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CONTINUITY OF STOCHASTIC CONVOLUTIONS

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Dedicated to Ivo Vrkoč on the occasion of his 70th birthday

Abstract. Let B be a Brownian motion, and let $\mathcal{C}_{\rm p}$ be the space of all continuous periodic functions $f\colon \mathbb{R} \to \mathbb{R}$ with period 1. It is shown that the set of all $f \in \mathcal{C}_{\rm p}$ such that the stochastic convolution $X_{f,B}(t) = \int_0^t f(t-s)\,\mathrm{d}B(s)$, $t \in [0,1]$ does not have a modification with bounded trajectories, and consequently does not have a continuous modification, is of the second Baire category.

Keywords: stochastic convolutions, continuity of Gaussian processes, Gaussian trigonometric series

MSC 2000: 60H05, 60G15, 60G17, 60G50

1. Introduction

Let B be a one dimensional Brownian motion defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and let $\mathcal{C}_{\rm p}$ be the space of all periodic continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with period 1. We endow $\mathcal{C}_{\rm p}$ with the uniform norm. Our aim is to prove the following theorem.

Theorem 1. *The set of all $f \in \mathcal{C}_{\rm p}$ such that the stochastic convolution*

$$X_{f,B}(t) = \int_0^t f(t-s)\,\mathrm{d}B(s), \qquad t \in [0,1]$$

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does not have a modification with bounded trajectories, and consequently does not have a continuous modification, is of the second Baire category.

Although the properties of a stochastic convolution have been investigated recently in [3] and [4], the natural problem of its continuity has not been treated. Namely, [4] gives necessary and sufficient conditions on f under which the stochastic convolution $X_{f,B}$ is a semimartingale. In [3] the covariation process (or the bracket) of two processes $X_{f,B}$ and $X_{g,B}$ is calculated under the assumption of their continuity.

Motivations for the study of stochastic convolutions come from mathematical finance where the so called forward interest rate curve (see [1], [5], [8]) is a solution to the Heath-Jarrow-Morton-Musiela equation

$$
\begin{aligned}
&\mathrm{d}u(t,x) = \Big(\frac{\partial u}{\partial x}(t,x) + a(x)\Big)\,\mathrm{d}t + \sigma(x)\,\mathrm{d}B(t), \quad t \in [0,T], \ \ x \geqslant 0, \\
&u(0,x) = \psi(x), \quad x \geqslant 0.
\end{aligned} \tag{1}
$$

In (1), σ is a continuous function and $a(x) = \sigma(x)\int_0^x \sigma(y)\,\mathrm{d}y$. Then the short rate $R(t) = u(t,0)$, $t \geqslant 0$ is given by the formula

$$
R(t) = \psi(t) + \int_0^t a(t-s)\,\mathrm{d}s + \int_0^t \sigma(t-s)\,\mathrm{d}B(s), \quad t \geqslant 0.
$$

The following result is a direct consequence of Theorem 1.

Corollary 1. *The set of all volatility functions $\sigma \in \mathcal{C}_{\mathrm{p}}$ such that the short rate process $R(t)$, $t \geqslant 0$, does not have a modification with bounded trajectories, and consequently does not have a continuous modification, is of the second Baire category.*

In particular, the continuity of the volatility function σ does not guarantee that the short rate process has bounded, or continuous trajectories.

2. Proof of Theorem 1

Theorem 1 will be deduced from the following proposition.

Proposition 1. *The set of all $f \in \mathcal{C}_{\mathrm{p}}$ such that the process*

$$
Y_{f,B}(t) = \int_0^1 f(t-s)\,\mathrm{d}B(s), \quad t \in [0,1]
$$

has a modification with bounded trajectories is of the first Baire category.

Proof. Let us begin the proof with the following simple observation. The process $Y_{f,B}$ can be defined for any Borel measurable square integrable function $f\colon [0,1] \to \mathbb{R}$. Note that if $f(t) = g(t)$ for almost all $t \in [0,1]$, then $Y_{f,B}$ is a modification of $Y_{g,B}$. In particular, $Y_{f,B}$ has a modification with bounded trajectories if and only if the same holds true for $Y_{g,B}$.

