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ON THE INSTABILITY OF LINEAR NONAUTONOMOUS DELAY SYSTEMS

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Abstract. The unstable properties of the linear nonautonomous delay system $x'(t) = A(t)x(t) + B(t)x(t - r(t))$, with nonconstant delay $r(t)$, are studied. It is assumed that the linear system $y'(t) = (A(t) + B(t))y(t)$ is unstable, the instability being characterized by a nonstable manifold defined from a dichotomy to this linear system. The delay $r(t)$ is assumed to be continuous and bounded. Two kinds of results are given, those concerning conditions that do not include the properties of the delay function $r(t)$ and the results depending on the asymptotic properties of the delay function.

Keywords: Liapounov instability, h-instability, instability of delay equations, nonconstant delays

MSC 2000: 34D20, 34D05

1. Introduction

In this paper sufficient conditions for the solutions of the equation

(1) $$x'(t) = A(t)x(t) + B(t)x(t - r(t)), \quad x \in \mathbb{C}^n, \ t \geqslant t_0,$$

to be unstable are given. The problem of stability and instability, in the case of constant matrices A, B and a constant delay, r has been studied by many authors [5], [10]. For a good acquaintance with the subject, as well as for the application to mathematical ecology, the reader is referred to the monograph [7]. The constant case is frequently studied by means of the location of the roots of the so called characteristic polynomial: $P(\lambda) = \det(\lambda I - A - Be^{-r\lambda})$. An outstanding result

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ensures that the solutions of the equation

$$x'(t) = Ax(t) + Bx(t-r)$$

are unstable if there exists a root of the characteristic polynomial with a positive real part. This algebraic method cannot be applied to the nonautonomous Eq. (1).

The unstable properties of Eq. (1) have not been sufficiently studied [5], [7], [10], [11]. This contrasts with the evolution of this theory for ordinary differential equations, to which we may mention the classical result of Perron [1] and Coppel's theorem on instability [3] for nonautonomous systems.

In this paper we will give results on instability and asymptotic instability of Eq. (1), relying on the ideas of the paper [12]. We will distinguish the case when B is integrable, where, under suitable conditions, the instability can be obtained for a general bounded delay $r(t)$. The case when B is not integrable is treated in our paper by means of conditions of boundedness and integrability of the function $r(t)B(t)$. In a section of examples we will apply the results obtained to different classes of delay differential equations.

2. Basic definitions and notation

The symbol $\mathbf{V}$ will denote the linear space $\mathbb{R}^n$ or $\mathbb{C}^n$; $|x|$ stands for a fixed norm of the vector x, and the corresponding matrix norm of a matrix A will be denoted by $|A|$. We will assume that the function $r(t)\colon [t_0,\infty) \to [0,\sigma]$, $\sigma > 0$, is continuous. Throughout, we will denote $J = [t_0,\infty)$, $J_\sigma = [t_0-\sigma,\infty)$; the functions $A, B\colon J_\sigma \to \mathbf{V}$, are assumed to be continuous; Φ will denote the fundamental matrix of

$$y'(t) = (A(t)+B(t))y(t) \tag{2}$$

satisfying $\Phi(t_0) = I$, where the matrix I denotes the identity; for an interval $I_{a,b}$ of real numbers, we will denote by $C(I_{a,b})$ the space of continuous and bounded functions defined on $I_{a,b}$ with values on $\mathbf{V}$; if $x \in C(J_\sigma)$ and $t \geqslant t_0$, then $x_t \in C([-\sigma,0])$ will denote the function $x_t(s) = x(t+s)$, $s \in [-\sigma,0]$; in the space $C([-\sigma,0])$, the following norm will be used:

$$|\varphi|_\sigma = \max_{s\in[-\sigma,0]} |\varphi(s)|;$$

$x(t;t_0,\varphi)$ will denote the unique solution of the problem

$$\begin{cases} x'(t) = A(t)x(t) + B(t)x(t-r(t)), & t \geqslant t_0, \\ x_{t_0} = \varphi, & \varphi \in C([-\sigma,0]); \end{cases} \tag{3}$$

the letters h, k, p, q will denote positive continuous functions, and $h^{-1}(t) = 1/h(t)$; for $f \in C(J_\sigma)$, let $|f|^\infty := \sup\{|f(t)|\colon t \in J_\sigma\}$, $C_h(J_\sigma) := \{f\colon J_\sigma \to \mathbf{V}\colon h^{-1}f \in C(J_\sigma)\}$; for $f \in C_h(J_\sigma)$, let $|f|_h := |h^{-1}f|^\infty$ and $B_h[0,\varrho] = \{f \in C_h(J_\sigma)\colon |f|_h \leqslant \varrho\}$; $C_h(J_\sigma)$, called the space of h-bounded functions, endowed with the norm $|\cdot|_h$ is a Banach space; $L^1(J)$ will denote the space of integrable functions defined on J, with norm $|f|^1 = \int_{t_0}^\infty |f(s)|\,\mathrm{d}s$; $L^1_h(J)$ is the space of h-integrable functions, that is $f \in L^1_h$ iff $|f|^1_h := \int_{t_0}^\infty h^{-1}(s)|f(s)|\,\mathrm{d}s < \infty$.

The instability of Eq. (2) will be characterized by means of the following notion of dichotomy [13]:

Definition 1. We say that Eq. (2) has a weak $([h,p],[k,q])$-dichotomy on J_σ, iff there exist a constant K and a projection matrix P such that

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leqslant Kh(t)p(s), \quad t_0-\sigma \leqslant s \leqslant t, \\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leqslant Kk(t)q(s), \quad t_0-\sigma \leqslant t \leqslant s, \end{aligned} \tag{4}$$

and

$$\begin{cases} h(t)p(s) \leqslant Ck(t)q(s), & t \geqslant s \geqslant t_0-\sigma, \\ k(t)q(s) \leqslant Ch(t)p(s), & s \geqslant t \geqslant t_0-\sigma, \end{cases} \tag{5}$$

where $C \geqslant 1$ is a constant.

