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SOME PROPERTIES OF RESIDUATED LATTICES

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Abstract. We investigate some (universal algebraic) properties of residuated lattices—algebras which play the role of structures of truth values of various systems of fuzzy logic.

Keywords: residuated lattice, fuzzy logic, variety, congruence

MSC 2000: 03B52, 06F05, 08A30, 08A40, 08B05

The interest in fuzzy logic has been rapidly growing recently (see [14], also [6, 19]). Several new algebras playing the role of the structures of truth values have been introduced and axiomatized (i.e. a system of logical axioms complete w.r.t. semantics defined over the class of corresponding algebras has been found). The aim of this paper is to investigate some (universal algebraic) properties of algebraic structures of fuzzy logic. We investigate, among others, some congruence properties which have shown to be important in universal algebra. Whenever possible, we apply some general condition known from universal algebra (we use Mal'cev-type conditions). Doing so, we obtain instances of the universal algebraic conditions for the algebraic structures of fuzzy logic in question. This approach, in a sense, makes visible the "very reason" of why a structure obeys a property.

The most general structure considered here is that of a residuated lattice. Residuated lattices were introduced in [9] (it should be, however, noted that the motivation was by far not logical) and reinvigorated in the context of fuzzy logic in [12]. The following definition stems from [16]:

Definition 1. A *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice (the corresponding order will be denoted by $\leqslant$) with the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds), (iii) $x \otimes y \leqslant z$ iff $x \leqslant y \rightarrow z$ holds (adjointness condition).

A residuated lattice satisfies the *prelinearity axiom* [15, 14] iff $(x \to y) \vee (y \to x) = 1$ holds. A residuated lattice is *divisible* [15] iff $x \wedge y = x \otimes (x \to y)$. It can be shown [15] that divisibility is equivalent to the following condition: for each $x \leqslant y$ there is z such that $x = y \otimes z$. A residuated lattice satisfies the law of *double negation* (and is called integral, commutative *Girard*-monoid [15]) iff $x = (x \to 0) \to 0$ holds. A residuated lattice has *square roots* if there is a unary operation $\frac{1}{2}$ satisfying (1) $x^{\frac{1}{2}} \otimes x^{\frac{1}{2}} = x$, and (2) $y \otimes y \leqslant x$ implies $y \leqslant x^{\frac{1}{2}}$. A *Heyting algebra* is a residuated lattice where $x \otimes y = x \wedge y$. A *BL-algebra* [14] is a residuated lattice which is divisible and satisfies the prelinearity axiom. An *MV-algebra* [4, 6, 14, 15] is a residuated lattice in which $x \vee y = (x \to y) \to y$ holds. Equivalently [15], MV-algebra is a residuated lattice which is divisible and satisfies the law of double negation. Thus, each BL-algebra satisfying the law of double negation is an MV-algebra (which is the way MV-algebras are defined in [14]). A Π-*algebra* (product algebra) [14] is a BL-algebra satisfying $(z \to 0) \to 0 \leqslant ((x \otimes z) \to (y \otimes z)) \to (x \to y)$ and $x \wedge (x \to 0) = 0$. A *G-algebra* (Gödel algebra) is a BL-algebra which satisfies $x \otimes x = x$ (i.e. a Heyting algebra satisfying the prelinearity axiom). A *Boolean algebra* is a residuated lattice which is both a Heyting algebra and an MV-algebra (relation to the usual axiomatization is $x \to y = x' \vee y$).

Emphasizing the monoidal structure, residuated lattices are called integral, commutative, residuated l-monoids [2, 15, 16]. The operations $\otimes$ (*multiplication*) and $\to$ (*residuum*) model conjunction and implication of the corresponding logical calculus. It is easy to see that w.r.t. $\leqslant$, $\otimes$ is isotonic and $\to$ is isotonic in the second and antitonic in the first argument. Moreover, the following are true in each residuated lattice (see e.g. [12]):

$$x = 1 \to x, \tag{1}$$

$$x \leqslant y \quad \text{iff} \quad x \to y = 1, \tag{2}$$

$$x \leqslant (x \to y) \to y \ \text{ and } \ x \otimes (x \to y) \leqslant y. \tag{3}$$

From the point of view of logic, the structure of residuated lattices is implied by some natural requirements, see [12] and also [14].

