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PRINCIPAL IDEALS OF FINITELY GENERATED COMMUTATIVE MONOIDS

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Abstract. We study the semigroups isomorphic to principal ideals of finitely generated commutative monoids. We define the concept of finite presentation for this kind of semigroups. Furthermore, we show how to obtain information on these semigroups from their presentations.

Keywords: monoid, ideal, cancellative, torsion free

MSC 2000: 20M14, 20M30

Introduction

In this paper we study the semigroups isomorphic to principal ideals of finitely generated commutative monoids. This study is made from the following perspective. In [11] it was shown that the semigroups which can be embedded in finitely generated commutative monoids are isomorphic to quotient semigroups of the form A/σ_A, with A a subsemigroup of $\mathbb{N}^p$ for some positive integer p and σ_A the restriction to A of the congruence σ on $\mathbb{N}^p$. In that paper the problem of finding algorithms for determining properties of the semigroup A/σ_A from (A, σ) was proposed. The work presented here has been developed following this line, since it covers the case when A is a principal ideal. The contents of this paper are organized as follows. In Section 1 we show that the pairs (m, ϱ) with m an element of $\mathbb{N}^p$ and ϱ a finite subset of $\mathbb{N}^p \times \mathbb{N}^p$ determine, up to isomorphisms, all semigroups isomorphic to a principal ideal of a finitely generated commutative monoid. So these semigroups are finitely presented whenever we admit a pair (m, ϱ) as a presentation of them. The rest of

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the paper is devoted to showing how we can obtain from (m, ϱ) information on the semigroup that this pair represents. In this sense, in Section 2 we see when they have an identity element. In Section 3 we show how it can be determined whether one of these semigroups is a group. Finally, Section 4 is devoted to the study of the properties of being cancellative and/or torsion free.

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1. Presentations of principal ideals

Let $(S, +)$ be a commutative monoid generated by $\{s_1, s_2, \ldots, s_p\}$. An ideal of S is a subset H of S fulfilling the following condition: if $h \in H$ and $s \in S$ then $h+s \in H$.

It is known (see [4]) that if H is an ideal of S, then there exists a finite subset B of H such that $H = B + S = \{b + s \mid b \in B,\ s \in S\}$. We say that H is a principal ideal of S if there exists $h \in H$ such that

$$H = h + S = \{h + s \mid s \in S\}.$$

Our goal in this section is to prove that a semigroup isomorphic to a principal ideal of a finitely generated commutative monoid is, up to isomorphism, perfectly determined by a pair (m, ϱ) with m an element of $\mathbb{N}^p$ for some positive integer p and $\varrho = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}$ a finite subset of $\mathbb{N}^p \times \mathbb{N}^p$.

We start constructing a principal ideal of a finitely generated commutative monoid from a pair (m, ϱ). Let $I = m+\mathbb{N}^p$, which is an ideal of $\mathbb{N}^p$; $\overline{\varrho} = \{(\alpha_1+m, \beta_1+m), \ldots, (\alpha_t+m, \beta_t+m)\}$, which is a subset of $I \times I$; let $\overline{\sigma}$ be the congruence on $\mathbb{N}^p$ generated by $\overline{\varrho}$ and $\overline{\sigma}_I = \overline{\sigma} \cap (I \times I)$, which clearly is a congruence on I.

We shall see that the semigroup $I/\overline{\sigma}_I$ is an ideal of $\mathbb{N}^p/\overline{\sigma}$. First we show a lemma and one of its consequences.

Lemma 1. *Let $x, y \in \mathbb{N}^p$, then $x \,\overline{\sigma}\, y$ if and only if one of the following conditions holds:*

- *$\{x, y\} \not\subseteq I$ and $x = y$,*
- *$\{x, y\} \subseteq I$ and $x \,\overline{\sigma}_I\, y$.*

P r o o f. The proof is easily deduced if we take into account the following remarks:

- Since $\alpha_1 + m, \beta_1 + m, \ldots, \alpha_t + m, \beta_t + m \in I$, we have $\overline{\varrho} \subseteq I \times I$.
- The congruence $\overline{\sigma} = \langle \overline{\varrho} \rangle$ can be constructed in three steps as follows (see for instance [2]):

i) $\overline{\varrho}_0 = \overline{\varrho} \cup \overline{\varrho}^{-1} \cup \tau$ with $\overline{\varrho}^{-1} = \{(\beta, \alpha) \mid (\alpha, \beta) \in \overline{\varrho}\}$ and $\tau = \{(\alpha, \alpha) \mid \alpha \in \mathbb{N}^p\}$.

ii) $\overline{\varrho}_1 = \{(\alpha + \gamma, \beta + \gamma) \mid (\alpha, \beta) \in \overline{\varrho}_0 \text{ and } \gamma \in \mathbb{N}^p\}$.

iii) $(\alpha, \beta) \in \overline{\sigma}$ if and only if there exist $v_0, \ldots, v_l \in \mathbb{N}^p$ such that $\alpha = v_0$, $\beta = v_l$ and $(v_i, v_{i+1}) \in \overline{\varrho}_1$ for all $i \in \{0, \ldots, l-1\}$. $\square$

As an immediate consequence of Lemma 1 we obtain the following result.

