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A NOTE ON SEMILOCAL GROUP RINGS

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Abstract. Let R be an associative ring with identity and let $J(R)$ denote the Jacobson radical of R. R is said to be *semilocal* if $R/J(R)$ is Artinian. In this paper we give necessary and sufficient conditions for the group ring RG, where G is an abelian group, to be semilocal.

Keywords: semilocal, group ring

MSC 2000: 16L30

1. Introduction

All rings considered in this paper are associative with identity. Given a ring R and a group G, we will denote the group ring of G over R by RG. If H is a subgroup of G then ωH will denote the right ideal of RG generated by $\{1-h \mid h \in H\}$. In particular, if H is a normal subgroup of G then ωH is an ideal of RG and $RG/\omega H \cong R(G/H)$. If $H = G$, then ωG is called the *augmentation ideal* of RG and is written as Δ. It is well-known that $R \cong RG/\Delta$. If I is an ideal of R then IG is the ideal of RG generated by the subset I and $(R/I)G \cong RG/IG$. These results and notation may be found in Connell's paper (see [2]).

For any ring R, the Jacobson radical of R will be denoted by $J(R)$ and the characteristic of R by char R. By an Artinian ring we mean a ring that is both left and right Artinian. If R is a ring such that $R/J(R)$ is Artinian then we say that R is *semilocal.* By $p > 0$ we mean that p is a prime number.

Our main result in this paper is as follows:

Theorem 1. *Let R be a ring and G an abelian group. Then RG is semilocal if and only if*

(i) *R is semilocal and G is finite, or*
(ii) *R is semilocal and $G \cong G_p \times H$, where G_p is an infinite p-group, H is finite, the order of H is relatively prime to p and $R/J(R)$ is of characteristic $p > 0$.*

We remark that if R is a commutative ring and G is an abelian group, Gulliksen, Ribenboim and Viswanathan [4] and Renault [8] have shown that conditions (i) and (ii) in Theorem 1 are necessary and sufficient for RG to be semilocal. Theorem 1 is thus an extension of their result. In the case when $R = \mathbb{F}$ is a field and G is an arbitrary group, the question on whether $\mathbb{F}G$ is semilocal implies that G is locally finite or a finite extension of a p-group (where $p = \operatorname{char} \mathbb{F} > 0$) has been of some interest. Theorem 1 shows that the answer to this question is in the affirmative if G is abelian. S. M. Woods has in fact shown in [9], Theorem 3.2 that G must be torsion if RG is semilocal. J. M. Goursaud [3] and D. S. Passman [7] independently proved that if $\mathbb{F}G$ is semilocal and G is finite, then G is a finite extension of a p-group. J. Lawrence [6] proved that if $\mathbb{F}$ is a field transcendental over the algebraic closure of its prime subfield and $\mathbb{F}G$ is semilocal, then G is a finite extension of a p-group where $p = \operatorname{char} \mathbb{F}$.

2. Preliminaries

For the sake of completeness we first deal with some preliminaries of the proof of Theorem 1.

Theorem 2.1 ([2])**.** *Let R be a ring and G a group. Then RG is Artinian if and only if R is Artinian and G is finite.*

Theorem 2.2 ([2], [1])**.** *Let R be a ring and G a group. Then RG is regular if and only if*
(a) *R is regular;*
(b) *G is locally finite;*
(c) *the order of every finite subgroup of G is a unit in R.*

Proposition 2.3. *Let R be a ring and G a group. If R is semilocal and G is finite then RG is semilocal.*

Proof. Since $R/J(R)$ is Artinian and G is finite, so $(R/J(R))G$ is Artinian (by Theorem 2.1). Since G is locally finite, it follows from [2], Proposition 9 that $J(R)G \subseteq J(RG)$. Now consider the mapping $\pi\colon RG/J(R)G \to RG/J(RG)$ defined as follows:

$$\pi(x + J(R)G) = x + J(RG), \quad x \in RG.$$

The mapping π is well-defined since $J(R)G \subseteq J(RG)$. It is easy to verify that π is a ring epimorphism. Then since $RG/J(R)G \cong (R/J(R))G$ is Artinian, so is $RG/J(RG)$. Hence RG is semilocal. □

The proof of the following proposition is straightforward and will be left to the reader.