Remark. There are other definitions of MV-algebras. Recall the original Chang's definition [4], see also [6], and the definition in terms of $\to$ and 0 only [11] (the corresponding algebras are called Wajsberg algebras).

Remark. Important examples of residuated lattices are induced by t-norms (recall that a t-norm is a mapping $\otimes\colon [0,1] \times [0,1] \to [0,1]$ which makes $\langle [0,1], \otimes, 1 \rangle$ a commutative monoid, i.e. $\otimes$ is commutative, associative and $x \otimes 1 = x$ holds). If $\otimes$ is a left-continuous t-norm then putting $x \to y = \bigvee \{z \mid z \otimes x \leqslant y\}$ makes

$\langle [0,1], \min, \max, \otimes, \to, 0, 1\rangle$ a residuated lattice (see e.g. [15]). It has been recently proved in [7] that the class of all algebras generated in the above way by continuous t-norms generates the variety of BL-algebras.

Logical systems corresponding to the above algebras can be found in [14, 16, 19].

Recall that a class of algebras of the same type is called a variety if it is the class of all algebras satisfying a certain set of identities. Equivalently (Birkhoff's result, see e.g. [2]), a variety is a class of algebras closed under subalgebras, direct products and homomorphic images.

Proposition 2. *The class of all residuated lattices is a variety of algebras.*

Proof. We give the set of defining identities for residuated lattices. Clearly, the conditions (i) and (ii) of the definition of residuated lattices are expressible by identities. We claim that these identities plus

$$(x \otimes y) \to z = x \to (y \to z) \tag{4}$$

$$(x \otimes (x \to y)) \vee y = y \tag{5}$$

$$x \to (x \vee y) = 1 \tag{6}$$

define residuated lattices. First, (4)–(6) hold in residuated lattices. Indeed, applying adjointness and associativity and commutativity we get $(x \otimes y) \to z \leqslant x \to (y \to z)$ iff $x \otimes ((x \otimes y) \to z) \leqslant (y \to z)$ iff $(x \otimes y) \otimes ((x \otimes y) \to z) \leqslant z$ which holds by (3). To obtain the converse inequality is equally easy. Hence (4) holds. (5) and (6) follow from (3) and (2), respectively.

To complete the proof we have to show that (i), (ii), and (4)–(6) imply the adjointness condition. First, notice that (5) and (6) imply (2). Indeed, if $x \leqslant y$ then (by (6)) $x \to y = x \to (x \vee y) = 1$. Conversely, if $x \to y = 1$ then (by (5)) $x \vee y = (x \otimes (x \to y)) \vee y = y$, i.e. $x \leqslant y$. Now, by (2) and (4) we have $x \otimes y \leqslant z$ iff $(x \otimes y) \to z = 1$ iff $x \to (y \to z) = 1$ iff $x \leqslant y \to z$. □

Remark. (1) The fact that residuated lattices form a variety of algebras was proved by J. Pavelka in his thesis.[1] It also follows from [20, II, 2.10. Remark] from which it is clear that the structure of residuated lattices is preserved by factorization (preservation under taking subalgebras and direct products is obvious).

(2) In a structure satisfying (i) and (ii), each of the three identities (4)–(6) is independent of the two remaining. Indeed, take any t-norm $\otimes$. First, put $a \to b = 1$ for every $a, b \in [0,1]$. Then (5) does not hold while (4) and (6) do. Second, put

[1] Thanks are due to the referee who pointed out this fact.

$a \to b = 0$ for every $a, b \in [0,1]$. Then (4) and (5) hold but (6) does not. Third, let $\otimes$ be the Łukasiewicz t-norm and put $a \to b = 1$ if $a \leqslant b$ and $a \to b = 0$ otherwise. In this case, (4) does not hold (e.g. for $x = 0.7$, $y = 0.6$, $z = 0.5$) while both (5) and (6) do.

