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PRIMARY ELEMENTS IN PRÜFER LATTICES

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Abstract. In this paper we study primary elements in Prüfer lattices and characterize α-lattices in terms of Prüfer lattices. Next we study weak ZPI-lattices and characterize almost principal element lattices and principal element lattices in terms of ZPI-lattices.

Keywords: principal element, primary element, Prüfer lattice

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An element e of a multiplicative lattice L is said to be principal if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $(a \vee be) : e = (a : e) \vee b$. Elements satisfying the first identity are called meet principal and elements satisfying the second identity are said to be join principal. An element e of L is said to be a cancellation element if, for any $a, b \in L$, $ae = be$ implies $a = b$. By a C-lattice we mean a (not necessarily modular) complete multiplicative lattice with least element 0 and compact greatest element 1 (a multiplicative identity), which is generated under joins by a multiplicatively closed subset C of compact elements. In a principally generated C-lattice, principal elements are compact [1, Theorem 1.3] and a finite product of principal elements is again a principal element [7].

Throughout this paper we assume that L is a C-lattice generated by principal elements. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \in C \mid x \nleqslant p\}$. For basic properties of localization, the reader is referred to [9]. We also note that in a C-lattice, $a = b$ if and only if $a_m = b_m$ for all maximal elements m of L. For any prime element p of L, p^{Δ} denotes the meet of all p-primary elements of L. A prime element p of L is said to be branched (unbranched) if $p > p^{\Delta}$ ($p = p^{\Delta}$). Let p, m be two prime elements of L. We say m covers p if $m > p$ and there is no prime element p_1 of L such that $m > p_1 > p$. An element q is said to be p-semiprimary if p is a prime element and $\sqrt{q} = p$.

L is said to be *reduced* if 0 is the only nilpotent element of L. A multiplicative lattice L in which every element is principal is called a *principal element lattice*. Similarly, L is said to be an *almost principal element lattice* if L_m is a principal element lattice for every maximal element m of L. For various characterizations of almost principal element lattices and principal element lattices, the reader is referred to [5], [8] and [10]. It is well known that L is a principal element lattice if and only if every prime element is principal. L is said to be a Prüfer lattice if every compact element is principal. It is well known that L is a Prüfer lattice if and only if L_p is totally ordered for every prime p of L. For more information on Prüfer lattices, the reader is referred to [1, Theorem 3.4] and [15]. π-lattices were introduced in [1]. A multiplicative lattice L_0 is said to be a π-lattice if L_0 is generated by a set S of elements (not necessarily principal) each of which is a product of prime elements. A ZPI-lattice is a multiplicative lattice in which every element is a product of prime elements. A C-lattice L_0 is said to be an M-normal lattice if every prime element contains a unique minimal prime element. For various characterizations of M-normal lattices the reader is referred to [3] and [13].

In this paper we study primary elements in Prüfer lattices and characterize α-lattices. In fact, it is established that a reduced lattice L is an α-lattice if and only if L is a Prüfer lattice in which every non minimal prime is non idempotent if and only if L is a Prüfer lattice in which the a.c.c. (ascending chain condition) for prime elements is valid and every idempotent prime is unbranched (see Theorem 6). Next we study weak ZPI-lattices. We prove that L is an almost principal element lattice if and only if L_m is a ZPI-lattice for every maximal element m of L. Using this result, we characterize principal element lattices and almost principal element lattices in terms of ZPI-lattices (see Theorem 8 and Theorem 9). For general background and terminology, the reader may consult [1], [4] and [9].

We shall begin with several lemmas.

Lemma 1. *Let q be p-semiprimary. Assume that, for each maximal element m of L, q_m is p_m-primary in L_m. Then q is p-primary.*

Proof. Since $\sqrt{q} = p$, it follows that q_p is p-primary ([9, Proposition 0.5]). We show that $q_m = (q_p)_m$ for every maximal element m of L. Let $p \leqslant m$ for some maximal element m of L and let $x_m \leqslant (q_p)_m$ for some principal element x of L. Then $xy \leqslant q_p$ for some $y \nleqslant m$. Again $xyz \leqslant q$ for some $z \nleqslant p$. Note that $x_m z_m \leqslant q_m$ and $z_m \nleqslant p_m$. Since q_m is p_m-primary, it follows that $x_m \leqslant q_m$ and hence $q_m = (q_p)_m$. Consequently, $q = q_p$ and hence q is p-primary. □