Corollary 2. *For every $x \in \mathbb{N}^p$,*

1. *if $x \notin I$, then $[x]_{\overline{\sigma}} = \{x\}$,*
2. *if $x \in I$, then $[x]_{\overline{\sigma}} = [x]_{\overline{\sigma}_I}$.*

Now, we are ready to prove the following result.

Theorem 3. *The semigroup $I/\overline{\sigma}_I$ is a principal ideal of $\mathbb{N}^p/\overline{\sigma}$.*

P r o o f. We show that $I/\overline{\sigma}_I = [m]_{\overline{\sigma}} + \mathbb{N}^p/\overline{\sigma}$.

- Let $[x]_{\overline{\sigma}_I} \in I/\overline{\sigma}_I$. Then $x \in I$ and therefore $[x]_{\overline{\sigma}_I} = [x]_{\overline{\sigma}}$. Furthermore, $x \in I$ implies that there exists $y \in \mathbb{N}^p$ such that $x = y + m$. Hence $[x]_{\overline{\sigma}} = [m]_{\overline{\sigma}} + [y]_{\overline{\sigma}}$, which leads to $[x]_{\overline{\sigma}_I} = [x]_{\overline{\sigma}} \in [m]_{\overline{\sigma}} + \mathbb{N}^p/\overline{\sigma}$.
- Let $[x]_{\overline{\sigma}} \in [m]_{\overline{\sigma}} + \mathbb{N}^p/\overline{\sigma}$. Then there exists $y \in \mathbb{N}^p$ such that $[x]_{\overline{\sigma}} = [m]_{\overline{\sigma}} + [y]_{\overline{\sigma}}$. Hence $[x]_{\overline{\sigma}} = [m + y]_{\overline{\sigma}}$. Since $m + y \in I$, we obtain that $[m + y]_{\overline{\sigma}} = [m + y]_{\overline{\sigma}_I}$ and conclude that $[x]_{\overline{\sigma}} = [m + y]_{\overline{\sigma}_I} \in I/\overline{\sigma}_I$. $\square$

From the above study we obtain the following statement.

Corollary 4. *Every pair (m, ϱ) with m an element of $\mathbb{N}^p$ and ϱ a finite subset of $\mathbb{N}^p \times \mathbb{N}^p$ determines a principal ideal of a finitely generated commutative monoid.*

Now, we see that this construction characterizes, up to isomorphisms, all principal ideals of finitely generated commutative monoids.

Let $(S, +)$ be a commutative monoid generated by $\{s_1, \ldots, s_p\}$, $H = h + S$ a principal ideal of S, $\varphi\colon \mathbb{N}^p \longrightarrow S$ the monoid homomorphism defined by

$$\varphi(x_1, \ldots, x_p) = x_1 s_1 + \ldots + x_p s_p$$

and R the kernel congruence of φ. Then S is isomorphic to the quotient monoid $\mathbb{N}^p/R$. Since $h \in S$, we have that there exists $m = (m_1, \ldots, m_p) \in \mathbb{N}^p$ such that $h = m_1 s_1 + \ldots + m_p s_p$. Let $I = m + \mathbb{N}^p$, $R_I = R \cap (I \times I)$, which is a congruence on I, and let $\overline{\sigma}$ be the congruence on $\mathbb{N}^p$ generated by R_I. Since $\overline{\sigma}$ is a congruence on $\mathbb{N}^p$, we obtain that it is finitely generated (see for instance [7]). Moreover, since $\overline{\sigma}$ is generated by R_I which is a subset of $I \times I$, we can assume that it is generated by a finite subset $\overline{\varrho}$ of $I \times I$. Note that then $\overline{\varrho}$ is of the form $\{(\alpha_1 + m, \beta_1 + m), \ldots, (\alpha_t + m, \beta_t + m)\}$ for a finite subset $\varrho = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}$ of $\mathbb{N}^p \times \mathbb{N}^p$.

In this way we have associated to H a pair (m, ϱ). For showing that the construction given at the beginning of this section characterizes all principal ideals of finitely generated commutative monoids, we shall prove Theorem 6, but before that we need a lemma.

Lemma 5. *The congruence $\overline{\sigma}_I$ is the same as R_I.*

P r o o f.