Corollary 3. (1) *Each class of all residuated lattices satisfying, in addition, a given set of conditions introduced in the paragraph following Definition* 1 *(prelinearity, divisibility etc.) forms a variety of algebras, namely a subvariety of the variety of all residuated lattices.* (2) *The class of all residuated lattices with square roots is a variety of type* $\langle \wedge, \vee, \otimes, \to, ^{\frac{1}{2}}, 0, 1 \rangle$.

Proof. (1) Each inequality $t_1 \leqslant t_2$ can be equivalently expressed by an identity, e.g. $t_1 \vee t_2 = t_2$. The first assertion then follows from the fact that each of the additional conditions is an inequality or an equality.

(2) We have to show that the condition $y \otimes y \leqslant x$ implies $y \leqslant x^{\frac{1}{2}}$ can be expressed by identities. Indeed, it is easy to verify that

$$((y \otimes y) \vee x)^{\frac{1}{2}} \wedge y = y$$

is an identity equivalent to the above condition. □

Remark. Note that the fact that BL-algebras form a variety is proved in [14, Lemma 2.3.10]. However, the proof there consists in proving that adjointness is equivalent to a set of five identities which includes our (5) and (6) (in fact, instead of (5), the identity $((x \to y) \otimes x) \vee y = y$ is used; they are equivalent in case of commutativity of $\otimes$). Moreover, one of them $((x \wedge y) \otimes z = (x \otimes z) \wedge (x \otimes y))$ does not hold in residuated lattices.

Residuated lattices are defined as algebras with lattice structure, monoidal structure, and an "additional" operation $\to$ bound by adjointness. The following proposition shows that an alternative definition is possible in which $\otimes$ plays the role of the "additional" operation.

Proposition 4. $\mathbf{L} = \langle L, \wedge, \vee, \otimes \to, 0, 1 \rangle$ *is a residuated lattice iff*

(i) $\langle L, \wedge, \vee, 0, 1 \rangle$ *is a bounded lattice,*

(ii′) $\langle L, \to, 1 \rangle$ *satisfies*

$$x = 1 \to x \tag{7}$$

$$x \to (y \to z) = y \to (x \to z), \tag{8}$$

(iii) $\otimes$ *and* $\to$ *satisfy the adjointness property.*

P r o o f. It is easy to see that (i), (ii′), and (iii) hold in any residuated lattice. Conversely, it suffices to show that (i), (ii′), and (iii) imply that $\langle L, \otimes, 1\rangle$ is a commutative monoid. We have $x \otimes 1 \leqslant t$ iff $x \leqslant 1 \to t$ iff (by (7)) $x \leqslant t$, which implies $x \otimes 1 = x$. Furthermore, $x \otimes y \leqslant t$ iff $(1 \to x =)x \leqslant y \to t$ iff $1 \leqslant x \to (y \to t)$ iff (by (8)) $1 \leqslant y \to (x \to t)$ iff ... iff $y \otimes x \leqslant t$, i.e. $x \otimes y = y \otimes x$. Finally, $(x \otimes y) \otimes z \leqslant t$ iff ... iff $(1 \to x =)x \leqslant y \to (z \to t)$ iff $1 \leqslant x \to (y \to (z \to t))$ iff (by (8)) $1 \leqslant z \to (y \to (x \to t))$ iff ... iff $x \otimes (y \otimes z) \leqslant t$, i.e. $(x \otimes y) \otimes z = x \otimes (y \otimes z)$. Therefore $\langle L, \otimes, 1\rangle$ is a commutative monoid. □

We are going to show (a little bit more than) that for every finite set of identities of residuated lattices there is an equivalent single identity. Let t_i, s_i, $i \in I$, be terms of residuated lattices. We say that **L** satisfies the generalized identity $\bigwedge_{i\in I} t_i = \bigwedge_{i\in I} s_i$ if both $\bigwedge_{i\in I} t_i$ and $\bigwedge_{i\in I} s_i$ evaluate to the same element whenever they make sense (i.e. infima of the elements to which t_i's and s_i's evaluate exist).

Proposition 5. *Let $\{p_i = q_i \mid i \in I\}$ be a set of identities of residuated lattices. Then there is a generalized identity $p = q$ such that for any class $\mathcal{K}$ of residuated lattices it holds that*

$$\mathcal{K} \models \{p_i = q_i \mid i \in I\} \quad \textit{iff} \quad \mathcal{K} \models p = q.$$

If I is finite then $p = q$ can be chosen to be an identity.