Lemma 2. *Let p be a prime element of L such that p is comparable with other elements. If $0_p \leqslant p^n$ $(n \in \mathbb{Z}^+)$, then p^n is p-primary.*

P r o o f. Suppose $xy \leqslant p^n$ and $y \nleqslant p$ for some principal elements $x, y \in L$. As $y \nleqslant p$, by hypothesis, $p \leqslant y$, so $p = yy'$ for some $y' \in L$. Note that $y' \leqslant p$, so that $xy \leqslant p^n = p^{n-1}p = p^{n-1}yy'$ and therefore $x \leqslant p^{n-1}y' \vee 0 : y$. Again since $y \nleqslant p$, it follows that $(0 : y) \leqslant 0_p \leqslant p^n$ and hence $x \leqslant p^n$. This shows that p^n is p-primary. □

Lemma 3. *Let L be totally ordered and let a be a non zero element of L. Then exactly one of the following cases occurs:*

(i) *$a = a^2$ is prime;*

(ii) *$a^{n+1} < a^n$ for all n, $\bigwedge\limits_{n=1}^{\infty} a^n$ is prime and $\bigwedge\limits_{n=1}^{\infty} a^n = \bigwedge\limits_{n=1}^{\infty} e^n$ for any principal element $e \leqslant a$ with $e \nleqslant a^2$;*

(iii) *$a^n = 0$ for some n.*

P r o o f. The proof of the lemma is similar to that of Theorem 3.1 of [2]. □

Lemma 4. *Let L be a totally ordered lattice and let p be a non idempotent prime element. If q is p-primary and $q < p$, then $q = p^k$ for some $k \in \mathbb{Z}^+$.*

P r o o f. Suppose $q < p$ is p-primary. Since $p \neq p^2$, by Lemma 3, either $\bigwedge\limits_{n=1}^{\infty} p^n$ is prime or $p^n = 0$ for some $n \in \mathbb{Z}^+$. Since q is p-primary, we can find an integer k such that $p^k \leqslant q < p^{k-1}$. We claim that $q = p^k$. Let $x \leqslant q$ and choose any principal element $y \leqslant p^{k-1}$ such that $y \nleqslant q$. Then $x \leqslant q < y \leqslant p^{k-1}$. Since y is principal, we have $x = yz$ for some $z \in L$. As $yz \leqslant q$, $y \nleqslant q$ and q is p-primary, it follows that $z \leqslant p$ and hence $x = yz \leqslant p^{k-1}p = p^k$. This shows that $q = p^k$. □

Theorem 1. *Suppose L is a Prüfer lattice. If p is a prime element and $0_p \leqslant p^n$, then p^n is p-primary.*

P r o o f. Suppose p is a prime element and $0_p \leqslant p^n$. Let m be a maximal element of L such that $p \leqslant m$. By hypothesis, L_m is totally ordered. It can be easily verified that $0_{p_m} \leqslant p_m^n$ and hence by Lemma 2, p_m^n is a p_m-primary element of L_m. Now the proof of the theorem follows from Lemma 1. □

Theorem 2. *Suppose L is a Prüfer lattice. Let p be a prime element and let $0_p \leqslant p^n$. Then there exists no primary element q such that $p^n < q < p^{n-1}$.*

P r o o f. By Theorem 1, p^i is p-primary for $i = 1, 2, \ldots, n$. If $p = p^2$, then we are through. Suppose $p \neq p^2$. Then $p_p \neq p_p^2$. As L_p is totally ordered, with p_p as the non idempotent maximal element, by Lemma 4, if q is p-primary, then $q = q_p = (p_p)^k$ for some $k \in \mathbb{Z}^+$. Hence $p^n < q < p^{n-1}$ fails to hold for any p-primary element q of L. □

Lemma 5. *Let L be a totally ordered lattice. Suppose p is a prime element for which p^n is not primary for some $n \in \mathbb{Z}^+$. Then $p^n = 0$.*

Proof. Let n be the least positive integer such that p^n is not primary. Then by Theorem 1, $p^n < 0_p \leqslant p^{n-1}$. As $p^n < 0_p$, it follows that p is a minimal prime, so 0_p is p-primary and hence by Lemma 4, $0_p = p^{n-1}$. Now $p^n = p^{n-1}p = 0_p p$. Observe that $0_p p = \bigvee\{xp \mid x \text{ is principal and } x \leqslant 0_p\}$. Again if x is principal and $x \leqslant 0_p$, then $(0 : x) \nleqslant p$, so $p \leqslant (0 : x)$, and hence $xp = 0$. This shows that $p^n = 0$. $\square$