- If $(x, y) \in R_I$, then $(x, y) \in \overline{\sigma}$ and $(x, y) \in I \times I$. Therefore $(x, y) \in \overline{\sigma}_I$.
- If $(x, y) \in \overline{\sigma}_I$, then $(x, y) \in \overline{\sigma}$ and $(x, y) \in I \times I$. We see that $(x, y) \in R$ and therefore $(x, y) \in R_I$. Since $\overline{\sigma}$ is a congruence on $\mathbb{N}^p$ generated by R_I, then $(x, y) \in \overline{\sigma}$ implies (see the proof of Lemma 1) that there exist $v_0, \ldots, v_l \in \mathbb{N}^p$ such that $x = v_0$, $y = v_l$ and $(v_i, v_{i+1}) = (a_i + c_i, b_i + c_i)$ with $(a_i, b_i) \in R_I$ and $c_i \in \mathbb{N}^p$ for all $i \in \{0, \ldots, l-1\}$. But if $(a_i, b_i) \in R_I$, then $(a_i, b_i) \in R$ and $(a_i + c_i, b_i + c_i) \in R$. By the transitivity of R we obtain that $(x, y) \in R$. □

Theorem 6. *The semigroups H and $I/\overline{\sigma}_I$ are isomorphic.*

P r o o f. Define $f\colon I/\overline{\sigma}_I \longrightarrow H$ by

$$f([(x_1, \ldots, x_p)]_{\overline{\sigma}_I}) = x_1 s_1 + \ldots + x_p s_p.$$

The map f is well defined as the following two remarks show.

- If $(x_1, \ldots, x_p) \in I$, then $(x_1, \ldots, x_p) = (m_1, \ldots, m_p) + (a_1, \ldots, a_p)$ for some $(a_1, \ldots, a_p) \in \mathbb{N}^p$. Hence

$$x_1 s_1 + \ldots + x_p s_p = m_1 s_1 + \ldots + m_p s_p + a_1 s_1 + \ldots + a_p s_p \in h + S = H.$$

- If $(x_1, \ldots, x_p)\ \overline{\sigma}_I\ (y_1, \ldots, y_p)$ then, by Lemma 5, $(x_1, \ldots, x_p) R_I (y_1, \ldots, y_p)$. Hence $(x_1, \ldots, x_p) R (y_1, \ldots, y_p)$, which means that

$$x_1 s_1 + \ldots + x_p s_p = y_1 s_1 + \ldots + y_p s_p.$$

We show that f is surjective. If $x \in H$, then there exists $s \in S$ such that $x = h + s$. Let $(a_1, \ldots, a_p) \in \mathbb{N}^p$ with $a_1 s_1 + \ldots + a_p s_p = s$. Clearly, $f([(m_1 + a_1, \ldots, m_p + a_p)]_{\overline{\sigma}_I}) = x$.

Now, we show that f is injective. If $f([x_1, \ldots, x_p)]_{\overline{\sigma}_I}) = f([(y_1, \ldots, y_p)]_{\overline{\sigma}_I})$, then $x_1 s_1 + \ldots + x_p s_p = y_1 s_1 + \ldots + y_p s_p$. Hence $(x_1, \ldots, x_p) R_I (y_1, \ldots, y_p)$ and by Lemma 5 we have $[(x_1, \ldots, x_p)]_{\overline{\sigma}_I} = [(y_1, \ldots, y_p)]_{\overline{\sigma}_I}$.

Now we only have to prove that f is a homomorphism, which is trivial. □

As a consequence of Corollary 4 and Theorem 6 we obtain the following result.

Corollary 7. *Every semigroup isomorphic to a principal ideal of a finitely generated commutative monoid is determined, up to isomorphisms, by a pair* (m, ϱ) *where* $m \in \mathbb{N}^p$ *and* ϱ *is a finite subset of* $\mathbb{N}^p \times \mathbb{N}^p$.

2. Principal ideals which are monoids

Let (m, ϱ) be a pair with $m = (m_1, \dots, m_p) \in \mathbb{N}^p$ and $\varrho = \{(\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t)\}$ a subset of $\mathbb{N}^p \times \mathbb{N}^p$. Let $I = m + \mathbb{N}^p$, let $\overline{\sigma}$ be the congruence on $\mathbb{N}^p$ generated by $\overline{\varrho} = \{(\alpha_1 + m, \beta_1 + m), \dots, (\alpha_t + m, \beta_t + m)\}$ and $\overline{\sigma}_I = \overline{\sigma} \cap (I \times I)$. The objective of this section is to characterize the pairs (m, ϱ) such that the semigroup $I/\overline{\sigma}_I$ has an identity element.

Let σ be the congruence on $\mathbb{N}^p$ generated by ϱ. The following result shows the relationship between the congruences σ and $\overline{\sigma}$.

Lemma 8. *Let* $x, y \in \mathbb{N}^p$. *Then* $x \,\sigma\, y$ *if and only if* $(x + m)\,\overline{\sigma}\,(y + m)$.