P r o o f. Let all variables which occur in $p_i = q_i$ be $x_1, \dots, x_n$, and let p'_i and q'_i be the terms which result from p_i and q_i, respectively, by replacing x_j by x_{ij}, $j = 1, \dots, n$ (x_{ij} are new variables). Doing so, the sets of variables of different identities $p'_i = q'_i$ are disjoint. Put $p := \bigwedge_{i\in I} (p'_i \vee x_i)$ and $q := \bigwedge_{i\in I} (q'_i \vee x_i)$. Clearly, $\mathcal{K} \models \{p_i = q_i \mid i \in I\}$ implies $\mathcal{K} \models p = q$. Conversely, let $\mathcal{K} \models p = q$. Take any $\mathbf{L} \in \mathcal{K}$ and take any evaluation e of variables such that $e(x_k) = 0$ for $k = i$ and $e(x_k) = 1$ for $k \neq i$. It is immediate that $p = p'_i$ and $q = q'_i$ hold in **L**, therefore $p'_i = q'_i$ and also $p_i = q_i$ hold in **L** which completes the proof. □

Each of the defining identities of residuated lattices of Proposition 2 and Proposition 4 contains at most three variables. Next we prove that residuated lattices cannot be defined by identities with two variables.

Proposition 6. *Residuated lattices cannot be defined by identities with at most two variables.*

P r o o f. *Consider the lattice in Fig. 1. Put $x \otimes y = x \wedge y$, $x \to y = 1$ for $x \leqslant y$, $x \to y = y$ for $x > y$, $a \to b = b$, and $b \to a = a$. Consider the operation $\vee'$*

which is defined the same way as $\vee$ *except for* $a \vee b = 1$ *and* $b \vee a = 1$. *It is easy to see that* $\mathbf{L} = \langle \{0, a, b, c, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ *is a residuated lattice (in fact, a Heyting algebra) but* $\mathbf{L}' = \langle \{0, a, b, c, 1\}, \wedge, \vee', \otimes, \rightarrow, 0, 1 \rangle$ *is not. However, every subalgebra of* $\mathbf{L}'$ *generated by an (at most) two-element subset is a residuated lattice (it is a subalgebra of* $\mathbf{L}$*). Therefore,* $\mathbf{L}'$ *(which is not a residuated lattice) satisfies every identity with at most two variables which is valid in all residuated lattices.* □

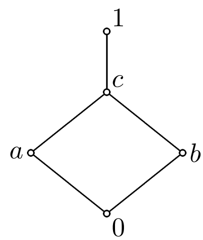

Figure 1. Residuated lattice from the proof of Proposition 6.

In the following we denote by $\mathrm{Con}\,\mathbf{A}$ the congruence lattice of the algebra $\mathbf{A} = \langle A, F \rangle$. In what follows we define some congruence properties of algebras. These properties apply to varieties in the following generic way: a variety has a property X if each of its algebras has X.

Congruence regularity and its forms. An algebra $\mathbf{A} = \langle A, F \rangle$ is *(congruence) regular* if for each $\theta, \varphi \in \mathrm{Con}\,A$ and any $a \in A$ we have that $[a]_\theta = [a]_\varphi$ implies $\theta = \varphi$, i.e. every congruence is fully determined by any of its classes. Regular varieties have been characterized by Mal'cev type conditions, see [8, 13, 22]: a variety $\mathcal{V}$ is regular iff there exist ternary terms $t_1, \ldots, t_n$ such that $[t_1(x, y, z) = z, \ldots, t_n(x, y, z) = z$ iff $x = y]$ holds in $\mathcal{V}$. It is often the case that an algebra satisfies only some weaker form of regularity. Suppose c is a constant from the type of algebras or c is equationally definable within the class of algebras under consideration. An algebra $\mathbf{A}$ is *(congruence) c-regular* if each $\theta \in \mathrm{Con}\,A$ is determined by the class containing c, i.e. if $[c]_\theta = [c]_\varphi$ implies $\theta = \varphi$ for every $\theta, \varphi \in \mathrm{Con}\,A$. It has been proved in [10] that a variety $\mathcal{V}$ is c-regular iff there exist binary terms $r_1, \ldots, r_n$ such that $[r_1(x, y) = c, \ldots, r_n(x, y) = c$ iff $x = y]$ holds in $\mathcal{V}$. An algebra $\mathbf{A}$ is *(congruence) c-locally regular* if for each $\theta, \varphi \in \mathrm{Con}\,A$ and any $a \in A$ we have that $[a]_\theta = [a]_\varphi$ implies $[c]_\theta = [c]_\varphi$. It has been proved in [3] that a variety $\mathcal{V}$ is c-locally regular iff there exist binary terms $q_1, \ldots, q_n$ such that $[q_1(x, y) = x, \ldots, q_n(x, y) = x$ iff $y = 0]$ holds in $\mathcal{V}$. It is clear that c-regularity and c-local regularity are complementary w.r.t. regularity in that an algebra (with c) is regular iff it is both c-regular and c-locally regular.

