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DIRECT LIMITS OF MONOUNARY ALGEBRAS

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1. Introduction

The direct limit construction is a well-known method for building up algebras from families of algebras, e.g. [2], §21.

In this paper we investigate direct limits of monounary algebras.

Several examples of direct limit classes of monounary algebras will be given. We will describe all monounary algebras A which satisfy the following condition:

(C) If an algebra B can be obtained as a direct limit of algebras which are isomorphic to A, then B is isomorphic to A.

Further, we will show that every direct limit class of monounary algebras contains at least one algebra A which satisfies the condition (C).

2. Preliminaries

As usual, by a monounary algebra we understand an algebra with a single unary operation; cf. e.g. [8], [9]. The notion of homomorphism is essentially applied in the construction of direct limits. Homomorphisms and endomorphisms of monounary algebras were thoroughly studied in [4], [7]–[9].

The class of all monounary algebras will be denoted by $\mathscr{U}$. We will use the symbol f for the operation in algebras of $\mathscr{U}$.

Let A be a monounary algebra.

The algebra A is said to be connected, if for each $x, y \in A$ there are positive integers m, n with $f^m(x) = f^n(y)$. A maximal connected subalgebra of A is said to be a component of A.

The class of all connected monounary algebras will be denoted by the symbol $\mathscr{U}^c$.

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Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let us denote by C_0 the connected monounary algebra which has $\aleph_0$ elements and a bijective operation. If $k \in \mathscr{N}$, then the connected monounary algebra which has k elements and f is a bijective operation will be denoted by C_k. We will say that an algebra A is a cycle of length k, if A is isomorphic to C_k. The class of all connected monounary algebras having a cycle of length k as its subalgebra will be denoted by $\mathscr{U}_k^c$. For the notation of the class of all connected monounary algebras without a cycle we will use the symbol $\mathscr{U}_0^c$, more precisely we put

$$\mathscr{U}_0^c = \mathscr{U}^c - \bigcup_{k \in \mathbb{N}} \mathscr{U}_k^c.$$

Let $A \in \mathscr{U}$. We will say that A has a cycle, if there exists $k \in \mathbb{N}$ such that a cycle of length k is a subalgebra of A.

These definitions immediately imply the following three lemmas:

Lemma 1. *Let $k \in \mathbb{N}$. If $x \in C_k$, then $f^k(x) = x$ and $|\{x, f(x), \ldots, f^{k-1}(x)\}| = k$.*

Lemma 2. *Let $A \in \mathscr{U}$ and $i, j \in \mathbb{N}$. Let $u \in A$. If $f^i(u) = u$ and $v = f^j(u)$, then $f^i(v) = v$.*

Lemma 3. *Let $A \in \mathscr{U}$. If there exist $n \in \mathbb{N}$ and $x \in A$ such that $f^n(x) = x$, then A has a cycle.*

Lemma 4. *Let A, B be monounary algebras, φ a homomorphism from A into B and $k \in \mathbb{N}$. If A has a cycle C of length k, then there exists $l \in \mathbb{N}$ such that $\varphi(C)$ is a cycle of length l and l divides k.*

Proof. Let $x \in C$. Then $f^k(\varphi(x)) = \varphi(f^k(x)) = \varphi(x)$. Therefore there exists $l \in \mathbb{N}$ such that l divides k and $\{\varphi(x), f(\varphi(x)), \ldots, f^{k-1}(\varphi(x))\}$ is a cycle of length l. Further, $\varphi(C) = \{\varphi(x), f(\varphi(x)), \ldots, f^{k-1}(\varphi(x))\}$. □

We recall the notion of the direct limit: in fact, we apply it to the case of monounary algebras.

Let $\langle P, \leqslant \rangle$ be a directed partially ordered set, $P \neq \emptyset$. For each $p \in P$ let A_p be a monounary algebra and assume that if $p, q \in P$, $p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$, a homomorphism φ_{pq} of A_p into A_q is defined such that $p < q < s$ implies that

$$\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}.$$

For each $p \in P$ let φ_{pp} be the identity on A_p. Then $\{A_p\}_{p\in P}$ is said to be a direct family of monounary algebras.

