] \to \mathbb{R}_e$ is *sap-Perron integrable* on $[a,b]$ if f has at least one sap-major function and one sap-minor function on $[a,b]$ and the numbers

$$\inf\{U_a^b \colon U \text{ is an sap-major function of } f \text{ on } [a,b]\};$$
$$\sup\{V_a^b \colon V \text{ is an sap-minor function of } f \text{ on } [a,b]\}$$

are equal. This common value is called the *sap-Perron integral* of f on $[a,b]$ and is denoted by $(\mathrm{SAP})\int_a^b f$. The function f is sap-Perron integrable on $E \subseteq [a,b]$ if $f\chi_E$ is sap-Perron integrable on $[a,b]$.

The following theorem is an immediate consequence of the definition.

Theorem 3.5. *A function $f\colon [a,b] \to \mathbb{R}_e$ is sap-Perron integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there exist an sap-major function U and an sap-minor function V of f on $[a,b]$ such that $U_a^b - V_a^b < \varepsilon$.*

An *approximate neighborhood* (or *ap-nbd*) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x and having density 1 at x. For every $x \in [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x\colon x \in [a,b]\}$ is a *choice* on $[a,b]$.

A (free) tagged interval $(x,[c,d])$ is said to be *subordinate* to the choice S if $c,d \in S_x$. Let $\mathscr{P} = \{(x_i,[c_i,d_i])\colon 1 \leqslant i \leqslant n\}$ be a finite collection of non-overlapping (free) tagged intervals. If $(x_i,[c_i,d_i])$ is subordinate to S for each i, then we say that $\mathscr{P}$ is subordinate to S. If $\mathscr{P}$ is subordinate to S and $[a,b] = \bigcup\limits_{i=1}^{n} [c_i,d_i]$, then we say that $\mathscr{P}$ is a (free) tagged partition of $[a,b]$ that is subordinate to S.

A function $f\colon [a,b] \to \mathbb{R}$ is said to be *ap-Henstock integrable* on $[a,b]$ if there exists a real number A with the following property: for each $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|f(\mathscr{P}) - A| < \varepsilon$ whenever $\mathscr{P}$ is a tagged partition of $[a,b]$ that is subordinate to S.

It is well-known [4] that the ap-Henstock integral is equivalent to the ap-Perron integral.

We now present the definition of the ap-McShane integral.

Definition 3.6. A function $f\colon [a,b] \to \mathbb{R}$ is *ap-McShane integrable* on $[a,b]$ if there exists a real number A with the following property: for every $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|f(\mathscr{P}) - A| < \varepsilon$ whenever $\mathscr{P}$ is a free tagged partition of $[a,b]$ that is subordinate to S. The real number A is called the *ap-McShane integral* of f on $[a,b]$ and is denoted by $(\mathrm{AM})\int_a^b f$. The function f is ap-McShane integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-McShane integrable on $[a,b]$.

It is clear from the definitions of both integrals that every McShane integrable function is ap-McShane integrable and every ap-McShane integrable function is ap-Henstock integrable.

The following two theorems show that the sap-Perron integral is equivalent to the ap-McShane integral.

Theorem 3.7. *If $f\colon [a,b]\to\mathbb{R}$ is sap-Perron integrable on $[a,b]$, then f is ap-McShane integrable on $[a,b]$ and the integrals are equal.*

Proof. Let $\varepsilon>0$. By the definition, there exist an sap-major function U and an sap-minor function V of f on $[a,b]$ such that

$$-\varepsilon < V_a^b - (\mathrm{SAP})\int_a^b f \leqslant 0 \leqslant U_a^b - (\mathrm{SAP})\int_a^b f < \varepsilon.$$