Remark 1. If $h = k$, $p = q$, then we say that Eq. (1) possesses an $[h,p]$-dichotomy. If Eq. (1) has an $([h,p,],[k,q])$-dichotomy, then condition (5) implies that Eq. (1) has an $[h,p]$-dichotomy, and also a $[k,q]$-dichotomy, each with the same projection matrix P and constant CK.

Remark 2. If $p(t) = h^{-1}(t)$, $q(t) = k^{-1}(t)$ [19], [15], [16], then we say that Eq. (1) possesses an (h,k)-dichotomy. In this case the hypotheses (5) reduce to the requirement:

$$h(t)h^{-1}(s) \leqslant Ck(t)k^{-1}(s), \quad t \geqslant s \geqslant t_0-\sigma. \tag{6}$$

The case of an (h,h)-dichotomy will be termed an h-dichotomy. If Eq. (1) possesses an (h,k)-dichotomy, then (6) implies that Eq. (1) has an h-dichotomy, and also has a k-dichotomy, each with the same projection matrix P and constant CK.

We call the attention to the use of square brackets to denote an $[h,p]$-dichotomy. This is made deliberately to distinguish this dichotomy from the notation of the (h,k)-dichotomies in the sense of Pinto [19], where we use parentheses instead.

We will use the following subspaces of initial conditions to Eq. (2):

$$\mathbf{V}_h = \{\xi \in \mathbf{V}\colon\ \Phi(t)\xi \in C_h\},\quad \mathbf{V}_{h,0} = \{\xi \in \mathbf{V}_h\colon\ \lim_{t\to\infty} h(t)^{-1}\Phi(t)\xi = 0\}.$$

The forthcoming Theorem A follows in a way similar to the proof of Proposition 2.2 in [4] (see also [15], [17], [14]), and Theorem B follows from a result on admissibility of a pair of functional spaces [4], [14].

Theorem A. *Let us assume that Eq.* (2) *has the weak dichotomy* (4)–(5), *then Eq.* (2) *has an* $([h,p],[k,q])$*-dichotomy with a projection* Q, *iff*

$$\mathbf{V}_{k,0} \subset Q[\mathbf{V}] \subset \mathbf{V}_h.$$

Theorem B. *If the system*

$$x'(t) = A(t)x(t)$$

has a weak $[h,p]$*-dichotomy and* $hpB \in L^1$, *then the system*

$$y'(t) = [A(t) + B(t)]y(t)$$

has a weak $[h,p]$*-dichotomy.*

Throughout, the functions h, k will be assumed to have a bounded growth.

Definition 2. We say that the function $h\colon\ J_\sigma \to (0,\infty)$ is of class $G_{\sigma,M}$, for a positive number M, iff

$$h(s)h^{-1}(t) \leqslant M,\quad s \in [t-\sigma, t+\sigma],\ \ t \geqslant t_0.$$

We will use the following definitions of instability:

Definition 3. We say that the null solution of Eq. (2) is h-stable on the interval J, iff for every $\varepsilon > 0$ there exists a $\delta > 0$ ($\delta = \delta(t_0,\varepsilon)$) such that if $\varphi \in C([-\sigma,0])$ and $|\varphi|_\sigma < \delta$, then the solution $x(t;t_0,\varphi)$ exists on all J and $h^{-1}(t)|x(t;t_0,\varphi)| < \varepsilon$ for all $t \geqslant t_0$. In addition to the above property, if for every $|\varphi|_\sigma < \delta$ we have

$$\lim_{t\to\infty} h^{-1}(t)x(t;t_0,\varphi) = 0, \tag{7}$$

then the null solution of Eq. (2) is called h-asymptotically stable.

Definition 4. We say that the null solution of Eq. (2) is h-unstable on the interval J, iff there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exist an initial value function $\varphi_\delta \in C([-\sigma,0])$, $|\varphi|_\sigma < \delta$, and a $\tau_\delta \geqslant t_0$ such that $|x(\tau_\delta;t_0,\varphi_\delta)| \geqslant \varepsilon$.

3. Instability under integrable conditions

It is convenient to write Eq. (1) in the equivalent form

$$x'(t) = (A(t) + B(t))x(t) + B(t)(x(t - r(t)) - x(t)). \tag{8}$$

Regarding this equation, for $t \geqslant t_0$ let us define the operator

$$\begin{aligned}\mathscr{U}[x](t) = &\int_{t_0}^{t} \Phi(t)P\Phi^{-1}(s)B(s)(x(s - r(s)) - x(s))\,\mathrm{d}s \\ &- \int_{t}^{\infty} \Phi(t)(I - P)\Phi^{-1}(s)B(s)(x(s - r(s)) - x(s))\,\mathrm{d}s.\end{aligned}$$

We call $\mathscr{U}$ the dichotomic operator associated to Eq. (1). Note that $\mathscr{U}$ applies to functions of the space $C(J_\sigma)$. Since the function $\mathscr{U}[y]$ is not defined on $[t_0 - \sigma, t_0]$, we complete this definition in the following manner:

$$\mathscr{T}[x](t) = \begin{cases} \mathscr{U}[x](t_0), & t \in [t_0 - \sigma, t_0], \\ \mathscr{U}[x](t), & t \geqslant t_0. \end{cases}$$

Lemma 1. *If Eq.* (2) *has the dichotomy* (4)–(5), *where* $h \in G_{\sigma,M}$, *then* $hpB \in L^1$ *implies that the operator* $\mathscr{T}\colon C_h(J_\sigma) \to C_h(J_\sigma)$ *is continuous:*

$$|\mathscr{T}[x]|_h \leqslant 2KCM^2 \int_{t_0}^{\infty} h(s)p(s)|B(s)|\,\mathrm{d}s|x|_h. \tag{9}$$

