Jan Kučera

Sequential completeness of LF-spaces

Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 1, 181–183

Persistent URL: http://dml.cz/dmlcz/127637

Czechoslovak Mathematical Journal, 51 (126) (2001), 181–183

SEQUENTIAL COMPLETENESS OF LF-SPACES

Jan Kučera, Pullman

(Received February 14, 1998)

Abstract. Any LF-space is sequentially complete iff it is regular.

Keywords: LB- and LF-space, regularity and sequential completeness of locally convex inductive limits

MSC 2000: primary 46A13, secondary 46A30

Throughout the paper $E_1 \subset E_2 \subset \ldots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $\mathrm{id}\colon E_n \to E_{n+1}, n \in N$. Their locally convex inductive limit is denoted by $\operatorname{ind} E_n$ or for brevity, just E. If all spaces E_n are Banach, resp. Fréchet, then we call E an LB-, resp. LF-space. We use the following notation: given a set $M \subset E$, then $\operatorname{co} M$, resp. $\mathrm{cl}_E\, M$ is its convex hull, resp. closure in the topology of E.

According to [3] or [1, § 5.2], the space $E = \operatorname{ind} E_n$ is called regular if every set bounded in E is also bounded in some constituent space E_n. By Makarov's Theorem, see [1, § 5.6], every quasi-complete LF-space is regular. It is natural to ask whether the reverse statement is true, at least for LB-spaces. By Raikov's Theorem, see [1, § 4.3], every LB-space is quasi-complete iff it is complete. So in [4] Mujica asks: Is every regular LB-space complete? The answer is negative as shown in [2] with an example of an incomplete regular LB-space. In this paper we slightly generalize Makarov's Theorem and receive an equivalence: An LF-space is regular iff it is sequentially complete.

Proposition 1. *Every sequentially complete LF-space is regular.*

P r o o f. Let B be a bounded set in an LF-space $E = \operatorname{ind} E_n$. Let A be the closure in E of the convex, balanced hull of B, and $F = \bigcup\{nA;\ n \in N\}$. We equip

F with the norm topology generated by the Minkowski functional of A and show that F is complete.

The set A is bounded in E. Hence for any 0-nbhd V in E there exists $\alpha > 0$ such that $A \subset \alpha V$. Thus the identity map id: $F \to E$ is continuous.

Let $\{x_n\}$ be a Cauchy sequence in F. Due to continuity of id: $F \to E$, it is also Cauchy in E and as such it converges to some $x_0 \in E$. The set $S = \{x_n;\ n \in N\}$ is bounded in F. Hence $S \subset \beta A$ for some $\beta > 0$. Since the set βA is closed in E, we have $x_0 \in \beta A \subset F$.

For any closed 0-nbhd λA, $\lambda > 0$ in F, there exists $k \in N$ such that $x_n - x_m \in \lambda A$ for m, $n \geqslant k$. If we let $m \to \infty$, we get $x_n - x_0 \in \lambda A$ for $n \geqslant k$, which implies $x_n \to x_0$ in F.

Now F is a Banach space and id: $F \to \operatorname{ind} E_n$ is continuous. Hence the graph of id: $F \to E$ in $F \times E$ is closed. By [5; cor. iv. 6.5] there exists $n \in N$ such that id: $E \to E_n$ is continuous. This implies that A, hence also B, is bounded in E_n, i.e., E is regular. □

Proposition 2. *Every regular LF-space is sequentially complete.*

Proof. Let $E = \operatorname{ind} E_n$ be a regular LF-space and $\{x_n;\ n \in N\}$ a Cauchy sequence in E. Put $B_n = \mathrm{cl}_E \operatorname{co}\{x_m;\ m > n\}$; $n = 0, 1, 2, \ldots$ Then B_0 is bounded in E and, by the regularity of E, it is bounded in some constituent space E_n. Without a loss of generality, we may assume $n = 1$.

The space E_1 is Fréchet, hence the canonical imbedding $E_1 \to E_1''$, where E_1'' is the second dual of E_1, equipped with its strong topology, is a topological isomorphism into E_1''. Since E_1 is complete, it is closed in E_1'' and each $f \in E_1'$ can be continuously extended to E_1''. Also, the set B_0 is closed and convex in E_1'', hence it is weakly closed in E_1''. Since each $f \in E_1'$ has an continuous extension in E_1'', the set B_0 is $\sigma(E_1'', E_1')$-closed in E_1''.

Further, the set B_0, bounded in E_1'', is equicontinuous on E_1'. Hence, by Alaoglu Theorem, it is relatively $\sigma(E_1'', E_1')$-compact. This, together with the $\sigma(E_1'', E_1')$-closedness implies that B_0 is $\sigma(E_1'', E_1')$-compact in E_1''.

Similarly, all sets B_n, $n \in N$, are $\sigma(E_1'', E_1')$-compact. Any finite intersection $\bigcap\{B_n;\ 0 \leqslant n \leqslant m\} = B_m$, $m \in N$, is non-empty, hence there exists $x_0 \in \bigcap\{B_n;\ n \geqslant 0\} \subset E_1$. This implies the existence of an upper-triangular matrix $\Lambda = (\lambda_{nm})$ with all $\lambda_{nm} \geqslant 0$, only finite number of non-zero entries in each row, and the sum of all entries in each row equal to 1, such that the sequence $\left\{y_n = \sum_{m=n}^{\infty} \lambda_{nm} x_m;\ n \in N\right\}$ converges to x_0 in the topology of E_1.

Evidently $y_n \to x_0$ also in the topology of E. Given a balanced, convex, 0-nbhd V in E, there exist p, $q \in N$ such that $y_n - x_0 \in V$ for $n \geqslant p$ and $x_m - x_n \in V$ for

$m \geqslant n \geqslant q$. Then for $n \geqslant \max(p,q)$, we have $x_0 - x_n = (x_0 - y_n) + (y_n - x_n) = (x_0 - y_n) + \sum\limits_{m=n}^{\infty} \lambda_{nm}(x_m - x_n) \in V + V$ and $x_n \to x_0$ in E. □

Remark. We have proved a little more: If a Cauchy sequence in E is bounded in a Fréchet space E_n, then it converges to an element in E_n in the topology of E, but not necessarily in the topology of E_n.

If we combine the two Propositions, we get:

Theorem. *Any LF-space is sequentially complete iff it is regular.*

References

[1] *K. Floret*: Lokalkonvexe Sequenzen mit kompakten Abbildungen. J. Reine Angew. Math. *247* (1971), 155–195.

[2] *J. Kučera and K. McKennon*: Quasi-incomplete regular LB-space. Internat. J. Math. *16* (1993), 675–678.

[3] *B. M. Makarov*: Pathological properties of inductive limits of Banach spaces. Uspekhi Mat. Nauk *18* (1963), 171–178.

[4] *J. Mujica*: Functional analysis, holomorphy and approximation theory II. North Holland, 1984.

[5] *M. de Wilde*: Closed graph theorems and webbed spaces. Pitman, London, 1978.

Author's address: Department of Mathematics, Washington State University, Pullman, Washington 99164-3113.