Let p and q be elements of P and let $x \in A_p$, $y \in A_q$. We put $x \equiv y$ if there exists $s \in P$ with $p \leqslant s$, $q \leqslant s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. For each $z \in \bigcup\limits_{p\in P} A_p$ put $\overline{z} = \{t \in \bigcup\limits_{p\in P} A_p\colon\ z \equiv t\}$. Denote $\overline{A} = \{\overline{z}\colon\ z \in \bigcup\limits_{p\in P} A_p\}$.

If z_1, z_2 are elements of $\bigcup\limits_{p\in P} A_p$ such that $\overline{z}_1 = \overline{z}_2$, then clearly $\overline{f(z_1)} = \overline{f(z_2)}$. Hence if we put $f(\overline{z}_1) = \overline{f(z_1)}$, then the operation f on $\overline{A}$ is correctly defined and with respect to this operation $\overline{A}$ is a monounary algebra. It is said to be the direct limit of the direct family $\{A_p\}_{p\in P}$. We express this situation by writing

$$\{A_p\}_{p\in P} \longrightarrow \overline{A}. \tag{1}$$

The definition of the direct limit yields the following four assertions.

Lemma 5. *Let* (1) *hold. Let $p \in P$ and φ_p be the mapping of A_p into $\overline{A}$ such that $\varphi_p(x) = \overline{x}$ for every $x \in A_p$. Then φ_p is a homomorphism of A_p into $\overline{A}$.*

Lemma 6. *Let $m \in \mathbb{N}$ and let* (1) *be valid. If $|A_p| \leqslant m$ for every $p \in P$, then $|\overline{A}| \leqslant m$.*

Lemma 7. *Let* (1) *be valid. If the operation of A_p is injective for every $p \in P$, then the operation of $\overline{A}$ is injective.*

Lemma 8. *Let* (1) *be valid and let $p \in P$. If $q \leqslant p$ for all $q \in P$, then $\overline{A} \cong A_p$.*

Lemma 9. *Let A be an algebra and let* (1) *be valid. If $A_p \cong A$ for all $p \in P$ and φ_{pq} is an isomorphism between A_p and A_q for all $p, q \in P$, $p \leqslant q$, then $\overline{A} \cong A$.*

It is obvious that Lemmas 5, 6, 8, 9 are not specific for monounary algebras, they are valid for direct limits of arbitrary type of algebraic systems.

Example. Suppose that P is the set of all finite subsets of the interval $(0,1)$. Let $\leqslant\ =\ \subseteq$. For $p \in P$, $p = \{p_1, \ldots, p_n\}$, where $n \in \mathbb{N}$, put $A_p = \{(0,p), (p_1,p), \ldots, (p_n,p)\}$. Further, put $f((0,p)) = f((p_i,p)) = (0,p)$ for $i = 1, \ldots, n$. If $p \subseteq q$, then let $\varphi_{pq}((z,p)) = (z,q)$ for every $z \in \{0, p_1, \ldots, p_n\}$. The family $\{A_p\}_{p\in P}$ is direct and its direct limit is isomorphic to the algebra $(\langle 0,1), f)$, where $f(x) = 0$ for each $x \in \langle 0,1)$.

This example shows that Lemma 6 cannot be generalized to the case when an infinite cardinal number will be put instead of m.

Lemma 10. *Let* (1) *be valid. The direct family* $\{A_p\}_{p\in P}$ *contains an algebra with a cycle if and only if* $\overline{A}$ *has a cycle. More precisely,* $\overline{A}$ *contains a cycle of length* k*, where* k *is the length of the shortest cycle in algebras of* $\{A_p\}_{p\in P}$.

P r o o f. Assume that $\{A_p\}_{p\in P}$ contains an algebra with a cycle. Let l be the length of the shortest cycle in algebras of $\{A_p\}_{p\in P}$. Every algebra of $\{A_p\}_{p\in P}$ can be homomorphically embedded into $\overline{A}$ according to Lemma 5. This implies that $\overline{A}$ is an algebra with a cycle. Moreover, $\overline{A}$ has at least one cycle with length less or equal to l.