Moreover, if

$$2KCM^2 \int_{t_0}^{\infty} h(s)p(s)|B(s)|\,\mathrm{d}s < 1, \tag{10}$$

then $\mathscr{T}$ *acts as a contraction.*

P r o o f. For $t \geqslant t_0$ we have the estimate

$$\begin{aligned}|h^{-1}(t)\mathscr{T}[x](t)| \leqslant K &\int_{t_0}^{t} p(s)|B(s)||x(s - r(s)) - x(s)|\,\mathrm{d}s \\ &+ K \int_{t}^{\infty} h^{-1}(t)k(t)q(s)|B(s)|\,|x(s - r(s)) - x(s)|\,\mathrm{d}s \\ &\leqslant KC(M + 1) \int_{t_0}^{\infty} h(s)p(s)|B(s)|\,\mathrm{d}s|x|_h.\end{aligned}$$

For $t \in [t_0 - \sigma, t_0]$ we may write

$$\begin{aligned}|h^{-1}(t)\mathscr{T}[x](t)| &= |h^{-1}(t)h(t_0)h^{-1}(t_0)\mathscr{T}[x](t_0)| \\ &\leqslant KCM(M + 1) \int_{t_0}^{\infty} h(s)p(s)|B(s)|\,\mathrm{d}s|x|_h.\end{aligned}$$

From these estimates the assertion of the lemma follows, because $M \geqslant 1$. □

Theorem 1. *If Eq.* (2) *has the dichotomy* (4)–(5), *where the function h is of class $G_{\sigma,M}$, and* (10) *is satisfied, then the null solution of Eq.* (1) *is h-unstable if* $\mathbf{V}_h \neq \mathbf{V}$.

Proof. Let us assume the contrary, then for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varphi \in C[-\sigma, 0]$, $|\varphi|_\sigma < \delta$ imply $|h^{-1}(t)x(t; t_0, \varphi)| < \varepsilon$, $\forall t \geqslant t_0$. Let φ be an initial value function satisfying

$$\varphi = \text{const.} = x_0, \quad 0 < |x_0| < \delta, \quad x_0 \in (I - P)[\mathbf{V}]. \tag{11}$$

We will show that $x(t; t_0, \varphi)$ is not h-bounded. This is enough to accomplish the proof of the theorem. We define

$$y(t) = x(t; t_0, \varphi) - \mathscr{T}[x(\cdot; t_0, \varphi)](t), \quad t \geqslant t_0 - \sigma.$$

By Lemma 1 the function $y(t)$ belongs to $C_h(J_\sigma)$. Besides, $y(t)$ is a solution of Eq. (2) on the interval $[t_0, \infty)$. Hence $y(t_0) \in \mathbf{V}_h$. Due to Theorem A we may assume that $y(t_0) \in P[\mathbf{V}]$. Moreover, we have

$$y(t_0) = x_0 + (I - P) \int_{t_0}^{\infty} \Phi^{-1}(s) B(s)(x(s - r(s)) - x(s)) \,\mathrm{d}s \in (I - P)[\mathbf{V}],$$

implying $y(t_0) = 0$, and consequently $y(t) = 0$, $\forall t \geqslant t_0$. But in this case $x(\cdot; t_0, \varphi)$ satisfies the integral equation

$$x(t; t_0, \varphi) = \mathscr{T}(x(\cdot, t_0, \varphi))(t), \quad t \geqslant t_0,$$

whence

$$x(t; t_0, \varphi) = \mathscr{T}(x(\cdot, t_0, \varphi))(t), \quad t \geqslant t_0 - \sigma.$$

Thus, any solution $x(\cdot; t_0, \varphi)$, where φ satisfies (11), is a fixed point of the dichotomic operator $\mathscr{T}\colon C_h(J_\sigma) \to C_h(J_\sigma)$. But condition (10) implies that the operator $\mathscr{T}$ is a contraction. Since $\mathscr{T}$ is linear, we have $x(\cdot; t_0, \varphi) = 0$, which yields the contradiction $\varphi(0) = x(t_0; t_0, \varphi) = x_0 = 0$. □

If the function h is bounded away from null ($h(t) \geqslant \alpha > 0$, $\forall t$), then the conditions of Theorem 1 imply that the solutions of Eq. (1) are unstable in the sense of Liapounov.

Corollary 1. *If Eq.* (2) *has the dichotomy* (4)–(5), *where the function k is of class $G_{\sigma,M}$, and*

$$2KCM^2 \int_{t_0}^{\infty} k(s) q(s) |B(s)| \,\mathrm{d}s < 1$$

is satisfied, then the null solution of Eq. (1) *is k-unstable if* $\mathbf{V}_k \neq \mathbf{V}$.

P r o o f. Condition (5) implies that Eq. (2) has a $[k, q]$-dichotomy. The rest of the proof follows in the same way as that of Theorem 1. $\square$

If the function $B(t)$ is integrable then the equation

$$y'(t) = (\operatorname{diag}\{-1, t^{-1}, t\} + B(t))y(t), \quad t \geqslant t_0 = 1,$$

has an (e^{-t}, t)-dichotomy. According to Theorem 1 the null solution of

$$x'(t) = \operatorname{diag}\{-1, t^{-1}, t\}x(t) + B(t)x(t - r(t)) \tag{12}$$

is e^{-t}-unstable for every bounded delay $r(t)$. This does not imply the Liapounov instability. Nevertheless, Eq. (12) has a k-dichotomy with $k(t) = t$. The condition $\mathbf{V}_k \neq \mathbf{V}$ is certainly satisfied, therefore Corollary 1 yields the Liapounov instability of Eq. (12).