P r o o f. *Necessity.* We know that $x \,\sigma\, y$ implies that there exist $v_0, \dots, v_l \in \mathbb{N}^p$ such that $x = v_0$, $y = v_l$ and $(v_i, v_{i+1}) = (a_i + c_i, b_i + c_i)$ for some $(a_i, b_i) \in \varrho \cup \varrho^{-1} \cup \tau$ and $c_i \in \mathbb{N}^p$ for all $i \in \{0, \dots, l-1\}$. Set $\overline{v_i} = v_i + m$ for all $i \in \{0, \dots, l\}$. Clearly, $x + m = \overline{v_0}$, $y + m = \overline{v_l}$ and $(\overline{v_i}, \overline{v_{i+1}}) = (a_i + m + c_i, b_i + m + c_i)$ with $(a_i + m, b_i + m) \in \overline{\varrho} \cup \overline{\varrho}^{-1} \cup \tau$ and $c_i \in \mathbb{N}^p$. Hence, $(x + m)\,\overline{\sigma}\,(y + m)$.

Sufficiency. We know that $(x+m)\,\overline{\sigma}\,(y+m)$ implies that there exist $\overline{v_0}, \dots, \overline{v_l} \in \mathbb{N}^p$ such that $x + m = \overline{v_0}$, $y + m = \overline{v_l}$ and $(\overline{v_i}, \overline{v_{i+1}}) = (a_i, b_i) + (c_i, c_i)$ for some $(a_i, b_i) \in \overline{\varrho} \cup \overline{\varrho}^{-1} \cup \tau$ and $c_i \in \mathbb{N}^p$ for all $i \in \{0, \dots, l-1\}$. Since we can assume that $\overline{v_i} \neq \overline{v_{i+1}}$, we have that $(a_i, b_i) \notin \tau$. Hence $(a_i, b_i) \in \overline{\varrho} \cup \overline{\varrho}^{-1}$ and therefore $(a_i - m, b_i - m) \in \varrho \cup \varrho^{-1}$. Let $v_i = \overline{v_i} - m$ for all $i \in \{0, \dots, l\}$. Then $v_0 = x$, $v_l = y$ and $(v_i, v_{i+1}) = (a_i - m, b_i - m) + (c_i, c_i)$ with $(a_i - m, b_i - m) \in \varrho \cup \varrho^{-1}$ and $c_i \in \mathbb{N}^p$. Clearly we obtain that $x \,\sigma\, y$. □

If $(A, +)$ is a commutative monoid, then we denote

$$\mathcal{U}(A) = \{a \in A \mid a + b = 0 \text{ for some } b \in A\}.$$

Usually, $\mathcal{U}(A)$ is called the group of units of A. The proof of the two following lemmas are left to the reader.

Lemma 9. *Let* $(A, +)$ *be a monoid and* $\mathcal{U}(A)$ *its group of units. Then the following conditions are fulfilled:*

- $\mathcal{U}(A)$ *is a submonoid of* A *and a group,*
- $a+b \in \mathcal{U}(A)$ *implies that* $\{a,b\} \subseteq \mathcal{U}(A)$,
- $A \setminus \mathcal{U}(A)$ *is an ideal of* A.

Lemma 10. *Let* $(A,+)$ *be a monoid and* $\mathcal{U}(A)$ *its group of units. If* A *is generated by* $\{a_1,\dots,a_p\}$ *and*

$$\mathcal{U}(A) \cap \{a_1,\dots,a_p\} = \{a_{i1},\dots,a_{ir}\},$$

then $\mathcal{U}(A)$ *is the submonoid of* A *generated by* $\{a_{i1},\dots,a_{ir}\}$.

Theorem 11. *The semigroup* $I/\overline{\sigma}_I$ *has an identity element if and only if* $[m]_\sigma \in \mathcal{U}(\mathbb{N}^p/\sigma)$.

P r o o f. *Necessity.* Let $u \in \mathbb{N}^p$ be such that $[m+u]_{\overline{\sigma}_I}$ is the identity element of $I/\overline{\sigma}_I$. Then $(m+m+u)\ \overline{\sigma}_I\ m$ and therefore $(m+m+u)\ \overline{\sigma}\ m$. By Lemma 8, we deduce that $(m+u)\ \sigma\ 0$. Then $[m]_\sigma + [u]_\sigma = [0]_\sigma$, which means $[m]_\sigma \in \mathcal{U}(\mathbb{N}^p/\sigma)$.

