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ON SOME COMMUTATIVITY THEOREMS FOR FINITE RINGS AND FINITE GROUPS

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In this note we investigate the relation between nilpotent elements in a ring and commutative conditions in both finite rings and finite groups.

Theorem 1. *Let R be a finite ring without nilpotent elements. Then R is commutative.*

Proof. Since R is finite and has no nilpotent element then for any $a \in R$ we have

$$a^{n(a)} = a^{m(a)} \tag{1}$$

where n and m are integers which depend on a. First, we will restrict ourselves to the case in which R is a division ring. In fact our condition (1) then implies that

$$a^{n(a)-m(a)+1} = a,$$

that is

$$a^{k(a)} = a$$

and this condition for our division ring clearly leads to commutativity by Jacobson's Commutativity Theorem.

Since our ring has no nilpotent element, Jacobson's radical is $\{0\}$ and the ring is semi-simple. Hence we can apply Wedderburn-Artin decomposition theorem for semi-simple artinian rings. Now suppose that we have a matrix ring $n \times n$ in this decomposition; the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

is clearly nilpotent. Therefore no matrix rings over division rings can occur in this decomposition; that is, all components of R are division rings, which by the first part of our proof are commutative. Hence R is commutative. □

Theorem 2. *Suppose R is a finite ring of characteristic p (a prime). The R is commutative if, and only if, the group of invertible elements of R is commutative.*

P r o o f. The theorem is an immediate consequence of the following one. □

De Morais Basis Theorem. *Let A be a finite dimensional algebra with unit over a field F. Then A has a base of invertible elements.*

Now we pass to a different category of results, namely to theorems about commutativity of groups. These theorems, however, are based on ideas which depend essentially on the assumptions and statements of the previous part of our paper.

Theorem 3. *Let G be a finite group of order n. Then G is non-commutative if, and only if, for any finite field F such that the characteristic of F does not divide the order of G, there exist m and a family $\{k_i\colon\ k_i \in F\}$ not all zero such that if $a = \sum k_i g_i$, then $a^m = 0$.*

P r o o f. In order to prove sufficiency and necessity of our condition we will consider the group algebra $F(G)$. Under our hypothesis $F(G)$ is semi-simple, hence if $F(G)$ has nilpotent elements then in the decomposition of $F(G)$ by Wedderburn's Structure Theorem some matrix algebra will occur and $F(G)$ is non-commutative and the same holds for G. Necessity follows easily by proceeding in the opposite direction. Indeed, if G and consequently $F(G)$ are non-commutative this implies that some matrix algebra occurs in the decomposition of $F(G)$ and hence we have nilpotent elements in $F(G)$ as we have show in the proof of Theorem 1. □

Theorem 4. *Let G be a finite group of order n. A necessary and sufficient condition for G to be commutative is that given any field F with $\operatorname{char} F \times n$, for any a with $a = \sum k_i g_i$, $k_i \in F$ we have*

$$a^{m(a)} = a^{l(a)} \tag{1}$$

with $a^{m(a)} \neq 0$, where m and 1 are integers which depend on a.

P r o o f. From (1) supposing $m > 1$ we have

$$a^l(1 - a^{m-l}) = 0. \tag{2}$$

Now if a is nilpotent then $1 - a^{m-l}$ is invertible and equation (2) is absurd. Hence $F(G)$ has no nilpotent element and the commutativity of both $F(G)$ and G follows from Theorem 1.

To prove necessity, we remark that $F(G)$ is semi-simple and the commutativity of G implies the commutativity of $F(G)$, therefore in the decomposition of $F(G)$ no matrix algebra and hence no nilpotent element occurs. Now $F(G)$ is finite and thus for all $a = \sum k_i g_i$, $k_i \in F$, we have

$$a^{m(a)} = a^{l(a)}.$$

□

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