The proof of Theorem 1 shows that no solution $x(t; t_0, \varphi)$ of Eq. (1) satisfying (11) is h-bounded. A natural question arises: For which initial value functions may we expect that the solution $x(t; t_0, \varphi)$ is not h-bounded? To answer this question, let us consider the set $\mathscr{I} \subset C[-\sigma, 0]$ of initial value functions defined by the following properties:

$$|h_{t_0}^{-1}\varphi|_\sigma = |h^{-1}(t_0)\varphi(0)| \neq 0, \quad \varphi(0) \in (I - P)[\mathbf{V}]. \tag{13}$$

Theorem 2. *Under the conditions of Theorem* 1*, every solution* $x(t; t_0, \varphi)$*, with* $\varphi \in \mathscr{I}$ *is h-unbounded.*

P r o o f. Assume that $\varphi \in \mathscr{I}$ and $x(t; t_0, \varphi)$ is h-bounded. By repeating the first lines of the proof of Theorem 1 we obtain

$$x(t; t_0, \varphi) = \mathscr{T}(x(\cdot; t_0, \varphi))(t), \quad t \geqslant t_0.$$

From this identity and properties (9), (13) we conclude

$$\sup_{[t_0, \infty)} |h^{-1}(t)x(\cdot; t_0, \varphi)| \leqslant 2KCM^2|hpB|^1|x(\cdot; t_0, \varphi)|_h.$$

Since $\varphi \in \mathscr{I}$, the last estimate implies

$$|x(\cdot; t_0, \varphi)|_h \leqslant 2KCM^2|hpB|^1|x(\cdot; t_0, \varphi)|_h,$$

whence $x(t; t_0, \varphi) = 0$, $t \geqslant t_0 - \sigma$, because of condition (10). Hence $\varphi = 0$, a contradiction with $\varphi \in \mathscr{I}$. $\square$

If the condition $\mathbf{V}_h \neq \mathbf{V}$ of Theorem 1 is not satisfied, then the following theorem provides the answer to the problem of instability:

Theorem 3. *If Eq.* (2) *has the dichotomy* (4)–(5), *where the function* h *is of class* $G_{\sigma,M}$, *and* (10) *is satisfied, then the null solution of Eq.* (1) *is not asymptotically* h*-stable if* $\mathbf{V}_{h,0} \neq \mathbf{V}_h$ *(respectively, the null solution of Eq.* (1) *is not asymptotically* k*-stable if* $\mathbf{V}_{k,0} \neq \mathbf{V}_k$ *and* $2KCM^2|kqB|^1 < 1$*).*

Proof. According to Remark 1, we may handle the dichotomy (4)–(5) as an $[h, p]$-dichotomy. Let us assume that the null solution of Eq. (1) is asymptotically h-stable. Then for $\varepsilon = 1$ there exists a positive δ such that $|\varphi|_\sigma < \delta$ implies $|h^{-1}(t)x(t; t_0, \varphi)| < 1$, $t \geqslant t_0$, and (7) is satisfied. Let ϱ be a positive number such that $\varrho|h_{t_0}|_\sigma < \delta$ and let γ be a positive number such that

$$\gamma + 2KCM^2|hpB|^1\varrho \leqslant \varrho.$$

Fixing a vector $y_0 \in \mathbf{V}_h \setminus \mathbf{V}_{h,0}$ with the property $|\Phi y_0|_h < \gamma$, we introduce an operator $\mathscr{F}$ defined by

$$\mathscr{F}[x](t) = \Phi(t)y_0 + \mathscr{T}[x](t), \quad t \geqslant t_0 - \sigma. \tag{14}$$

Due to the choice of γ we have the property

$$\mathscr{F}\colon\ B_h[0, \varrho] \to B_h[0, \varrho].$$

By virtue of (10) the operator $\mathscr{F}$ is contractive in this ball. Let x be the unique fixed point of $\mathscr{F}$ in $B_h[0, \varrho]$. This function is a solution of Eq. (1). Due to Theorem A we may assume that the projection P defining the $[h, p]$-dichotomy satisfies

$$\lim_{t\to\infty} h^{-1}(t)\Phi(t)P = 0.$$

This property implies the identity

$$x(t) = \Phi(t)y_0 + o(h)(t), \tag{15}$$

where $o(h)$ denotes a function satisfying $\lim\limits_{t\to\infty} h^{-1}(t)o(h)(t) = 0$. As $x = \mathscr{F}[x]$, the estimate (9), for $t \in [t_0 - \sigma, t_0]$, yields

$$|x(t)| \leqslant h(t)|\Phi y_0|_h + h(t)|\mathscr{T}[x]|_h \leqslant (\gamma + 2KCM^2|hpB|^1\varrho)|h_{t_0}|_\sigma \leqslant \varrho|h_{t_0}|_\sigma < \delta.$$

Therefore $|x_{t_0}| < \delta$, implying $x(t) = o(h)(t)$. However

$$\lim_{t\to\infty} h^{-1}(t)\Phi(t)y_0 \neq 0.$$

The last relation and (15) are contradictory. □

The forthcoming Theorem 4 and Theorem 5 are consequences of Theorem 1 and Theorem 3, respectively.

Theorem 4. *If Eq.* (2) *has an* (h,k)*-dichotomy, where the function* h *is of class* $G_{\sigma,M}$*, and*

$$2KCM^2 \int_{t_0}^{\infty} |B(s)|\,\mathrm{d}s < 1, \tag{16}$$

then the null solution of Eq. (1) *is* h*-unstable if* $\mathbf{V}_h \neq \mathbf{V}$.