Proposition 2.4. *Any homomorphic image of a semilocal ring is semilocal.*

Proposition 2.5. *Let R be a ring and G a group. If RG is semilocal then R is semilocal and G is a torsion group.*

Proof. Since $R \cong RG/\Delta$ and RG is semilocal, it follows from Proposition 2.4 that R is semilocal. The assertion that G is a torsion group follows from the proof of Theorem 3.2 in [9]. □

Lemma 2.6. *Let $R_1, \ldots, R_n$ be rings. Then $R = \prod_{i=1}^{n} R_i$ is semilocal if and only if each R_i is semilocal.*

Proof. We first note that

$$R/J(R) = \prod_{i=1}^{n} R_i \Big/ J\left(\prod_{i=1}^{n} R_i\right) = \prod_{i=1}^{n} R_i \Big/ \prod_{i=1}^{n} J(R_i) \cong \prod_{i=1}^{n} R_i \Big/ J(R_i). \tag{2.1}$$

Now if R is semilocal then $R/J(R)$ is Artinian and so is $\prod_{i=1}^{n} R_i/J(R_i)$ (by (2.1)). Therefore $R_i/J(R_i)$ is Artinian and hence R_i is semilocal ($i = 1, \ldots, n$).

Conversely, if each R_i is semilocal, then each $R_i/J(R_i)$ is Artinian. Hence $\prod_{i=1}^{n} R_i/J(R_i)$ is Artinian and so is $R/J(R)$ (by (2.1)). Therefore R is semilocal and this completes the proof. □

Lemma 2.7. *Let R be a ring. If $\overline{R} = R/J(R)$ is semilocal, so is R.*

Proof. Since $\overline{R}$ is semiprimitive and semilocal, so $\overline{R} \cong \overline{R}/J(\overline{R})$ is Artinian; hence R is semilocal. □

For any ring R and positive integer n, we let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

Lemma 2.8. *A ring R is semilocal if and only if $M_n(R)$ is semilocal.*

Proof. It is well-known that

$$M_n(R/J(R)) \cong M_n(R)/M_n(J(R)) = M_n(R)/J(M_n(R)).$$

The result then follows immediately from the fact that a ring R is Artinian if and only if $M_n(R)$ is Artinian (see [5], p. 71). □

Lemma 2.9. *Let R be a ring and G a group. Then*

$$M_n(R)G \cong M_n(RG).$$

Proof. Let $\theta\colon M_n(R)G \to M_n(RG)$ be the mapping defined as follows: For any $A_1g_1 + \ldots + A_sg_s \in M_n(R)G$, let

$$\theta(A_1g_1 + \ldots + A_sg_s) = (b_{ij}),$$

where $b_{ij} = a_{ij}^{(1)}g_1 + \ldots + a_{ij}^{(s)}g_s$ and $a_{ij}^{(m)}$ is the entry in the i-th row and j-th column of A_m, $m = 1, \ldots, s$. It may be verified routinely that θ is a ring isomorphism. Hence $M_n(R)G \cong M_n(RG)$. □

Remark. It is known that if R is a completely reducible ring, then R is isomorphic to a finite direct product of full matrix rings over division rings, that is,

$$R \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k)$$

where D_i is a division ring ($i = 1, \ldots, k$). We shall refer to the D_i's ($i = 1, \ldots, k$) as division rings associated with R.

Proposition 2.10. *Let R be a ring and G a group. If RG is semilocal, so is DG for each division ring D associated with $R/J(R)$.*

Proof. Assume that RG is semilocal. By Proposition 2.4 we have that $(R/J(R))G \cong RG/J(R)G$ and $R \cong RG/\Delta$ are semilocal. Therefore, $R/J(R)$ is completely reducible and hence, isomorphic to a finite direct product of full matrix rings over division rings, that is,

$$R/J(R) \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k) = \prod_{i=1}^{k} M_{n_i}(D_i)$$

for some division rings $D_1, \ldots, D_k$. It follows that

$$\begin{aligned}(R/J(R))G &\cong \Big(\prod_{i=1}^{k} M_{n_i}(D_i)\Big)G \cong \prod_{i=1}^{k} M_{n_i}(D_i)G \\ &\cong \prod_{i=1}^{k} M_{n_i}(D_iG) \quad \text{(by Lemma 2.9)}.\end{aligned}$$

Since $(R/J(R))G$ is semilocal, so is $\prod_{i=1}^{k} M_{n_i}(D_iG)$ (by Proposition 2.4). It follows from Lemma 2.6 that each $M_{n_i}(D_iG)$ is semilocal and hence by Lemma 2.8, D_iG is semilocal ($i = 1, \ldots, k$). □

Proposition 2.11. *If D is a division ring of characteristic $p > 0$ and G is an abelian group which is a finite extension of a p-group, then DG is semilocal.*

Proof. By assumption we have that G/G_p is finite for some p-subgroup G_p of G. If $G_p = \{1\}$, then G is finite and it follows easily that DG is semilocal. Now assume that $G_p \neq \{1\}$ and let $g \in G_p$, $g \neq 1$. Then $g^{p^n} = 1$ for a positive integer n and therefore, $(1-g)^{p^n} = 0$. It follows that $1-g$ is a nilpotent element. Since $1-g$ lies in the centre of DG, so the ideal generated by $1-g$ is nilpotent (hence nil). Then, since all nil ideals of DG are contained in $J(DG)$, so $1-g \in J(DG)$. It thus follows that $\omega G_p \subseteq J(DG)$. Now consider the mapping $\pi\colon\ DG/\omega G_p \to DG/J(DG)$ defined as follows:

$$\pi(x + \omega G_p) = x + J(DG), \quad x \in DG.$$

Since $\omega G_p \subseteq J(DG)$, π is well-defined. It is easily verified that π is a ring epimorphism. Note that $D(G/G_p)$ is Artinian since D is Artinian and G/G_p is finite. Then since $DG/\omega G_p \cong D(G/G_p)$ is Artinian, so is $DG/J(DG)$. Hence DG is semilocal.