Now let $\overline{A}$ have a cycle of length n, $n\in\mathbb{N}$. Assume that $p\in P$ and $x\in A_p$ are such that $f^n(\overline{x})=\overline{x}$. We have $f^n(x)\in\overline{x}$ because $f^n(\overline{x})=\overline{f^n(x)}$. This means that there exists $q\in P$ such that $\varphi_{pq}(x)=\varphi_{pq}(f^n(x))$. We obtain $\varphi_{pq}(x)=f^n(\varphi_{pq}(x))$. Thus A_q has a cycle with length less or equal to n. □

Lemma 11. *Suppose that* (1) *is valid. Let* $\overline{A}$ *have no cycle and let the direct family* $\{A_p\}_{p\in P}$ *contain an algebra with a subalgebra isomorphic to* C_0*. Then* $\overline{A}$ *has a subalgebra isomorphic to* C_0.

P r o o f. Let $p\in P$ be such that C is a subalgebra of A_p isomorphic to C_0. Consider the homomorphism φ_p from Lemma 5. Then $\varphi_p(C)$ is a homomorphic image of C and $\varphi_p(C)$ is a subalgebra of $\overline{A}$. The algebra $\overline{A}$ has no cycle by the assumption and thus $\varphi_p(C)\cong C_0$. □

3. Direct limit classes

The operator $\underrightarrow{\mathbf{L}}$ on classes of algebras was introduced in the textbook [2], §23. By this definition, if $\mathscr{K}$ is a class of algebras, then $\underrightarrow{\mathbf{L}}(\mathscr{K})$ is the class of all direct limits of algebras of $\mathscr{K}$.

Let $\mathscr{K}$ be a class of algebras. We denote by $[\mathscr{K}]$ the class of all isomorphic copies of algebras of $\mathscr{K}$. Further, we denote by $\underrightarrow{\mathbf{L}}'(\mathscr{K})$ the class of all isomorphic copies of direct limits of algebras of $\mathscr{K}$, i.e., $\underrightarrow{\mathbf{L}}'(\mathscr{K})=[\underrightarrow{\mathbf{L}}(\mathscr{K})]$.

We will use $\underrightarrow{\mathbf{L}}'\mathscr{K}$ instead of $\underrightarrow{\mathbf{L}}'(\mathscr{K})$. For an algebra A we will use $[A]$ instead of $[\{A\}]$.

Lemma 12. *Let* $\mathscr{K}$ *be a class of algebras. Then* $\mathscr{K}\subseteq\underrightarrow{\mathbf{L}}'[\mathscr{K}]$.

P r o o f. It follows from Lemma 9. □

Lemma 13. *Let $\mathscr{K}_1$, $\mathscr{K}_2$ be classes of algebras. If $\mathscr{K}_1 \subseteq \mathscr{K}_2$, then $\underrightarrow{\mathbf{L}}'[\mathscr{K}_1] \subseteq \underrightarrow{\mathbf{L}}'[\mathscr{K}_2]$.*

P r o o f. Let $A \in \underrightarrow{\mathbf{L}}'[\mathscr{K}_1]$. Then there exists a direct family $\{A_p\}_{p\in P}$ such that $A_p \in [\mathscr{K}_1]$ for every $p \in P$ and $\{A_p\}_{p\in P} \longrightarrow A$. Since $A_p \in [\mathscr{K}_2]$ for every $p \in P$, we have $A \in \underrightarrow{\mathbf{L}}'[\mathscr{K}_2]$. □

Definition. Let $\mathscr{K}$ be a class of algebras. If $\underrightarrow{\mathbf{L}}'[\mathscr{K}] = [\mathscr{K}]$ is satisfied, then we will say that $\mathscr{K}$ is a direct limit class.

The next lemma we will often use without any notice.