Theorem 5. *If Eq.* (2) *has an* (h,k)*-dichotomy, where the function* h *is of class* $G_{\sigma,M}$ *and* (16) *is satisfied, then the null solution of Eq.* (1) *is not asymptotically* h*-stable if* $\mathbf{V}_{h,0} \neq \mathbf{V}_h$.

4. Instability for nonintegrable coefficients

The previous result is of a limited interest, since the condition (10) does not involve the time lag function $r(t)$. Let $h \in G_{\sigma,M}$. In order to incorporate the properties of $r(t)$ into the statements of our theorems, following the ideas in [8], we introduce the set $\mathscr{M}_h$ consisting of the functions belonging to $C_h(J_\sigma)$ such that

$$(\mathbf{M}_h) \qquad h^{-1}(t)|x(t) - x(t')| \leqslant M\beta(t)(t - t')|x|_h,\ \ t - \sigma \leqslant t' \leqslant t,\ \ t \geqslant t_0,\ \ t' \geqslant t_0,$$

where, for convenience, the constant M is the same as in Definition 2, and the function β is defined by

$$\beta(t) = \max\{1, |(A+B)_t|_\sigma\}.$$

By standard arguments, we can prove that $\mathscr{M}_h$ is a closed set in $C_h(J_\sigma)$.

Lemma 2. *Assume that Eq.* (2) *has the dichotomy* (4)–(5)*,* $h \in G_{\sigma,M}$*. If*

$$2KCM^2\{|\beta rhpB|^1 + |\beta rhpB|^\infty\} < 1, \tag{17}$$

then

$$\mathscr{T}\colon \mathscr{M}_h \to \mathscr{M}_h. \tag{18}$$

P r o o f. The definition of the operator $\mathscr{T}$ implies, for $t \geqslant t_0$ and $x \in C_h(J_\sigma)$, that

$$
\begin{aligned}
|h^{-1}(t)\mathscr{T}[x](t)| &\leqslant KC\int_{t_0}^{\infty} p(s)|B(s)||x(s-r(s))-x(s)|\,\mathrm{d}s \\
&\leqslant KCM\int_{t_0}^{\infty} \beta(s)r(s)h(s)p(s)|B(s)|\,\mathrm{d}s|x|_h.
\end{aligned}
$$

On the other hand, for $t \in [t_0-\sigma, t_0]$ we have

$$
\begin{aligned}
|h^{-1}(t)\mathscr{T}[x](t)| &= |h^{-1}(t)h(t_0)h^{-1}(t_0)\mathscr{T}[x](t_0)| \\
&\leqslant KCM^2\int_{t_0}^{\infty} \beta(s)r(s)h(s)p(s)|B(s)|\,\mathrm{d}s|x|_h.
\end{aligned}
$$

Hence, we have proved that condition (17) implies $\mathscr{T}\colon C_h(J_\sigma) \to C_h(J_\sigma)$.

In order to verify the property $[\mathbf{M}_h]$, we write

$$
\mathscr{T}[x](t) - \mathscr{T}[x](t') = I_1(t) + I_2(t) + I_3(t) - I_4(t), \quad t-\sigma \leqslant t' \leqslant t,
$$

where

$$
\begin{aligned}
I_1(t) &= \int_{t'}^{t} \Phi(t)P\Phi^{-1}(s)B(s)\big(x(s)-x(s-r(s))\big)\,\mathrm{d}s, \\
I_2(t) &= \int_{t_0}^{t'} [\Phi(t)-\Phi(t')]P\Phi^{-1}(s)B(s)\big(x(s)-x(s-r(s))\big)\,\mathrm{d}s, \\
I_3(t) &= \int_{t'}^{t} \Phi(t')(I-P)\Phi^{-1}(s)B(s)\big(x(s)-x(s-r(s))\big)\,\mathrm{d}s, \\
I_4(t) &= \int_{t}^{\infty} [\Phi(t)-\Phi(t')](I-P)\Phi^{-1}(s)B(s)\big(x(s)-x(s-r(s))\big)\,\mathrm{d}s.
\end{aligned}
$$

From condition (4) we have

$$
\begin{aligned}
|h^{-1}(t)I_1(t)| &\leqslant KM\int_{t'}^{t} h(s)p(s)|B(s)|\beta(s)r(s)\,\mathrm{d}s|x|_h \\
&\leqslant KM\beta(t)|\beta rhpB|^{\infty}(t-t')|x|_h.
\end{aligned}
\tag{19}
$$

On the interval $s \leqslant t' \leqslant t$, $t-t' \leqslant \sigma$, we obtain the estimate

$$
\begin{aligned}
|(\Phi(t)-\Phi(t'))P\Phi^{-1}(s)| &= \left|\int_{t'}^{t} (A+B)(u)\Phi(u)\,\mathrm{d}uP\Phi^{-1}(s)\right| \\
&\leqslant K\beta(t)\int_{t'}^{t} h(u)p(s)\,\mathrm{d}u \\
&\leqslant KM\beta(t)h(t)p(s)(t-t').
\end{aligned}
\tag{20}
$$

Inserting the estimate (20) into the definition of function I_2, we may write

$$(21)\qquad |h^{-1}(t)I_2(t)| \leqslant KM^2\beta(t)\int_{t_0}^{\infty}\beta(s)h(s)p(s)|B(s)|r(s)\,\mathrm{d}s(t-t')|x|_h.$$

Further, we have

$$(22)\qquad \begin{aligned}|h^{-1}(t)I_3(t)| &\leqslant KCM\left(\int_{t'}^{t}h(t')h^{-1}(t)h(s)p(s)\beta(s)r(s)|B(s)|\,\mathrm{d}s\right)|x|_h\\ &\leqslant KCM\beta(t)|hp\beta rB|^{\infty}(t-t')|x|_h.\end{aligned}$$