Sufficiency. If $[m]_\sigma \in \mathcal{U}(\mathbb{N}^p/\sigma)$, then there exists $x \in \mathbb{N}^p$ such that $[m]_\sigma + [x]_\sigma = [0]_\sigma$, which leads to $(m+x)\ \sigma\ 0$. By Lemma 8 we obtain that $(m+m+x)\ \overline{\sigma}\ m$. We show that $[m+x]_{\overline{\sigma}_I}$ is the identity element of $I/\overline{\sigma}_I$. Let $[m+y]_{\overline{\sigma}_I} \in I/\overline{\sigma}_I$. Since $(m+m+x)\ \overline{\sigma}\ m$, we deduce that $(m+m+x+y)\ \overline{\sigma}\ (m+y)$. Furthermore, $\{m+m+x+y, m+y\} \subseteq I \times I$ and then $(m+m+x+y)\ \overline{\sigma}_I\ (m+y)$. Hence $[m+y]_{\overline{\sigma}_I} + [m+x]_{\overline{\sigma_I}} = [m+y]_{\overline{\sigma}_I}$. □

We close this section by explaining that from the computational point of view it is possible to determine algorithmically from a pair (m,ϱ) whether the semigroup $I/\overline{\sigma}_I$ has an identity element. We know that $\mathbb{N}^p/\sigma$ is the monoid generated by $\{[e_1],\dots,[e_p]\}$ with e_i the element of $\mathbb{N}^p$ whose i-th coordinate equals 1 and the other coordinates equal 0. In [8] an algorithm is given for determining from ϱ the set $\{i \mid [e_i] \in \mathcal{U}(\mathbb{N}^p/\sigma)\}$. The following result which is an immediate consequence of Lemma 10 and Theorem 11 gives us a method to check whether $I/\overline{\sigma}_I$ is a monoid.

Corollary 12. *The semigroup* $I/\overline{\sigma}_I$ *is a monoid if and only if* $\mathrm{Supp}(m) \subseteq \{i \mid [e_i] \in \mathcal{U}(\mathbb{N}^p/\sigma)\}$ *with* $\mathrm{Supp}(m) = \{i \mid m_i \neq 0\}$.

3. Principal ideals which are groups

Assume that $I/\overline{\sigma}_I$ is a monoid. Here we determine the group of units of this monoid. In particular, we are able to determine when $I/\overline{\sigma}_I$ is a group.

As in the preceding section, (m,ϱ) is a pair with $m=(m_1,\dots,m_p)\in\mathbb{N}^p$ and $\varrho=\{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}\subseteq\mathbb{N}^p\times\mathbb{N}^p$. Denote by $\overline{\varrho}$ the set $\{(\alpha_1+m,\beta_1+m),\dots,(\alpha_t+m,\beta_t+m)\}$, by σ and $\overline{\sigma}$ the congruences on $\mathbb{N}^p$ generated by ϱ and $\overline{\varrho}$, respectively, and by $\overline{\sigma}_I$ the congruence $\overline{\sigma}\cap(I\times I)$ on I. Furthermore, we assume that $\{i_1,\dots,i_r\}=\{i\mid[e_i]\in\mathcal{U}(\mathbb{N}^p/\sigma)\}$ and that $m+u$ is the identity element of $I/\overline{\sigma}_I$.

Proposition 13. *The set $\mathcal{U}(I/\overline{\sigma}_I)$ is the same as the set*

$$\{[x]_{\overline{\sigma}_I}\in I/\overline{\sigma}_I\mid \operatorname{Supp}(x)\subseteq\{i_1,\dots,i_r\}\}.$$

Proof. Assume that $x\in I$ and $\operatorname{Supp}(x)\subseteq\{i_1,\dots,i_r\}$. Then $x=m+y$ for some $y\in\mathbb{N}^p$ and x is a unit of $\mathbb{N}^p/\sigma$ (see Lemma 10). Hence there exists $z\in\mathbb{N}^p$ such that $[m+y]_\sigma+[z]_\sigma=[0]$. Then $(m+y+z)\,\sigma\,0$ and applying Lemma 8, we obtain that $(m+y+z+m)\,\overline{\sigma}\,m$. So we deduce that $(m+y+z+m+u)\,\overline{\sigma}\,(m+u)$ and therefore $[m+y]_{\overline{\sigma}_I}+[z+m+u]_{\overline{\sigma}_I}=[m+u]_{\overline{\sigma}_I}$. We conclude that $[x]_{\overline{\sigma}_I}=[m+y]_{\overline{\sigma}_I}\in\mathcal{U}(I/\overline{\sigma}_I)$.

Now, assume that $[m+y]_{\overline{\sigma}_I}$ is a unit of $I/\overline{\sigma}_I$, then there exists $[m+z]_{\overline{\sigma}_I}\in I/\overline{\sigma}_I$ such that $(m+y+m+z)\,\overline{\sigma}_I\,(m+u)$. Hence $[m+y+m+z]_{\overline{\sigma}_I}=[m+u]_{\overline{\sigma}_I}$ is the identity element of $I/\overline{\sigma}_I$. Clearly, we obtain that $[m]_{\overline{\sigma}_I}+[m+y+m+z]_{\overline{\sigma}_I}=[m]_{\overline{\sigma}_I}$ and therefore $(m+m+y+m+z)\,\overline{\sigma}\,m$. By Lemma 8, we have that $(m+y+m+z)\,\sigma\,0$ and then $[m+y]_\sigma$ is a unit of $\mathbb{N}^p/\sigma$. By Lemma 10, we conclude that $\operatorname{Supp}(m+y)\subseteq\{i_1,\dots,i_r\}$. □

As a consequence of Proposition 13 we obtain the following corollary.