Theorem 3. *Suppose L is a Prüfer lattice. Let p be a non minimal prime element. Then p^n is p-primary for all $n \in \mathbb{Z}^+$.*

Proof. Suppose p^k is not p-primary for some $k \in \mathbb{Z}^+$. By Lemma 1, there exists a maximal element m such that $p \leqslant m$ and p_m^k is not p_m-primary. As L_m is totally ordered, by Lemma 5, $p_m^k = 0_m$. As p is non minimal, it follows that $p_0 < p$ for some prime element p_0 of L. But $p^k \leqslant 0_m \leqslant p_0 < p$, which is a contradiction. Hence p^n is p-primary for all $n \in \mathbb{Z}^+$ and the proof is complete. $\square$

Theorem 4. *Suppose L is a Prüfer lattice. Let p be a non minimal prime element of L. Then $\bigwedge\limits_{n=1}^{\infty} p^n$ is a prime element.*

Proof. If $p = p^2$, then we are through. Suppose $p \neq p^2$. By Theorem 3, $p^n = p_p^n$ for all $n \in \mathbb{Z}^+$. As L_p is totally ordered, by Lemma 3, $\overline{\bigwedge\limits_{n=1}^{\infty}} p_p^n$ is prime ($\overline{\bigwedge}$ is the meet in L_p) in L_p. Now it can be easily verified that $\bigwedge\limits_{n=1}^{\infty} p^n$ is a prime element in L. $\square$

Theorem 5. *Suppose L is a Prüfer lattice and $p \neq p^2$ is a prime element. If p is non minimal, then $\{p^n\}_{n=1}^{\infty}$ is the set of all p-primary elements. If p is minimal and if $q > 0_p$ is a p-primary element, then q is a power of p.*

Proof. Suppose p is non minimal. Then by Theorem 3, p^n is p-primary for all $n \in \mathbb{Z}^+$. Also by Theorem 4, $p^n \neq p^{n+1}$ for all $n \in \mathbb{Z}^+$. Again by Lemma 4, every p-primary element is a power of p. Therefore $\{p^n\}_{n=1}^{\infty}$ is the set of all p-primary elements. Suppose p is minimal. If p^n is p-primary for all $n \in \mathbb{Z}^+$, then p_p is a non idempotent prime element of L_p. Therefore by Lemma 4, if q is p-primary, then $q = q_p = p_p^k = p^k$. Now assume that p^n is not p-primary for some $n \in \mathbb{Z}^+$. By Lemma 1, there exists a maximal element m of L such that $p \leqslant m$ and p_m^n is not p_m-primary in L_m. Let k be the least positive integer for which there exists a maximal element m such that p_m^k is not p_m-primary. By Lemma 5, $p_m^k = 0_m$. Again if $t < k$, then p^t is p-primary. Therefore we have $p^k \leqslant 0_m \leqslant 0_p \leqslant p^{k-1}$. Note that as L is a

Prüfer lattice, the set of all p-primary elements is linearly ordered. If q is p-primary and $p^{k-1} \leqslant q$, then by Theorem 2, q is a power of p. Suppose $0_p < q \leqslant p^{k-1}$. Then $0_p < q_p \leqslant p_p^{k-1}$ in L_p and hence by Lemma 4, $q_p = p_p^{k-1}$ in L_p. Consequently, $q = p^{k-1}$. This completes the proof of the theorem. $\square$

L is said to satisfy the condition (α) if every primary element is a power of its radical. Multiplicative lattices satisfying the condition (α) have been studied in [11] to characterize principal element lattices.

Definition 1. L is said to be an α-lattice if the ascending chain condition for prime elements is valid in L and every primary element is a power of its radical.

If R is an α-ring (see [6]), then the lattice of all ideals of R is an α-lattice. Also almost principal element lattices are examples of α-lattices (see [8, Theorem 5] and [10, Theorem 1]). Using the properties 0.7 and 0.8 of [9], it is not hard to show that if L is an α-lattice, then L_p is again an α-lattice for every prime element p of L.