Finally, relying on (20) we conclude

$$(23)\qquad |h^{-1}(t)I_4(t)| \leqslant KCM^2\beta(t)|hp\beta rB|^1(t-t')|x|_h.$$

From (19), (21), (22) and (23) we obtain

$$|\mathscr{T}[x](t)-\mathscr{T}[x](t')| \leqslant 2KCM^2\beta(t)h(t)(|hp\beta rB|^{\infty}+|hp\beta rB|^1)(t-t')|x|_h.$$

Therefore condition (17) implies (18). □

Lemma 3. *If $h\in G_{\sigma,M}$ and $y_0\in \mathbf{V}_h$, then $\Phi y_0\in \mathscr{M}_h$.*

Proof. We have to verify the property $[\mathbf{M}_h]$. If $t-\sigma\leqslant t'\leqslant t$, $t\geqslant t_0$, then

$$\begin{aligned}h^{-1}(t)|\Phi(t)y_0-\Phi(t')y_0| &= \left|h^{-1}(t)\int_{t'}^{t}(A(s)+B(s))\Phi(s)y_0\,\mathrm{d}s\right|\\ &\leqslant h^{-1}(t)\int_{t'}^{t}|A(s)+B(s)|h(s)h^{-1}(s)\Phi(s)y_0|\,\mathrm{d}s\\ &\leqslant M\beta(t)(t-t')|\Phi y_0|_h,\end{aligned}$$

whence the proof of the lemma follows. □

Theorem 6. *Let us assume that Eq.* (1) *has the dichotomy* (4)–(5) *and the functions h, k are of class $G_{\sigma,M}$. If*

$$(24)\qquad 2KCM^2\{|kq\beta rB|^1+|kq\beta rB|^{\infty}\}<1$$

and $\mathbf{V}_h\neq\mathbf{V}_k$, then the null solution of Eq. (1) *is h-unstable.*

Proof. Let us assume that the null solution of Eq. (1) is h-stable. Then for $\varepsilon=1$ there exists a $\delta>0$ such that $|x(\cdot,t_0,\varphi)|_h<1$ if $|\varphi|_\sigma<\delta$. Let ϱ be a small number such that

$$(25)\qquad \varrho|k_{t_0}|_\sigma<\delta.$$

For a small $\gamma > 0$ satisfying

$$\gamma + 2KCM^2\{|kq\beta rB|^1 + |qk\beta rB|^\infty\}\varrho \leqslant \varrho$$

we fix an initial condition $y_0 \in \mathbf{V}_k \setminus \mathbf{V}_h$ such that

$$|y(\cdot, t_0, y_0)|_k \leqslant \gamma.$$

Let us consider the integral equation $x = \mathscr{F}[x]$, where the operator $\mathscr{F}$ is defined by (14). From Lemma 2 and Lemma 3 (where instead of h we put k) we have $\mathscr{F}\colon \mathscr{M}_k \to \mathscr{M}_k$. Let $B_k^*[0,\varrho]$ be the closed ball with center $x = 0$ and radius ϱ contained in $\mathscr{M}_k$. In view of the choice of the number γ we have

$$\mathscr{F}\colon\ B_k^*[0,\varrho] \to B_k^*[0,\varrho].$$

In virtue of (17), the operator $\mathscr{F}$ contracts the points of the ball $B_k^*[0,\varrho]$. Let x be a fixed point of the operator $\mathscr{F}$. A straightforward calculation shows that x is a solution of Eq. (1). Moreover, $t \in [t_0 - \sigma, t_0]$ and (25) yield

$$|x(t)| \leqslant k(t)(|\Phi y_0|_k + |\mathscr{T}[x]|_k) \leqslant (\gamma + 2KCM^2|kqB|^1\varrho)|k_{t_0}|_\sigma \leqslant \varrho|k_{t_0}|_\sigma < \delta,$$

implying that x is an h-bounded function. Therefore $\mathscr{T}[x]$ is h-bounded. The last assertion is proved in the following way: the second estimate in the proof of Lemma 2 shows that this is certainly satisfied if $\beta rhpB \in L^1[t_0,\infty)$, but this follows from (24), because $hp \leqslant Ckq$. Since

$$x(t) = y(t, t_0, y_0) + \mathscr{T}[x](t),$$

we obtain that the function $y(\cdot, t_0, y_0)$ must be h-bounded. However, this contradicts the choice of y_0. □

The forthcoming Lemma 4 and Theorem 7 follow from Lemma 2 and Theorem 6 in the particular case of a (h,k)-dichotomy.

Lemma 4. *Assume that Eq. 2 has an (h,k)-dichotomy and $h \in G_{\sigma,M}$. If*

$$2KCM^2\{|\beta rB|^1 + |\beta rB|^\infty\} < 1, \tag{26}$$

then

$$\mathscr{T}\colon\ \mathscr{M}_h \to \mathscr{M}_h.$$

Theorem 7. *Let us assume that Eq.* (2) *has an* (h,k)*-dichotomy and the functions* h*,* k *belong to* $G_{\sigma,M}$*. If condition* (26) *is satisfied and* $\mathbf{V}_h \neq \mathbf{V}_k$*, then the null solution of Eq.* (1) *is* h*-unstable.*

Finally, before we start the section of examples, we emphasize that the stability analysis of Eq. (1) via an appropriate ordinary differential equation is not new. It was used, for example, by Cooke [2], and recently by Győri and Pituk in an interesting paper [8].

5. Applications

We present three examples of independent interest.