Corollary 14. *The semigroup $I/\overline{\sigma}_I$ is a group if and only if $\{i_1,\dots,i_r\}=\{1,\dots,p\}$.*

An alternative restatement of Corollary 14 is: The semigroup $I/\overline{\sigma}_I$ is a group if and only if $\mathbb{N}^p/\sigma$ is a group. Hence the ideals associated with the presentations (m,ϱ) which are groups are those for which $\mathbb{N}^p/\langle\varrho\rangle$ is a group.

4. Torsion free cancellative principal ideals

The aim of this section is to characterize the pairs (m, ϱ) such that the semigroup $I/\overline{\sigma}_I$ is cancellative and/or torsion free.

Take $m = (m_1, \dots, m_p) \in \mathbb{N}^p$ and let $\varrho = \{(\alpha_1, \beta_1), \dots, (\alpha_l, \beta_l)\}$ be a finite subset of $\mathbb{N}^p \times \mathbb{N}^p$. Denote by $\overline{\varrho}$ the set $\{(\alpha_1 + m, \beta_1 + m), \dots, (\alpha_t + m, \beta_t + m)\}$, by σ and $\overline{\sigma}$ the congruences on $\mathbb{N}^p$ generated by ϱ and $\overline{\varrho}$, respectively, and let $\overline{\sigma}_I = \overline{\sigma} \cap (I \times I)$.

Proposition 15. *The semigroup $I/\overline{\sigma}_I$ is cancellative if and only if $\mathbb{N}^p/\sigma$ is cancellative.*

P r o o f. *Necessity.* Assume that $[x]_\sigma + [z]_\sigma = [y]_\sigma + [z]_\sigma$. Then $(x+z)\ \sigma\ (y+z)$. By Lemma 8 we obtain that $(m+x+z)\ \overline{\sigma}\ (m+y+z)$ and therefore $(m+x+m+z)\ \overline{\sigma}$ $(m+y+m+z)$. Hence,

$$[m+x]_{\overline{\sigma}_I} + [m+z]_{\overline{\sigma}_I} = [m+y]_{\overline{\sigma}_I} + [m+z]_{\overline{\sigma}_I}.$$

By the cancellativity of $I/\overline{\sigma}_I$, we obtain that $[m+x]_{\overline{\sigma}_I} = [m+y]_{\overline{\sigma}_I}$ and thus $(m+x)\ \overline{\sigma}$ $(m+y)$. By Lemma 8 we deduce that $x\ \sigma\ y$ and therefore $[x]_\sigma = [y]_\sigma$.

Sufficiency. Assume that $[m+x]_{\overline{\sigma}_I} + [m+z]_{\overline{\sigma}_I} = [m+y]_{\overline{\sigma}_I} + [m+z]_{\overline{\sigma}_I}$, then $(m+x+m+z)\ \overline{\sigma}\ (m+y+m+z)$. Applying Lemma 8 we obtain that $(x+m+z)\ \sigma$ $(y+m+z)$. Since $\mathbb{N}^p/\sigma$ is cancellative, we deduce that $x\ \sigma\ y$ and by Lemma 8 we have $(m+x)\ \overline{\sigma}\ (m+y)$. Hence $[m+x]_{\overline{\sigma}_I} = [m+y]_{\overline{\sigma}_I}$. □

An algorithm in [9] allows us to determine from ϱ whether $\mathbb{N}^p/\sigma$ is cancellative or not. Hence we have a algorithm for deciding from (m, ϱ) whether its associated semigroup $I/\overline{\sigma}_I$ is cancellative.

Recall that a semigroup $(S, +)$ is torsion free if and only if $kx = ky$ with $k \in \mathbb{N}\backslash\{0\}$ implies $x = y$.

Proposition 16. *The semigroup $I/\overline{\sigma}_I$ is cancellative and torsion free if and only if $\mathbb{N}^p/\sigma$ is cancellative and torsion free.*

P r o o f. *Necessity.* It is enough to show that $\mathbb{N}^p/\sigma$ is torsion free. Suppose that $k[x]_\sigma = k[y]_\sigma$ with $k \in \mathbb{N}\backslash\{0\}$. Then $kx\ \sigma\ ky$ and by Lemma 8, $(kx+m)\ \overline{\sigma}\ (ky+m)$. Hence $(kx+m+(k-1)m)\ \overline{\sigma}\ (ky+m+(k-1)m)$ and therefore $k(x+m)\ \overline{\sigma}\ k(y+m)$. Applying the fact that $I/\overline{\sigma}_I$ is torsion free, we deduce that $(x+m)\ \overline{\sigma}\ (y+m)$ and, by Lemma 8, $x\ \sigma\ y$. Hence, $[x]_\sigma = [y]_\sigma$.