Lemma 6. *If L is a quasi-local α-lattice, then the prime elements are comparable.*

Proof. By using [11, Theorem 3] and by imitating the proof of Theorem 4.3 of [6] we can get the result. $\square$

Lemma 7. *Let L be a reduced quasi-local α-lattice. Then L is totally ordered.*

Proof. By Lemma 6, L is a domain. Suppose there exist non comparable principal elements. Let

$$\Psi = \{p \in L \mid p \text{ is the radical of two non comparable principal elements}\}.$$

By our assumption, $\Psi \neq \varphi$. As the prime elements are linearly ordered, it follows that every $p \in \Psi$ is a prime element. Again by the a.c.c. for prime elements, Ψ contains a maximal element, say m. Let $m = \sqrt{x \vee y}$, where x and y are non comparable principal elements. We claim that m is branched. As $m \neq 0$, by the a.c.c. for prime elements, there exists a prime element p such that m covers p. So either $x \nleqslant p$ or $y \nleqslant p$. Without loss of generality, assume that $x \nleqslant p$. Then m is minimal over $p \vee x$. Again by [9, Property 0.5] and [10, Lemma 12], m is branched. Also by [11, Theorem 3], m^{Δ} is prime and m covers m^{Δ}. Therefore either $x \nleqslant m^{\Delta}$ or $y \nleqslant m^{\Delta}$, so either x_m or y_m is m-primary and hence x_m and y_m are comparable. Since L is quasi-local, every principal element is (completely) join irreducible. Using this fact, it can be easily shown that there exist two principal elements $z, z_1 \in L$ such that $xz = yz_1$ and either $z \nleqslant m$ or $z_1 \nleqslant m$ (see also [4, Theorem 9]). Note that z and z_1 are non comparable as x and y are non-comparable. Let $m_0 = \sqrt{z \vee z_1}$. Then $m_0 \in \Psi$ and $m < m_0$, a contradiction. Therefore L is totally ordered and the proof is complete. $\square$

Theorem 6. *Let L be a reduced lattice. Then the following statements are equivalent:*

(i) *L is an α-lattice.*

(ii) *L is a Prüfer lattice in which every non minimal prime element is non idempotent.*

(iii) *L is a Prüfer lattice in which every idempotent prime is unbranched and the a.c.c. for prime elements is valid.*

P r o o f. (i) $\Rightarrow$ (ii). Suppose (i) holds. By Lemma 7, L is a Prüfer lattice. Let p be a non minimal prime. By the a.c.c. for prime elements, there exists a prime element $q < p$ such that p covers q. So p is minimal over $q \vee x$ for any principal element $x \leqslant p$ such that $x \nleqslant q$. Again by [10, Lemma 12] and [9, Property 0.5], p is branched and hence by [11, Theorem 3], p can not be an idempotent element.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. We show that L satisfies the a.c.c. for prime elements. By localizing if necessary, we may assume that L is totally ordered. Let $p_1 < p_2 < \ldots < p_n < \ldots$ be an infinite strictly increasing chain of prime elements. Then $p = \bigvee\limits_{i=1}^{\infty} p_i \leqslant p^2$ is an idempotent prime element, a contradiction. Therefore L satisfies the a.c.c. for prime elements. Let p be an idempotent prime element. By (ii), p is minimal. As L is reduced, it follows that p is unbranched. Therefore (iii) holds.

(iii) $\Rightarrow$ (i) follows from Theorem 5 and the fact that in a reduced lattice L the minimal prime elements are unbranched. $\square$

Next we define weak ZPI-lattices and characterize almost principal element lattices and principal element lattices in terms of ZPI-lattices.

Definition 2. A multiplicative lattice L_0 is said to be a weak ZPI-lattice, if L_0 is generated by a set S of elements (not necessarily principal) such that for every $x, y \in S$, $x \vee y$ is a finite product of prime elements.

Observe that principal element lattices and ZPI-lattices are examples of weak ZPI-lattices. Obviously, every weak ZPI-lattice is a π-lattice.

Lemma 8. *Let L be quasi-local with a maximal element m. Suppose the join of any two principal elements is a finite product of prime elements. Then L is a principal element lattice.*

P r o o f. Observe that L is a π-lattice, and so by [1, Lemma 4.1] and [4, Proposition 2], every minimal prime of L is principal. Let $p < m$ be a non-zero principal prime and let y be a principal element satisfying $y \leqslant m$ and $y \nleqslant p$. Then L/p is generated under joins by elements of the form $e \vee p$, with e principal in L, and every such element is a product of primes. As e is join principal in L, $e \vee p$ is weak join principal ([1], Proposition 1.1) and a product of primes in L/p. As a factor of a cancellation element is a cancellation element in a domain, the factorization of $e \vee p$ as a product of primes is unique. As in the proof of [1, Lemma 4.8], it follows that $p \leqslant p^2 \vee y$, from which it follows that $1 = (p^2 \vee y) : p = p \vee (y : p)$. As L is quasi-local, it follows that $1 = y : p$. Therefore $p \leqslant y$ and hence $p = py$. Again since p is join principal, it follows that $1 = y \vee (0 : p)$. This shows that either $p = 0$ or m is the only prime of L. If m is minimal, then we are through. If m is non minimal, then L is a one dimensional π-domain and hence by [5, Theorem 2.3 and Theorem 2.4], L is a principal element lattice. □