5.1. Instability of $x^{(n)}(t) = b(t)x(t-r(t))$.

To start, let us begin with the second order equation

$$x''(t) = b(t)x(t-r(t)), \quad 0 \leqslant r(t) \leqslant 1, \;\; t \geqslant t_0 = 1, \tag{27}$$

whose vectorial form is

$$\begin{pmatrix} y(t) \\ x(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ x(t) \end{pmatrix} + b(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y(t-r(t)) \\ x(t-r(r)) \end{pmatrix}.$$

If $|(u,v)| = |u| + |v|$ is the norm in $\mathbb{R}^2$, then the linear system

$$\begin{pmatrix} y(t) \\ x(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}$$

with the fundamental matrix $\Phi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ has the weak dichotomy

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leqslant 2s, \quad 1-\sigma \leqslant s \leqslant t, \\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leqslant 2t, \quad 1-\sigma \leqslant t \leqslant s, \end{aligned} \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfying (5) with functions $h(t) = 1$, $p(t) = t$, $k(t) = t$, $q(t) = 1$, and constant $C = 1$. If the function $tb(t)$ is integrable, then, according to Theorem B, the perturbed equation

$$x''(t) = b(t)x(t)$$

has a weak dichotomy of type $([1,t],[t,1])$. According to Theorem 1, the null solution of Eq. (27) is unstable if $tb(t) \in L^1$, because $\mathbf{V}_h \neq \mathbf{V}$. Since $\mathbf{V}_k \neq \mathbf{V}_{k,0}$, even Theorem 3 is applicable to this example if $tb(t) \in L^1$; the null solution of Eq. (27)

is t-asymptotic unstable. In this example, the property $\mathbf{V}_h \neq \mathbf{V}_k$ suggests the possibility of applying Theorem 6, for which we require the sufficient condition

$$\lim_{t\to\infty}\left(\int_t^\infty |s\max\{1,|b(s)|\}r(s)b(s)|\,\mathrm{d}s + |t\max\{1,|b(t)|\}r(t)b(t)|\right) = 0 \tag{28}$$

equivalent to

$$g \in L^1, \quad \lim_{t\to\infty} g(t) = 0, \quad g(t) := \max\{1, b(t)\}r(t)b(t),$$

in order to establish property (24). Under this condition the null solution of Eq. (27) is unstable in the sense of Liapounov.

The general equation

$$x^{(n)}(x) = b(x)x(t - r(t))$$

can be treated similarly, the linear system

$$x'(t) = (I + I_1)x(t), \quad I_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

having the family of weak dichotomies

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leqslant Kt^{m-1}s^m, \quad t \geqslant s \geqslant 1, \\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leqslant Kt^m s^{m-1}, \quad s \geqslant t \geqslant 1, \end{aligned}$$

where

$$P = \mathrm{diag}\{\overbrace{1,\dots,1}^{m}, 0, \dots, 0\}, \quad 0 < m < n.$$

5.2 Euler equations.

As a second example we study the instability of the Euler equation

$$y'' + \frac{\mu}{t^2}y = b(t)y(t - r(t)), \tag{29}$$

where μ is a real parameter satisfying $|\mu| < 1/4$ (the case $|\mu| \geqslant 1/4$ can be dealt with similarly as we do in the text). In this case the fundamental matrix corresponding to the linear equation

$$y'' + \frac{\mu}{t^2}\,y = 0$$

has the form

$$(30)\qquad \Phi(t)=\begin{pmatrix} t^{\lambda_-} & t^{\lambda_+}\\ \lambda_- t^{\lambda_- -1} & \lambda_+ t^{\lambda_+ -1}\end{pmatrix},\quad \lambda_\mp=\frac{1}{2}\mp\sqrt{\frac{1}{4}-\mu}.$$

For the projection matrix $P=\operatorname{diag}\{1,0\}$ we have the following estimates satisfying (4)–(5):

$$(31)\qquad \begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leqslant Kt^{\lambda_-}s^{\lambda_+}, \quad t\geqslant s,\\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leqslant Kt^{\lambda_+}s^{\lambda_-}, \quad s\geqslant t.\end{aligned}$$

We aim at applying Theorem 3. For $h(t)=t^{\lambda_-}$, $k(t)=t^{\lambda_+}$, we observe that $\mathbf{V}_k\neq\mathbf{V}_{k,0}$. By Theorem B, the equation

$$y''+\frac{\mu}{t^2}y=b(t)y$$

has a $[t^{\lambda_+},t^{\lambda_-}]$-dichotomy if $tb(t)\in L^1$. According to Theorem 3 we have that the null solution of Eq. (29), under condition $tb(t)$, is not asymptotically t^{λ_+}-stable (this implies the Liapounov instability). Such a consequence cannot be obtained if $b(t)=O(t^{-2})$. Let us study this case more carefully. Assume the existence of a constant ν such that the function $(\nu-t^2b(t))t^{-1}$ is integrable. Then Eq. (29) can be written in the form

$$(32)\qquad x''(t)+\frac{\mu-\nu}{t^2}x(t)+\frac{\nu-t^2b(t)}{t^2}x(t)=b(t)(x(t-r(t))-x(t)).$$

If $4(\mu-\nu)<1$, then the equation

$$\begin{pmatrix} y(t)\\ x(t)\end{pmatrix}'=\frac{\mu-\nu}{t^2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} y(t)\\ x(t)\end{pmatrix}$$

has the dichotomy (31), with a fundamental matrix (30) defined with values $\lambda_\mp$, where instead of μ it is necessary to write $\mu-\nu$. By Theorem B, the dichotomy (31) is preserved if $(\nu-t^2b(t))t^{-1}$ is integrable. Now Theorem 6 will apply to Eq. (32) if the condition (24) is fulfilled. In the present example a sufficient condition to obtain (24) is (28).

An example of a function satisfying these conditions is $b(t)=1/(1+t^2)$, with $\nu=1$. Note that $tb(t)$ is not integrable.

In examples regarding Euler equations it is worth mentioning that the classical change of the time scale $t=\mathrm{e}^\tau$ in Eq. (29) would not work, since it is not clear how to define the function $y(\tau)=x(\mathrm{e}^\tau)$ from $x(t-r(t))$.