Sufficiency. Assume that $k[m+x]_{\overline{\sigma}_I} = k[m+y]_{\overline{\sigma}_I}$ with $k \in \mathbb{N} \setminus \{0\}$. Then $k(m+x)\ \overline{\sigma}\ k(m+y)$ and by Lemma , $k(m+x)\sigma k(m+y)$. Since $\mathbb{N}^p/\sigma$ is torsion free, we deduce that $(m+x)\sigma(m+y)$ and since $\mathbb{N}^p/\sigma$ is cancellative, we have that $x\sigma y$. Using Lemma 8, we obtain that $[m+x]_{\overline{\sigma}_I} = [m+y]_{\overline{\sigma}_I}$. □

In [8] and [10] an algorithm which allows us to determine from ϱ whether a cancellative monoid $\mathbb{N}^p/\sigma$ is torsion free is given. Hence we have a method for determining from (m, ϱ) whether its associated semigroup $I/\overline{\sigma}_I$ is cancellative and torsion free.

The fact that $I/\overline{\sigma}_I$ satisfies these two properties is interesting because it was proved in [11] that if a semigroup is torsion free, cancellative and can be embedded in a finitely generated monoid (as in this case), then it is isomorphic to a subsemigroup of $(\mathbb{Z}^n, +)$ for some $n \in \mathbb{N}$.

We conclude this section by characterizing the pairs (m, ϱ) for which the semigroup $I/\overline{\sigma}_I$ is torsion free.

As a consequence of the proof of Proposition 16 we obtain the following result.

Lemma 17. *If $I/\overline{\sigma}_I$ is torsion free, then $\mathbb{N}^p/\sigma$ is torsion free.*

Unfortunately, the converse of Lemma 17 is not true. Take $(m, \varrho) = ((1,1), \{(1,1), (1,0)\})$. We show that $\mathbb{N}^p/\sigma$ is torsion free and $I/\overline{\sigma}_I$ is not torsion free.

- Assume that $k(x,y)\ \sigma\ k(\overline{x},\overline{y})$. Then $(kx, ky)\ \sigma\ (k\overline{x}, k\overline{y})$ and since $\sigma = \langle\varrho\rangle$, we deduce that $kx = k\overline{x}$. Hence $x = \overline{x}$ and by the form of ϱ, it is deduced that $(x,y)\ \sigma\ (\overline{x},\overline{y})$. This proves that $\mathbb{N}^p/\sigma$ is torsion free.
- Now, we prove that $I/\overline{\sigma}_I$ is not torsion free. Recall that $I = (1,1) + \mathbb{N}^2$ and that $\overline{\sigma}$ is the congruence on $\mathbb{N}^2$ generated by $\overline{\varrho} = \{((2,2),(2,1))\}$. Clearly $((1,2),(1,1)) \notin \overline{\sigma}$ and $(2(1,2), 2(1,1)) \in \overline{\sigma}$, which shows that $I/\overline{\sigma}_I$ is not torsion free.

For proving a converse to Lemma 17 we need to add an extra condition on m. Let $(S,+)$ be a semigroup. An element $x \in S$ is cancellable if $a + x = b + x$ implies that $a = b$.

Theorem 18. *The semigroup $I/\overline{\sigma}_I$ is torsion free if and only if $\mathbb{N}^p/\sigma$ is torsion free and $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$.*

P r o o f. *Necessity.* By Lemma 17, it suffices to prove that $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$. Assume that $[x]_\sigma + [m]_\sigma = [y]_\sigma + [m]_\sigma$. Then $(x+m)\ \sigma\ (y+m)$ and by Lemma 8 we have $(x+m+m)\ \overline{\sigma}\ (y+m+m)$. Using this we obtain $2(m+x) = (2m+2x)\ \overline{\sigma}\ (2m+y+x)\ \overline{\sigma}\ (2m+2y) = 2(m+y)$ and since $I/\overline{\sigma}_I$ is torsion free, we deduce that $(m+x)\ \overline{\sigma}\ (m+y)$. From Lemma 8 we obtain that $x\ \sigma\ y$, which means that $[x]_\sigma = [y]_\sigma$.

Sufficiency. Assume that $k[m+x]_{\overline{\sigma}_I} = k[m+y]_{\overline{\sigma}_I}$ with $k \in \mathbb{N}\setminus\{0\}$. Then $k(m+x)\ \overline{\sigma}\ k(m+y)$ and by Lemma 8 also $k(m+x)\ \sigma\ k(m+y)$. Since $\mathbb{N}^p/\sigma$ is torsion free, we have $(m+x)\ \sigma\ (m+y)$ and since $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$, we obtain that $x\ \sigma\ y$. Using Lemma 8, we have that $(m+x)\ \overline{\sigma}\ (m+y)$ and therefore $[m+x]_{\overline{\sigma}_I} = [m+y]_{\overline{\sigma}_I}$. □

Now, we explain how it can be determined from a system of generators ϱ of a congruence σ whether an element $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$.