□

3. Proof of Theorem 1

We are now ready for the proof of the main theorem.

Proof of Theorem 1. ($\Rightarrow$): Suppose that RG is semilocal. Since $R \cong RG/\Delta$, it follows from Proposition 2.4 that R is semilocal. By Proposition 2.10 we have that DG is semilocal for each division ring D associated with $R/J(R)$. Let D be one of those division rings and let $p = \operatorname{char} D$. We consider the following cases:

Case 1: $p = 0$. In this case, the order of every finite subgroup of G is a unit in D. By Proposition 2.5 we know that G is torsion. Then, since G is abelian, it follows that G is locally finite. Since D is regular, it follows from Theorem 2.2 that DG is regular. Hence $J(DG) = \{0\}$ and therefore $DG \cong DG/J(DG)$ is Artinian. We thus have that G is finite, that is (i) occurs.

Case 2: $p > 0$. Since G is an abelian torsion group we may write $G \cong G_p \times H$ where G_p is the Sylow p-subgroup of G and the order of every element of H is prime to p. Clearly, the order of every finite subgroup of H is a unit in D. Furthermore,

H is locally finite (since G is locally finite) and D is regular. It follows that DH is regular (by Theorem 2.2); hence $J(DH) = \{0\}$. Since

$$DH \cong D(G/G_p) \cong DG/\omega G_p$$

is semilocal (by Proposition 2.4), so $DH \cong DH/J(DH)$ is Artinian and hence H is finite. If G_p is finite, then (i) occurs.

Now suppose that G_p is infinite. We show that each of the division rings associated with the completely reducible ring $R/J(R)$ has the characteristic p. Suppose that there exists a division ring D' associated with $R/J(R)$ such that char $D' = q$ and $q \neq p$. If $q = 0$, then by the same argument as in Case 1 we have that G is finite. But this is impossible since $G_p \subseteq G$ and G_p is infinite. If $q > 0$, then since G is an abelian torsion group, we may write $G \cong G_q \times H'$, where G_q is the Sylow q-subgroup of G and the order of every element of H' is prime to q. By the same argument as in the preceding paragraph we can show that H' is finite. But since $G_p \subseteq H'$, this implies that G_p is finite; a contradiction. Hence if G_p is infinite, then each of the division rings associated with $R/J(R)$ is of the characteristic $p > 0$. It follows then that char $R/J(R) = p$ and hence, (ii) occurs.

($\Leftarrow$): Suppose that (i) occurs. It follows readily from Proposition 2.3 that RG is semilocal.

Now suppose that (ii) occurs. Then $R/J(R)$ is a completely reducible ring and therefore it is isomorphic to a finite direct product of full matrix rings over division rings, that is,

$$R/J(R) \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k) = \prod_{i=1}^{k} M_{n_i}(D_i)$$

for some division rings $D_1, \ldots, D_k$. Therefore,

$$\begin{aligned} RG/J(R)G &\cong (R/J(R))G \cong \prod_{i=1}^{k} M_{n_i}(D_i)G \\ &\cong \prod_{i=1}^{k} M_{n_i}(D_iG) \quad \text{(by Lemma 2.9).} \end{aligned} \tag{3.1}$$

Since char $R/J(R) = p > 0$, so each D_i is a division ring of the characteristic $p > 0$. Then since G is a finite extension of a p-group, it follows from Proposition 2.11 that each D_iG is semilocal. From Lemma 2.8 we have that $M_{n_i}(D_iG)$ is semilocal for each i. Therefore $\prod_{i=1}^{k} M_{n_i}(D_iG)$ is semilocal (by Lemma 2.6) and it follows from (3.1) that $RG/J(R)G$ is semilocal. Now since G is locally finite, so $J(R)G \subseteq J(RG)$ (by [2], Proposition 9). Consider the mapping $\pi\colon RG/J(R)G \to RG/J(RG)$ defined as follows:

$$\pi(\alpha + J(R)G) = \alpha + J(RG), \quad \alpha \in RG.$$

By routine verification, π is a well-defined ring epimorphism. Since $RG/J(R)G$ is semilocal, so is $RG/J(RG)$ (by Proposition 2.4). It then follows from Lemma 2.7 that RG is semilocal.

This completes the proof of the theorem. □

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