Lemma 14. *A class $\mathscr{K}$ is a direct limit class if and only if the following condition is valid:*

whenever (1) *holds and $A_p \in [\mathscr{K}]$ for each $p \in P$, then $\overline{A} \in [\mathscr{K}]$.*

P r o o f. It follows from definitions and Lemma 12. □

Lemma 15. a) *Let J be a nonempty set and for $j \in J$ let $\mathscr{K}_j$ be a direct limit class. Then $\bigcap\limits_{j\in J} \mathscr{K}_j$ is a direct limit class.*

b) *If $\mathscr{K}_1$ and $\mathscr{K}_2$ are direct limit classes, then $\mathscr{K}_1 \cup \mathscr{K}_2$ is a direct limit class.*

P r o o f. The assertion a) follows from definitions.

b) Suppose that (1) is valid and $A_p \in [\mathscr{K}_1 \cup \mathscr{K}_2]$ for all $p \in P$. Denote $Q = \{q \in P\colon\ A_q \in [\mathscr{K}_2]\}$.

Let there exist $p \in P$ such that for every $q \in Q$ the relation $p \nleqslant q$ holds. Put $R = \{r \in P\colon\ p \leqslant r\}$. The set R is directed. Further, if $r \in R$, then $A_r \in [\mathscr{K}_1]$. If $s \in P$, then we can choose $s' \in P$ such that $s \leqslant s'$, $p \leqslant s'$. We have $s' \in R$. This means that R is cofinal with P. Thus $\{A_r\}_{r\in R} \longrightarrow \overline{A}$ and $\overline{A} \in [\mathscr{K}_1]$.

Now for every $p \in P$ let there exists $q \in Q$ such that $p \leqslant q$. Then Q is cofinal with P and $\{A_q\}_{q\in Q} \longrightarrow \overline{A}$. Since $\mathscr{K}_2$ is a direct limit class, we obtain $\overline{A} \in [\mathscr{K}_2]$. □

Direct limit classes of cyclically ordered groups have been dealt with by J. Jakubík and G. Pringerová, [3].

Example. Let O_ω be a monounary algebra such that $\mathbb{N}_0$ is the underlying set of O_ω and $f(x) = 0$ for all $x \in \mathbb{N}_0$. Let $k \in \mathbb{N}$. Let $O_k = \{0, 1, \ldots, k\}$ and $f(x) = 0$ for all $x \in \{0, 1, \ldots, k\}$. Put $\mathscr{K}_k = \{C_1, O_1, \ldots, O_k\}$.

Assume that (1) is valid and $A_p \in [\mathscr{K}_k]$ for all $p \in P$. Let $o_p = f(x)$ for every $p \in P$ and $x \in A_p$. We have $|\overline{A}| \leqslant k + 1$ according to Lemma 6. Suppose that $p, q \in P$. Then there is $s \in P$ such that $p, q \leqslant s$. Since $\varphi_{ps}(o_p) = o_s = \varphi_{qs}(o_q)$, we obtain $\overline{o_p} = \overline{o_q}$. Further, $f(\overline{x}) = \overline{f(x)} = \overline{o_p} = \overline{o_q} = \overline{f(y)} = f(\overline{y})$ for every $x \in A_p$ and $y \in A_q$. We conclude $\overline{A} \in [\mathscr{K}_k]$ and $\mathscr{K}_k$ is a direct limit class.

Consider $\mathscr{K} = \bigcup_{k\in\mathbb{N}} \mathscr{K}_k$. Then $\mathscr{K} = \{C_1\} \cup \{O_i, i \in \mathbb{N}\}$. For every $i \in \mathbb{N}$ let E_i be the trivial monounary algebra on the set $\{i\}$ and let $A_i = O_i \times E_i$. Let $\varphi_{i,i+1}$ be an embedding of A_i into A_{i+1}. Then $\{A_i\}_{i\in\mathbb{N}}$ is a direct family which has the direct limit isomorphic to O_ω. Since $O_\omega \notin [\mathscr{K}]$, we have $\mathscr{K}$ is not a direct limit class.

This example shows that the union of direct limit classes need not be a direct limit class. The following lemma and Proposition 4 give some sufficient conditions which yield that the union of direct limit classes is a direct limit class.

Lemma 16. *Let $\mathscr{K}_k \subseteq \mathscr{U}_k^c$ be a direct limit class for all $k \in \mathbb{N}_0$. Then $\bigcup_{k\in\mathbb{N}_0} \mathscr{K}_k$ is a direct limit class.*

P r o o f. Let $\mathscr{K} = \bigcup_{k\in\mathbb{N}_0} \mathscr{K}_k$. Suppose that $\mathscr{K} \neq \emptyset$. Let (1) be valid, $A_p \in [\mathscr{K}]$ for all $p \in P$.