The following result is well known.

Proposition 7. *The variety of all residuated lattices is* 1*-regular.*

Proof. For the above characterization put $r(x, y) = (x \to y) \wedge (y \to x)$. Clearly, $r(x, x) = 1$. On the other hand, if $r(x, y) = 1$ then $1 \leqslant x \to y$, $1 \leqslant y \to x$ i.e. $x \leqslant y$ and $y \leqslant x$, thus $x = y$. □

Proposition 8. *The variety of all residuated lattices satisfying the law of double negation is* 0*-regular.*

Proof. Put $r(x, y) = ((x \to y) \wedge (y \to x)) \to 0$. Then $r(x, x) = 1 \to 0 = 0$. If $r(x, y) = 0$ then $(x \to y) \wedge (y \to x) = (((x \to y) \wedge (y \to x)) \to 0) \to 0 = r(x, y) \to 0 = 0 \to 0 = 1$, i.e. $x \to y = 1$ and $y \to x = 1$, thus $x \leqslant y$ and $y \leqslant x$ which yields $x = y$. □

However, residuated lattices are not 0-regular in general. Consider the following (counter)examples.

Example 9. Take any chain L with $|L| > 2$, put $\otimes = \min$, $x \to y = 1$ if $x \leqslant y$ and $x \to y = y$ otherwise. $\mathbf{L}$ is a G-algebra. Take any $a \in L$, $0 < a < 1$, and let θ_a be the equivalence having $[a) = \{x \in L \mid a \leqslant x\}$ and $\{b\}$ (for $b < a$) as its classes. One easily verifies that $\theta_a \in \operatorname{Con} L$. We have $[0]_{\theta_a} = [0]_\omega = \{0\}$ (where $\omega \in \operatorname{Con} L$ is the identity) but $\theta_a \neq \omega$. Thus, $\mathbf{L}$ is not 0-regular.

Example 10. Let $\mathbf{L}$ be the standard product algebra, i.e. $L = [0, 1]$, $\otimes$ is the (number-theoretic) product, $a \to b = 1$ if $a \leqslant b$, $a \to b = b/a$ if $a > b$. One easily checks that $\operatorname{Con} L$ contains the following three congruence relations: two trivial congruences (the identity ω and the full congruence $L \times L$) plus the congruence φ with two classes $\{0\}$ and $(0, 1]$. This shows that $\mathbf{L}$ is not 0-regular. Morevover, one can check that there are no other congruences on $\mathbf{L}$. (Indeed, if θ is a congruence on $\mathbf{L}$ then $[1]_\theta \neq \{1\}$ on account of 1-regularity.) Thus there is some $a < 1$ such that $a \in [1]_\theta$. If $a < b < 1$ then also $b \in [1]_\theta$ (the same argument as for lattices shows this fact). Therefore, if $0 \in [1]_\theta$ then $\theta = L \times L$. If $0 \notin [1]_\theta$, take any $b > 0$. Clearly, there is some $n > 0$ such that $a^n < b$. Compatibility of θ yields $a^n \in [1]_\theta$ ($\langle a, 1 \in \theta$ implies $\langle a^2, 1\rangle \in \theta$ implies ... implies $\langle a^n, 1\rangle \in \theta$), hence $b \in [1]_\theta$. Since b is arbitrary we conclude $[1]_\theta = (0, 1]$, i.e. $\theta = \varphi$.)