5.3. Instability of $x'(t) = Ax(t) + B(t)x(t - r(t))$.

Let us consider the system

$$x'(t) = Ax(t) + B(t)x(t - r(t)), \tag{33}$$

where the function A is constant. Throughout, $\sigma(A)$ will denote the set of eigenvalues of the matrix A; $\sigma_+ = \{\lambda \in \sigma(A)\colon\ \operatorname{Re}\lambda > 0\}$. If $\sigma_+(A) \neq \emptyset$, we define numbers α, ν such that

$$0 < \alpha < \mu - \nu < \nu < \mu = \min\{\operatorname{Re}\lambda\colon\ \operatorname{Re}\lambda > 0\}.$$

The change of the variable

$$x(t) = \mathrm{e}^{\nu t} z(t) = \mathrm{e}^{(\nu-\alpha)t}\mathrm{e}^{\alpha t} z(t), \tag{34}$$

reduces Eq. (33) to the form

$$z'(t) = (A - \nu I + \mathrm{e}^{-r(t)}B(t))z(t) + \mathrm{e}^{-r(t)}B(t)(z(t - r(t)) - z(t)), \tag{35}$$

the system

$$u'(t) = (A - \nu I)u(t) \tag{36}$$

having an $(\mathrm{e}^{-\alpha t}, \mathrm{e}^{\beta t})$-dichotomy. If $B \in L^1$, then the system

$$u'(t) = (A - \nu I + \mathrm{e}^{-r(t)}B(t))u(t) \tag{37}$$

has the same $(\mathrm{e}^{-\alpha t}, \mathrm{e}^{\beta t})$-dichotomy [4], [15]. It is clear that the function $\mathrm{e}^{-\alpha t}$ is of class $G_{\sigma,M}$ for some positive constant M. All conditions of Theorem 1 are fulfilled for large values of the initial moment t_0. Thus, the null solution $z = 0$ is $\mathrm{e}^{-\alpha t}$-unstable, which implies, in view of (34), the Liapounov instability of the solutions of (33). We may write the above outcomes in

Theorem 8. *If $\sigma_+(A)$ is not empty and $B(t)$ is integrable, then for any bounded delay function $r(t)$, the null solution of Eq.* (1) *is unstable.*

A more interesting result is obtained for a nonintegrable and bounded function $B(t)$. First we recall a known result on the stability of exponential dichotomies [4], [16]:

Theorem C. *Let us assume that Eq.* (36) *allows the exponential dichotomy*

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leqslant K_1 \mathrm{e}^{-\alpha(t-s)}, \quad t_0 - \sigma \leqslant s \leqslant t,\ \alpha > 0,\\ |\Phi(t)(I - P)\Phi^{-1}(s)| &\leqslant K_2 \mathrm{e}^{\beta(t-s)}, \quad t_0 - \sigma \leqslant t \leqslant s,\ \beta > 0. \end{aligned} \tag{38}$$

Then the condition

$$|e^{-r(\cdot)}B|^{\infty}\left\{\frac{K_1}{\alpha}+\frac{K_2}{\beta}\right\}<\frac{1}{2} \tag{39}$$

implies that Eq. (37) *has the exponential dichotomy*

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leqslant Ke^{-\alpha_1(t-s)}, \quad && t_0-\sigma\leqslant s\leqslant t,\\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leqslant Ke^{\beta_1(t-s)}, \quad && t_0-\sigma\leqslant t\leqslant s, \end{aligned} \tag{40}$$

where K is a constant and $0<\alpha_1<\alpha$, $0<\beta_1<\beta$.

We desire to apply Theorem 6 to Eq. (35), where Eq. (36) has the dichotomy (38). If (39) is fulfilled, then Eq. (37) has the dichotomy (40). The condition (17) requires the estimate

$$|\beta r(\cdot)e^{-r(\cdot)}B|^{\infty}\leqslant |r(\cdot)e^{-r(\cdot)}B|^{\infty}(1+|A|+|e^{-r(\cdot)}B|^{\infty}).$$

For the functions $h(t)=e^{-\alpha_1 t}$, $k(t)=e^{\beta_1 t}$, the constant M in Definition 2 can be chosen to be $M=e^{\max\{\alpha,\beta\}\sigma}$. The property (5) is established with $C=1$. Thus, condition (17) is satisfied if

$$2Ke^{2\alpha_1\sigma}(1+|A|+|e^{-r(\cdot)}B|^{\infty})(|re^{-r(\cdot)}B|^{1}+|r(\cdot)e^{-r(\cdot)}B|^{\infty})<1.$$

The condition $\mathbf{V}_h\neq\mathbf{V}_k$ of Theorem 6 is certainly satisfied. Note that α_1 can be taken as close to null as desired. Thus, this last condition follows from

$$2K(1+|A|+|e^{-r(\cdot)}B|^{\infty})(|r(\cdot)e^{-r(\cdot)}B|^{1}+|r(\cdot)e^{-r(\cdot)}B|^{\infty})<1. \tag{41}$$

From Theorem 6 we obtain that conditions (39), (41) imply the $e^{-\alpha_1 t}$-instability of Eq. (35). Since $\alpha_1<\alpha$, then the null solution of Eq. (35) is $e^{-\alpha t}$-unstable, implying, according to (34), the instability of Eq. (33). From these conclusions we obtain

Theorem 9. *If the matrix A allows an eigenvalue with a positive real part and $B(t)\sim C_1t^{-\gamma_1}$, $r(t)\sim C_2t^{-\gamma_2}$, where γ_1, γ_2 are nonnegative constants such that $\gamma_1+\gamma_2>1$, then the null solution of the equation with bounded delay* (33) *is unstable.*

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