Assume that $\{A_p\}_{p\in P}$ contains an algebra with a cycle. Let i be the length of the shortest cycle in the algebras of $\{A_p\}_{p\in P}$. Put $Q = \{q \in P\colon A_q \in \mathscr{U}_i^c\}$. A homomorphic image of a cycle of length i can be only a cycle of length less or equal than i, in our case thus a cycle of length i. If $q \in Q$ and $p \in P$ are such that $q \leqslant p$, then $p \in Q$. Thus Q is directed and cofinal with P. We have $\{A_q\}_{q\in Q} \longrightarrow \overline{A}$ and $\overline{A} \in \mathscr{U}_i^c$ by Lemma 10. Because $\mathscr{K}_i$ is a direct limit class, we have $\overline{A} \in [\mathscr{K}_i]$ and $\overline{A} \in [\mathscr{K}]$.

Now assume that $\{A_p\}_{p\in P}$ contains no algebra with a cycle. Then $A_p \in [\mathscr{K}_0]$ for every $p \in P$. Since $\mathscr{K}_0$ is a direct limit class, we obtain $\overline{A} \in [\mathscr{K}_0] \subseteq [\mathscr{K}]$. □

Proposition 1. *The classes $\mathscr{U}$, $\mathscr{U}^c$, $\mathscr{U}_k^c$ and $\{C_k\}$ are direct limit classes for every $k \in \mathbb{N}_0$.*

P r o o f. It is obvious that $\mathscr{U}$ is a direct limit class.

Let (1) be valid.

a) We will prove that $\mathscr{U}^c$ is a direct limit class. Suppose that $A_p \in \mathscr{U}^c$ for all $p \in P$. Let $\overline{x}, \overline{y} \in \overline{A}$. There exist $p \in P$ and $x_1 \in \overline{x}$, $y_1 \in \overline{y}$ such that $x_1, y_1 \in A_p$. We can find $m, n \in \mathbb{N}_0$ such that $f^m(x_1) = f^n(y_1)$ by the connectivity of A_p. This means that $f^m(\overline{x}) = \overline{f^m(x_1)} = \overline{f^n(y_1)} = f^n(\overline{y})$. We obtain $\overline{A} \in \mathscr{U}^c$ and $\mathscr{U}^c$ is a direct limit class by Lemma 14 .

b) Suppose that $k \neq 0$. Let $A_p \in \mathscr{U}_k^c$ for all $p \in P$. We have $\overline{A} \in \mathscr{U}^c$ according to a) and $\overline{A}$ has a cycle of length k by Lemma 10. So, $\mathscr{U}_k^c$ is a direct limit class.

Now assume that $A_p \in \mathscr{U}_0^c$ for all $p \in P$. We have $\overline{A} \in \mathscr{U}_0^c$ according to a) and Lemma 10. Therefore $\mathscr{U}_0^c$ is a direct limit class.

c) Let $k \neq 0$ and let $A_p \cong C_k$ for all $p \in P$. The algebra $\overline{A}$ is connected by a) and $\overline{A}$ contains a cycle of length k according to Lemma 10. The operation f of $\overline{A}$ is injective according to Lemma 7. We have $\overline{A} \cong C_k$. Conclude $\{C_k\}$ is a direct limit class.

Let $A_p \cong C_0$ for all $p \in P$. The algebra $\overline{A}$ is connected by a) , $\overline{A}$ has no cycle by Lemma 10 and $\overline{A}$ possesses a subalgebra isomorphic to C_0 by Lemma 11 . In view of Lemma 7 we have $\overline{A} \cong C_0$. □

Let A be a monounary algebra. Let A satisfy the following condition: If $C \subseteq A$ and C is a cycle of A, then $C \cong C_1$. Then A is called a cycle-free algebra. Cycle-free algebras have been dealt with by G. Bordalo [1].

Proposition 2. *The class of all cycle-free monounary algebras is a direct limit class.*

Proof. It follows from Lemma 10. □

4. Algebras of type τ

Let A be a monounary algebra and let $\{A_j\}_{j\in J}$ be a component partition of A.