Proposition 11. *The variety of all MV-algebras is* 0*-locally regular.*

Proof. Take $n = 2$, $q_1(x, y) = (x \to y) \to 0$, $q_2(x, y) = x \vee y$. Clearly, $q_1(x, 0) = (x \to 0) \to 0 = x$, $q_2(x, 0) = x$. Conversely, let $q_1(x, y) = x$ and $q_2(x, y) = x$. The law of double negation and $q_1(x, y) = x$ yields $x \to y = x \to 0$, $q_2(x, y) = x$ yields $y \leqslant x$. We therefore get by divisibility $y = x \wedge y = x \otimes (x \to y) = x \otimes (x \to 0) = 0$. □

Proposition 12. *The variety of all Heyting algebras is* 0*-locally regular.*

Proof. Take $n = 2$, $q_1(x,y) = x \otimes (y \to 0)$, $q_2(x,y) = x \vee y$. Clearly, $q_1(x,0) = x$, $q_2(x,0) = x$. Conversely, if $q_1(x,y) = x$ and $q_2(x,y) = x$, then $x \leqslant y \to 0$ and $y \leqslant x$, thus $y \leqslant y \to 0$. We therefore have $y = y \wedge (y \to 0) = 0$. □

Proposition 13. *The variety of all* Π*-algebras is* 0*-locally regular.*

Proof. Take $n = 1$, $q_1(x,y) = (x \otimes (y \to 0)) \vee (((y \to 0) \to 0) \otimes (x \to 0))$. On the one hand, $q_1(x,0) = x$. Conversely, suppose $q_1(x,y) = x$ in some Π-algebra $\mathbf{L}$ and let us prove $y = 0$. Since each Π-algebra is a subdirect product of linearly ordered Π-algebras (see Theorem 4.8 in [15], also Lemma 2.3.16 and the proof of Theorem 2.3.22 in [14]) we may safely assume that $\mathbf{L}$ is linearly ordered. Distinguish two cases, $x = 0$ and $x > 0$. If $x = 0$ then $0 = q(0,y) = (y \to 0) \to 0$, therefore $1 = 0 \to 0 = ((y \to 0) \to 0) \to 0 = y \to 0$, i.e. $y = 0$. If $x > 0$ then $x = q(x,y) = x \otimes (y \to 0)$ and thus $y \to 0 = 1$ (linearly ordered Π-algebras allow cancellation by elements > 0 [14, Lemma 4.1.7]) which yields $y = 0$. □

Remark. Since the variety of all G-algebras is a subvariety of the variety of all Heyting algebras, Proposition 12 implies that G-algebras are also 0-locally regular. However, there is a single 0-local regularity term for G-algebras. Indeed, take $q_1(x,y)$ as in the proof of Proposition 13, follow the proof up to "If $x > 0$", and continue as follows. If $x > 0$ then $x = q_1(x,y) = x \wedge (y \to 0)$, i.e. $0 < x \leqslant (y \to 0)$ which yields $y = 0$. We don't know whether there exists a single term in case of Heyting algebras (note that a Heyting algebra is a subdirect product of linearly ordered Heyting algebras iff it is a G-algebra).