Denote by $\hat{f}_k$, $k \in \mathbb{Z}$ the Fourier coefficients $\hat{f}_k = \int_0^1 \mathrm{e}^{-2\pi \mathrm{i} kx} f(x)\,\mathrm{d}x$ of a function $f \in \mathcal{C}_{\mathrm{p}}$. Then for almost all $t \in [0,1]$,

$$\begin{aligned} Y_{f,B}(t) &= \int_0^1 \Big(\sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{2\pi\mathrm{i}k(t-s)}\Big)\,\mathrm{d}B(s) \\ &= \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{2\pi\mathrm{i}kt} \int_0^1 \mathrm{e}^{-2\pi\mathrm{i}ks}\,\mathrm{d}B(s) \\ &= \hat{f}_0 B(1) + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\Big(\hat{f}_k \int_0^1 \mathrm{e}^{-2\pi\mathrm{i}ks}\,\mathrm{d}B(s)\Big)\cos 2\pi kt \\ &\quad - \sum_{k=1}^{+\infty} 2\,\mathrm{Im}\Big(\hat{f}_k \int_0^1 \mathrm{e}^{-2\pi\mathrm{i}ks}\,\mathrm{d}B(s)\Big)\sin 2\pi kt, \end{aligned}$$

with the series converging in $L^2(\Omega, \mathfrak{F}, \mathbb{P}; L^2(0,1;\mathbb{R}))$. It is therefore easy to see that there exist independent standard Gaussian random variables ξ_k and η_k, $k = 0,1,2,\ldots$, such that

$$Y_{f,B}(t) = |\hat{f}_0|\xi_0 + \sqrt{2}\sum_{k=1}^{+\infty} |\hat{f}_k|(\xi_k \cos 2\pi kt + \eta_k \sin 2\pi kt).$$

Indeed, the two sequences $(\alpha_k)_{k=1}^{\infty}$ and $(\beta_k)_{k=1}^{\infty}$ defined by

$$\alpha_k = \sqrt{2}\int_0^1 \cos(2k\pi s)\,\mathrm{d}B(s), \quad \beta_k = \sqrt{2}\int_0^1 \sin(2k\pi s)\,\mathrm{d}B(s)$$

are independent standard Gaussian random variables. Thus the random variables ξ_k and η_k, $k \in \mathbb{N}$ defined by

$$\begin{aligned} \alpha_k \,\mathrm{Re}\,\hat{f}_k + \beta_k \,\mathrm{Im}\,\hat{f}_k &= \xi_k |\hat{f}_k|, \\ -\alpha_k \,\mathrm{Im}\,\hat{f}_k + \beta_k \,\mathrm{Re}\,\hat{f}_k &= \eta_k |\hat{f}_k| \end{aligned}$$

if $|\hat{f}_k| \neq 0$ and $\xi_k = \alpha_k$, $\eta_k = \beta_k$ otherwise, have the required properties.

Therefore $Y_{f,B}$ is a real Gaussian trigonometric series, see [6, p. 197]. By [6, Theorem 1, p. 99], see also [6, p. 199], if $Y_{f,B}$ has a modification with bounded trajectories, then

$$\sum_{j=0}^{+\infty}\Big(\sum_{2^j\leqslant k<2^{j+1}}|\hat{f}_k|^2\Big)^{1/2}<\infty. \tag{2}$$

Thus the proof will be completed as soon as we show that the set of all $f\in\mathcal{C}_{\mathrm{p}}$ such that (2) holds true is of the first Baire category. To do this define $T_n(f)=(\hat{f}_1,\hat{f}_2,\dots,\hat{f}_{2^{n+1}-1},0,0,\dots)$ for $n\in\mathbb{N}$ and $f\in\mathcal{C}_{\mathrm{p}}$. Then $\{T_n\}$ are bounded linear operators from $\mathcal{C}_{\mathrm{p}}$ into the Banach space Z of all complex sequences (a_k) such that

$$\|(a_k)\|_Z=\sum_{j=0}^{+\infty}\Big(\sum_{2^j\leqslant k<2^{j+1}}|a_k|^2\Big)^{1/2}<\infty.$$

Now, let $\mathcal{X}$ be the set of all $f\in\mathcal{C}_{\mathrm{p}}$ such that the sequence $\|T_nf\|_Z$, $n=1,2,\dots$ is bounded, or equivalently, the set of all $f\in\mathcal{C}_{\mathrm{p}}$ such that (2) holds true. Thus, if $f\in\mathcal{X}$, then $\sup\limits_n\|T_nf\|_Z<\infty$. We infer from the Banach-Steinhaus theorem that, if $\mathcal{X}$ were of the second Baire category, then $\mathcal{X}=\mathcal{C}_{\mathrm{p}}$.

Hence it is enough to show that there is a function $f\in\mathcal{C}_{\mathrm{p}}$ such that (2) does not hold. The existence of such a function is a simple consequence of the de Leeuw-Kahane-Katznelson theorem (see [2] or Theorem 4, p. 64 in [6]) which says that for any sequence (a_k) satisfying $\sum|a_k|^2<\infty$ there is an $f\in\mathcal{C}_{\mathrm{p}}$ such that $|\hat{f}_k|\geqslant|a_k|$ for any k. Taking (a_k) such that $\|(a_k)\|_Z=\infty$ we obtain an $f\in\mathcal{C}_{\mathrm{p}}$ for which (2) is violated. □