Theorem 7. *L is an almost principal element lattice if and only if L_m is a* ZPI-*lattice for every maximal element m.*

Proof. Suppose L is an almost principal element lattice. Let m be a maximal element. Then every element of L_m is principal. By [14, Theorem 5], L_m is a ZPI-lattice. The converse follows from Lemma 8. □

Globally, we have the following

Theorem 8. *The following statements on L are equivalent:*

(i) *L is a principal element lattice.*

(ii) *L is a* ZPI-*lattice.*

(iii) *The join of any two principal elements of L is a finite product of prime elements.*

(iv) *L is a weak* ZPI-*lattice.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. By [4, Theorem 1], L is an almost principal element lattice and every element of L is compact. Again by [8, Theorem 5], L is distributive and hence L is a Noether lattice. Now the result follows from [14, Theorem 5].

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). Suppose (iv) holds. By definition, there exists a set S such that S generates L under joins and for every $x, y \in S$, $x \vee y$ is a finite product of prime elements. Observe that each $x \in S$ has only finitely many minimal primes. Next we show that each principal element has only finitely many minimal primes. Let a be a principal element of L. As a is compact, we have $a = \bigvee\limits_{i=1}^{n} x_i$, $x_i'^{s} \in S$. Let p be a minimal prime over a. Then $a_p = (x_i)_p$ for some i in L_p. By [9, Property 0.5], p is a minimal prime over x_i. Consequently, a has only finitely many minimal primes. Again by [10, Theorem 8], it is enough if we show that L is an almost principal element lattice. Let m be a maximal element of L. Let a_m, b_m be two principal elements of L_m. Then by [4, Proposition 2], $a_m = x_m$ and $b_m = y_m$ for some $x, y \in S$. Therefore $a_m \vee b_m = (x \vee y)_m$ is a finite product of prime elements in L_m. So by Lemma 8, L is an almost principal element lattice. $\square$

Theorem 9. *Suppose $L \neq \{0,1\}$ is a reduced lattice. Then the following statements on L are equivalent:*

(i) *L is an almost principal element lattice.*

(ii) *L_m is a* ZPI-*lattice, for every maximal element m of L.*

(iii) $\dim L \leqslant 1$ *and every primary element is a power of its radical.*

(iv) *L is a Prüfer lattice,* $\dim L \leqslant 1$ *and every non minimal prime is non idempotent.*

(v) *L is an* M-*normal lattice and every element of L is locally join principal.*

P r o o f. (i) $\Leftrightarrow$ (ii) follows from Theorem 7, (i) $\Rightarrow$ (iii) follows from [10, Lemma 2 and Theorem 1] and (iii) $\Rightarrow$ (iv) follows from Theorem 6. Now we prove that (iv) $\Rightarrow$ (v). Suppose (iv) holds. Note that by [12, Lemma 2], L is an M-normal lattice. Let m be a maximal element of L. If m is minimal, then by [12, Lemma 3], L_m is a two element chain. If m is non minimal, then L_m is a one dimensional totally ordered domain. Again by [8, Lemma 7], L_m is a principal element domain. This shows that (v) holds.

(v) $\Rightarrow$ (i). Suppose (v) holds. As L is a reduced M-normal lattice, by [12, Lemma 1 and Theorem 1], L_m is a domain for every maximal element m of L. By the proof of [1, Theorem 1.5] and by [4, Theorem 1], every compact element is principal in L. We show that every non minimal prime is maximal. By localizing if necessary, we can assume that L is a quasi-local totally ordered domain. Let p be a non zero prime and let e be a principal element such that $e \nleqslant p$. Then $ep \leqslant e^2$, so $ep = e^2x$ for some $x \in L$ (e^2 is principal). Again since $e^2x \leqslant p$ and $e^2 \nleqslant p$, it follows that $x \leqslant p$. Therefore $ep = e^2x \leqslant e^2p$ and hence $e \leqslant e^2$ (as p is join principal). As e is join principal, we have $1 = e$. Consequently, $\dim L = 1$ and so the maximal element is non idempotent. Now the result follows from [8, Lemma 7]. $\square$

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