From Proposition 8 and Proposition 11 we get

Proposition 14. *The variety of all MV-algebras is congruence regular.*

Remark. (1) A moment's reflection shows that if $r_1(x,y)$ is a term characterizing 0-regularity and $q_1(x,y)$, $q_2(x,y)$ are terms characterizing 0-local regularity then $t_1(x,y,z) = q_1(z, r_1(x,y))$ and $t_2(x,y,z) = q_2(z, r_1(x,y))$ are terms characterizing regularity. Therefore, from the proofs of Proposition 8 and Proposition 11 we get that $t_1(x,y,z) = z \otimes ((x \to y) \wedge (y \to x))$ (since $z \otimes ((((x \to y) \wedge (y \to x)) \to 0) \to 0) = z \otimes ((x \to y) \wedge (y \to x)))$ and $t_2(x,y,z) = z \vee (((x \to y) \wedge (y \to x)) \to 0)$ are terms characterizing regularity of MV-algebras. Using Chang's axiomatization of MV-algebras [4] and Chang's subdirect representation theorem [5, Lemma 3], another two-element set of regularity terms for MV-algebras was obtained in [1]. The result in [1] says even more: there is no single regularity term $t_1(x,y,z)$ for MV-algebras, i.e. one cannot take $n = 1$ in the characterization of regularity. From this result it follows directly that there is no single term $q_1(x, y)$ characterizing 0-local regularity of MV-algebras. It might be a bit surprising that a single term for 0-local regularity exists in case of both Π-algebras and G-algebras (see the proof of Proposition 13 and the preceding remark).

(2) It follows that for the three most important cases of BL-algebras—MV-algebras, G-algebras, and Π-algebras—we have the following: MV-algebras are regular; both G-algebras and Π-algebras are 1-regular but not 1-locally regular (otherwise they would be regular, contrary to Example 9 and Example 10), and 0-locally regular but not 0-regular.

Permutability and its forms. An algebra $\mathbf{A} = \langle A, F \rangle$ is *(congruence) permutable* if for every $\theta, \varphi \in \operatorname{Con} A$, $\theta \circ \varphi = \varphi \circ \theta$. It was proved in [18] that a variety $\mathcal{V}$ is permutable iff there is a term $p(x, y, z)$ such that $x = p(x, z, z)$ and $p(x, x, z) = z$ hold in $\mathcal{V}$.

Proposition 15. *The variety of all residuated lattices is permutable.*

Proof. Consider the term

$$p(x, y, z) = x \otimes ((y \to z) \wedge (z \to y)) \vee z \otimes ((x \to y) \wedge (y \to x)).$$

We have $p(x, z, z) = x \vee z \otimes ((x \to z) \wedge (z \to x))$. Since $z \otimes ((x \to z) \wedge (z \to x)) \leqslant z \otimes (z \to x) \leqslant x$ we have $p(x, z, z) = x$. Similarly, $p(x, x, z) = z$. Therefore, $p(x, y, z)$ is a term which guarantees permutability. □

Remark. (1) If the congruence lattice is permutable, it is also Arguesian [17]. Therefore congruence lattices of residuated lattices are Arguesian (note that the Arguesian identity is the lattice theoretic form of Desargues' theorem from projective geometry).

(2) An algebra $\mathbf{A} = \langle A, F \rangle$ is *c-permutable* if for every $\theta, \varphi \in \operatorname{Con} A$ we have $[c]_{\theta \circ \varphi} = [c]_{\varphi \circ \theta}$ (note that neither $\theta \circ \varphi$ nor $\varphi \circ \theta$ have to be congruence relations). A variety with an equationally defined constant c is c-permutable iff there exists a binary term s such $s(x, x) = c$ and $s(x, c) = x$ (see e.g. [21]). Sometimes one works only with c-permutability (let us mention Ursini's work on c-permutable varieties (subtractive varieties in Ursini's terminology), see e.g. [21] and the references therein). Clearly, if $p(x, y, z)$ is a term characterizing permutability then $s(x, y) = p(x, y, c)$ is a term characterizing c-permutability. Therefore, from Proposition 15 it follows by an easy computation that $s(x, y) = x \to y$ is a term charaterizing 1-permutability and $s(x, y) = x \otimes (y \to 0)$ is a term characterizing 0-permutability of residuated lattices.

Distributivity. From general lattice theory we obtain the following assertion.

Proposition 16. *The congruence lattice of any residuated lattice* **L** *is a compete Brouwerian lattice, i.e.* $x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i)$ *holds in* $\mathrm{Con}\, L$.

P r o o f. The congruence lattice of any lattice is a complete Brouwerian lattice [2, p. 138]. The proposition thus follows from the fact that for any residuated lattice, Con **L** is a complete sublattice of the congruence lattice of its lattice reduct $\langle L, \wedge, \vee \rangle$. □

It follows directly that the congruence lattice of a residuated lattice is distributive (and hence also modular).

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