Since $\overline{SAD}V \leqslant f \leqslant \underline{SAD}U$, for each $x\in[a,b]$ there exists an ap-nbd S_x such that

$$f(x)-\varepsilon < \frac{U(d)-U(c)}{d-c} \quad\text{and}\quad \frac{V(d)-V(c)}{d-c} < f(x)+\varepsilon$$

for all $c,d\in S_x$ with $c\neq d$. Let $\mathscr{P}=\{(x_i,[c_i,d_i])\colon 1\leqslant i\leqslant n\}$ be a free tagged partition of $[a,b]$ that is subordinate to the choice $\{S_x\}$. Then for each i,

$$\begin{aligned} V(d_i)-V(c_i)-\varepsilon(d_i-c_i) &< f(x_i)(d_i-c_i) \\ &< U(d_i)-U(c_i)+\varepsilon(d_i-c_i). \end{aligned}$$

Summing over i,

$$V_a^b-\varepsilon(b-a) < f(\mathscr{P}) < U_a^b+\varepsilon(b-a).$$

Hence, we have

$$\begin{aligned} -\varepsilon(b-a+1) &< \{f(\mathscr{P})-V_a^b\} + \Big\{V_a^b-(\mathrm{SAP})\int_a^b f\Big\} \\ &= f(\mathscr{P})-(\mathrm{SAP})\int_a^b f \\ &= \{f(\mathscr{P})-U_a^b\} + \Big\{U_a^b-(\mathrm{SAP})\int_a^b f\Big\} \\ &< \varepsilon(b-a+1). \end{aligned}$$

This shows that

$$\Big|f(\mathscr{P})-(\mathrm{SAP})\int_a^b f\Big| < \varepsilon(b-a+1)$$

for every free tagged partition $\mathscr{P}$ of $[a,b]$ that is subordinate to $\{S_x\}$. It follows that f is ap-McShane integrable on $[a,b]$ and $(\mathrm{AM})\int_a^b f = (\mathrm{SAP})\int_a^b f$. $\square$

Theorem 3.8. *If $f\colon [a,b] \to \mathbb{R}$ is ap-McShane integrable on $[a,b]$, then f is sap-Perron integrable on $[a,b]$.*

Proof. Let $\varepsilon > 0$. By the definition, there exists a choice $S = \{S_x\}$ on $[a,b]$ such that $\bigl|f(\mathscr{P}) - (\mathrm{AM})\int_a^b f\bigr| < \varepsilon$ whenever $\mathscr{P}$ is a free tagged partition of $[a,b]$ that is subordinate to S. Without loss of generality, we may assume that each point of S_x is a point of density of S_x. For each $x \in (a,b]$, let

$$U(x) = \sup\{f(\mathscr{P})\colon\ \mathscr{P} \text{ is a free tagged partition of } [a,x] \text{ that is subordinate to } S\};$$
$$V(x) = \inf\{f(\mathscr{P})\colon\ \mathscr{P} \text{ is a free tagged partition of } [a,x] \text{ that is subordinate to } S\};$$

and let $U(a) = 0 = V(a)$. By the Saks-Henstock Lemma, the functions U and V are finite-valued on $[a,b]$. We prove that U is an sap-major function of f on $[a,b]$; the proof that V is an sap-minor function of f on $[a,b]$ is quite similar.

From the proof of [4, Theorem 17.15], it follows that U is a measurable function. Fix $c \in [a,b]$. Let $[p,q]$ be any interval with $p, q \in S_c$. For each free tagged partition $\mathscr{P}$ of $[a,p]$ that is subordinate to S we have

$$U(q) \geqslant f(\mathscr{P}) + f(c)(q-p)$$

and it follows that

$$U(q) \geqslant U(p) + f(c)(q-p);$$
$$\frac{U(q)-U(p)}{q-p} \geqslant f(c).$$

Since p and q are arbitrary points of S_c with $p<q$, we have $\inf\limits_{p,q\in S_c} \frac{U(q)-U(p)}{q-p} \geqslant f(c)$ and hence $\underline{SAD}U(c) \geqslant f(c) > -\infty$. This shows that U is an sap-major function of f on $[a,b]$.