Remark 1. Let $\mathcal{C}_{\mathrm{p}}(\mathbb{C})$ be the space of all periodic complex-valued continuous functions with period 1, and let $\mathcal{A}$ be the algebra of all f from $\mathcal{C}_{\mathrm{p}}(\mathbb{C})$ whose Fourier series converges absolutely. Let $\{\zeta_k\}$ be a sequence of standard independent complex-valued Gaussian random variables, and let $\mathcal{P}$ be the class of all functions $f\in\mathcal{C}_{\mathrm{p}}(\mathbb{C})$ such that the random series $\sum\limits_k\hat{f}_k\mathrm{e}^{2\pi\mathrm{i}kt}\zeta_k$ represents a continuous function. Pisier [9] proved that $\mathcal{P}$ is a homogeneous Banach algebra strictly contained in $\mathcal{C}_{\mathrm{p}}(\mathbb{C})$ and strictly containing $\mathcal{A}$. One can show that $\mathcal{P}$ is equal to the set of all $f\in\mathcal{C}_{\mathrm{p}}(\mathbb{C})$ such that the process $Y_{f,B}$ has a continuous modification.

Given $f\in\mathcal{C}_{\mathrm{p}}$ we set $f_{(\mathrm{s})}(t)=f(-t)$, $t\in\mathbb{R}$. Note that the process $B_{(\mathrm{R})}(t)=B(1)-B(1-t)$, $t\in[0,1]$ is a Brownian motion.

Proposition 2. *For any $f\in\mathcal{C}_{\mathrm{p}}$ one has*

$$Y_{f,B}(t)=X_{f,B}(t)+X_{f_{(\mathrm{s})},B_{(\mathrm{R})}}(1-t),\quad t\in[0,1].$$

P r o o f. We need to show that

$$\int_t^1 f(t-s)\,\mathrm{d}B(s) = \int_0^{1-t} f_{(\mathrm{s})}(1-t-s)\,\mathrm{d}B_{(\mathrm{R})}(s). \tag{3}$$

Clearly it is enough to show that (3) holds true for any continuously differentiable $f \in \mathcal{C}_\mathrm{p}$. To do this note that because of the 1-periodicity of f the right hand side of (3) is equal to

$$\begin{aligned}\int_0^{1-t} f(t+s-1)\,\mathrm{d}B_{(\mathrm{R})}(s) &= \int_0^{1-t} f(t+s)\,\mathrm{d}B_{(\mathrm{R})}(s)\\ &= f(1)B_{(\mathrm{R})}(1-t) - \int_0^{1-t} f'(t+s)B_{(\mathrm{R})}(s)\,\mathrm{d}s\\ &= f(t)B(1) - f(1)B(t) + \int_0^{1-t} f'(t+s)B(1-s)\,\mathrm{d}s\\ &= f(t)B(1) - f(1)B(t) + \int_t^1 f'(t-s+1)B(s)\,\mathrm{d}s\\ &= \int_t^1 f(t-s+1)\,\mathrm{d}B(s) = \int_t^1 f(t-s)\,\mathrm{d}B(s),\end{aligned}$$

which is the desired equality. □

P r o o f of Theorem 1. Denote by S (resp. $S_{(\mathrm{s})}$) the set of all $f \in \mathcal{C}_\mathrm{p}$ such that the process $X_{f,B}$ (resp. $X_{f_{(\mathrm{s})},B_{(\mathrm{R})}}$) does not have a modification with bounded trajectories. Similarly, denote by A the set of all $f \in \mathcal{C}_\mathrm{p}$ such that the process $Y_{f,B}$ does not have a modification with bounded trajectories. Our aim is to show that S is of the second Baire category. Suppose by contradiction that S is of the first Baire category. Then, since $f \to f_{(\mathrm{s})}$ is a homeomorphism of $\mathcal{C}_\mathrm{p}$, it follows that also $S_{(\mathrm{s})}$ is of the first Baire category. On the other hand, it follows from Proposition 2 that $A \subset S \cup S_{(\mathrm{s})}$, and hence A is of the first category as a subset of the union of two sets of the first category. This is in an obvious contradiction with Proposition 1. □

Remark 2. From the proofs of Propositions 1 and 2 one can easily deduce that for any function $f \in \mathcal{C}_\mathrm{p}$ satisfying $f = f_{(\mathrm{s})}$ and violating (2) the stochastic convolution $X_{f,B}$ does not have a modification with a bounded trajectories on $[0,1]$.

Remark 3. From the proof of Proposition 1 one can see that the set of all $f \in \mathcal{C}_\mathrm{p}$ such that $Y_{f,B}$ has a continuous modification is of the first Baire category. However, it is still an open problem to show that the set of all $f \in \mathcal{C}_\mathrm{p}$ such that $X_{f,B}$ has a continuous modification is of the first Baire category.

Remark 4. The main result of the paper was proved in August 1999. Some time later the authors learnt about a recent paper by Kahane [7] in which the Baire category methods were applied in a related but different context.

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