Proposition 19. *Let γ be a congruence defined by $x\ \gamma\ y$ if and only if $(x+m)\ \sigma\ (y+m)$. Then the element $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$ if and only if $\sigma = \gamma$.*

Proof. *Necessity.* Clearly $\sigma \subseteq \gamma$. We show that $\gamma \subseteq \sigma$. Since $x\ \gamma\ y$, $(x+m)\ \sigma\ (y+m)$. Hence $[x]_\sigma + [m]_\sigma = [y]_\sigma + [m]_\sigma$. Since $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$, we deduce that $[x]_\sigma = [y]_\sigma$ and therefore $x\ \sigma\ y$.

Sufficiency. Assume that $[x]_\sigma + [m]_\sigma = [y]_\sigma + [m]_\sigma$. Then $(x+m)\ \sigma\ (y+m)$, which means that $x\ \gamma\ y$. Since $\sigma = \gamma$, we obtain that $x\ \sigma\ y$ and then $[x]_\sigma = [y]_\sigma$. $\square$

The problem of deciding if $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$ is equivalent to determining whether $\sigma = \gamma$. This problem can be solved using the Gröner basis, so we need to introduce some concepts.

If S is a monoid and $\mathbb{K}$ is a field, then we can construct the semigroup ring $\mathbb{K}[S] = \bigoplus_{s\in S} \mathbb{K}y^s$, where the addition is defined componentwise and the multiplication using the rule $y^s \cdot y^{s'} = y^{s+s'}$ (for a detailed description of $\mathbb{K}[S]$ see for instance [4]). If S is generated by $\{s_1,\dots,s_p\}$, then we know that S is isomorphic to $\mathbb{N}^p/\sigma$, where σ is the kernel congruence of the monoid homomorphism

$$\varphi\colon\ \mathbb{N}^p \longrightarrow S,\ \ \varphi(a_1,\dots,a_p) = a_1s_1 + \ldots + a_ps_p.$$

Associated to φ we define the ring homomorphism

$$\psi\colon\ \mathbb{K}[x_1,\dots x_p] \longrightarrow \mathbb{K}[S],\ \ \psi(x_i) = y^{s_i}.$$

Denote the kernel of ψ by I_σ (note that $\mathbb{K}[x_1,\dots,x_p]/I_\sigma$ is isomorphic to $\mathbb{K}[S]$). From the papers by Herzog ([5]) and Preston ([6]) we deduce that the set $\{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}$ is a system of generators of the congruence σ if and only if the set $\{X^{\alpha_1} - X^{\beta_1},\dots,X^{\alpha_t} - X^{\beta_t}\}$ is a system of generators of the ideal I_σ, where X^a denotes $x_1^{a_1}\dots x_p^{a_p}$ for $a = (a_1,\dots,a_p)$.

Now, we consider the ideal quotient $I_\sigma : X^m$. In [3] it is proved that this ideal is binomial and from this fact it is easily deduced that $I_\gamma = I_\sigma : X^m$. In [1] an algorithm for computing the Gröbner basis of the ideal $I_\sigma : X^m$ from a system of generators of the ideal I_σ is presented. Finally, in [1] an algorithm is also given for determining whether I_σ is equal to I_γ from a system of generators of I_σ and a system of generators of I_γ, which is equivalent to deciding whether $\gamma = \sigma$.

Hence, we conclude that we have an algorithmic method for determining whether $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$ or not, since $\sigma = \gamma$ if and only if $I_\sigma = I_\gamma$.

We close this section with a result which asserts that the converse of Lemma 17 is true when $I/\overline{\sigma}_I$ is a monoid.

Proposition 20. *If $I/\overline{\sigma}_I$ is a monoid, then the following statements are equivalent:*

1. *The monoid $I/\overline{\sigma}_I$ is torsion free.*
2. *The monoid $\mathbb{N}^p/\sigma$ is torsion free.*

Proof. Assume that $I/\overline{\sigma}_I$ is torsion free. By Lemma 17 we obtain that $\mathbb{N}^p/\sigma$ is torsion free.

Now, suppose that $\mathbb{N}^p/\sigma$ is torsion free. For proving that $I/\overline{\sigma}_I$ is torsion free it suffices to show that $[m]_\sigma$ is a cancellable element of $\mathbb{N}^p/\sigma$. Since $I/\overline{\sigma}_I$ is a monoid we have that $[m]_\sigma$ is a unit of $\mathbb{N}^p/\sigma$ (see Theorem 11). This concludes the proof since the units of a monoid clearly are its cancellable elements. $\square$

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