Since

$$|f(\mathscr{P}_1) - f(\mathscr{P}_2)| \leqslant \left|f(\mathscr{P}_1) - (\mathrm{AM})\int_a^b f\right| + \left|(\mathrm{AM})\int_a^b f - f(\mathscr{P}_2)\right| < 2\varepsilon$$

for any two free tagged partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $[a,b]$ that are subordinate to S, it follows that $U_a^b - V_a^b \leqslant 2\varepsilon$. Hence the function f is sap-Perron integrable on $[a,b]$ by Theorem 3.5. □

Let $f\colon [a,b] \to \mathbb{R}$ be ap-McShane integrable on $[a,b]$ and let $F(x) = (\mathrm{AM})\int_a^x f$ for each $x \in [a,b]$. At this time, we do not know whether or not the function F is strongly approximately differentiable on $[a,b]$. But we can get the following result.

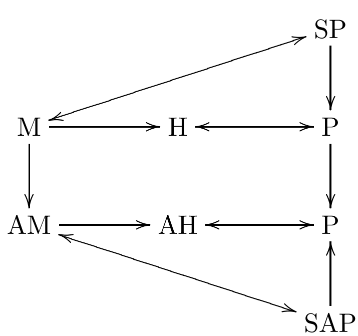
Theorem 3.9. *Let $f\colon [a,b] \to \mathbb{R}$ be ap-McShane integrable on $[a,b]$ and let $F(x) = (\mathrm{AM})\int_a^x f$ for each $x \in [a,b]$. If there exists a measurable set $E \subseteq [a,b]$ with $\mu([a,b]-E) = 0$ such that f is relatively continuous on E (i.e., for each $c \in E$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x)-f(c)| < \varepsilon$ if $|x-c| < \delta$ and $x \in E$), then F is strongly approximately differentiable a.e. on $[a,b]$ and $F'_{\mathrm{sap}} = f$ a.e. on $[a,b]$.*

Proof. Without loss of generality, we may assume that each point of E is a point of density of E. Let $c \in E$ and let $\varepsilon > 0$. Suppose that f is relatively continuous on E. Then there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ if $|x - c| < \delta$ and $x \in E$. Let $S_c = E \cap (c-\delta, c+\delta)$. If $x, y \in S_c$ and $x < y$, then

$$\begin{aligned}
f(c) - \varepsilon &= \frac{1}{y-x}\,(\mathrm{AM})\int_x^y \{f(c)-\varepsilon\} = \frac{1}{y-x}\,(\mathrm{AM})\int_{E\cap[x,y]} \{f(c)-\varepsilon\} \\
&\leqslant \frac{1}{y-x}\,(\mathrm{AM})\int_{E\cap[x,y]} f = \frac{1}{y-x}\,(\mathrm{AM})\int_x^y f = \frac{F(y)-F(x)}{y-x} \\
&\leqslant \frac{1}{y-x}\,(\mathrm{AM})\int_{E\cap[x,y]} \{f(c)+\varepsilon\} = \frac{1}{y-x}\,(\mathrm{AM})\int_x^y \{f(c)+\varepsilon\} \\
&= f(c) + \varepsilon.
\end{aligned}$$

It follows that $\left|\frac{F(y)-F(x)}{y-x} - f(c)\right| < \varepsilon$. Hence F is strongly approximately differentiable at c and $F'_{\mathrm{sap}}(c) = f(c)$. This completes the proof. □

Now we present a diagram relating the integrals we have been discussing: McShane integral (M), Henstock integral (H), Perron integral (P), s-Perron integral (SP), ap-McShane integral (AM), ap-Henstock integral (AH), ap-Perron integral (AP), and sap-Perron integral (SAP).



In the above diagram, an arrow stands for implication. For example, the implication ⌈AM → AH⌋ represents that if a function f is ap-McShane integrable, then it is ap-Henstock integrable and ⌈AH ↔ AP⌋ represents that a function f is ap-Henstock integrable if and only if f is ap-Perron integrable.

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