We will say that A is of type τ if the following two conditions are valid:

1. If $j \in J$, then there exists $k \in \mathbb{N}$ such that $A_j \cong C_k$;
2. if $i, j \in J$, $i \neq j$ and $k, l \in \mathbb{N}$ are such that $A_i \cong C_k$, $A_j \cong C_l$, then k does not divide l.

Denote by $\mathscr{T}$ the class of all algebras of type τ.

We will prove that $\mathscr{T}$ is a direct limit class, and some special subclasses of $\mathscr{T}$ are direct limit classes.

The definition of algebras of type τ yields that $C_k \in \mathscr{T}$ for every $k \in \mathbb{N}$. Further, if A is of type τ and C_1 is a subalgebra of A, then $A \cong C_1$. Further, if $A \in \mathscr{T}$ and B is a subalgebra of A, then $B \in \mathscr{T}$.

Lemma 17. *If $A \in \mathscr{T}$, then the set $\{A\}$ is a direct limit class.*

Proof. Suppose that (1) is valid and $A_p \cong A$ for each $p \in P$. Let $p, q \in P$. The algebra A is of type τ and thus φ_{pq} is an isomorphism between A_p and A_q in view of Lemma 4. This implies $\overline{A} \cong A$ according to Lemma 9. □

Lemma 18. *Let* (1) *be valid and let $k \in \mathbb{N}$. If $\overline{A}$ contains a cycle of length k, then there exists $p \in P$ such that A_q contains a cycle of length k for each $q \in P$ with $p \leqslant q$.*

P r o o f. We prove this assertion indirectly. Suppose that for each $p \in P$ there exists $q \in P$, $p \leqslant q$ such that A_q does not contain a cycle of length k.

We will show that for every $\overline{x} \in \overline{A}$ either $f^k(\overline{x}) \neq \overline{x}$ or

$$|\{\overline{x}, f(\overline{x}), \ldots, f^{k-1}(\overline{x})\}| < k.$$

Then $\overline{A}$ does not contain a cycle of length k by virtue of Lemma 1.

Assume that $\overline{x} \in \overline{A}$ and $f^k(\overline{x}) = \overline{x}$. Let $p \in P$ be such that $x \in A_p$. In view of the relation $\overline{f^k(x)} = \overline{x}$, there exists $q \in P$, $p \leqslant q$ such that $\varphi_{pq}(x) = \varphi_{pq}(f^k(x))$. We obtain $f^k(\varphi_{pq}(x)) = \varphi_{pq}(x)$. Thus A_q has a cycle of length m, where $m \leqslant k$.

Let $m < k$. The equality

$$\{\overline{x}, f(\overline{x}), \ldots, f^{k-1}(\overline{x})\} = \{\overline{\varphi_{pq}(x)}, f(\overline{\varphi_{pq}(x)}), \ldots, f^{k-1}(\overline{\varphi_{pq}(x)})\}$$

is valid. Therefore $|\{\overline{x}, f(\overline{x}), \ldots, f^{k-1}(\overline{x})\}| < k$.

Let $m = k$. Choose $s \in P$, $q \leqslant s$ such that A_s does not contain a cycle of length k. Then the element $\varphi_{qs}(\varphi_{pq}(x))$ belongs to a cycle of A_s which has length n, $n < k$. Analogously as in the previous case we obtain $|\{\overline{x}, f(\overline{x}), \ldots, f^{k-1}(\overline{x})\}| \leqslant n < k$. □

Proposition 3. *The class $\mathscr{T}$ is a direct limit class.*

P r o o f. Let (1) be valid and $A_p \in \mathscr{T}$ for all $p \in P$. According to Lemma 7 and Lemma 11, every component of $\overline{A}$ is isomorphic to C_k for some $k \in \mathbb{N}$.

Assume that $\overline{B}, \overline{C}$ are components of $\overline{A}$ such that $\overline{B} \cong C_k$, $\overline{C} \cong C_l$, $k, l \in \mathbb{N}$. In view of Lemma 18 there exist $p, r \in P$ such that for each $q \in P$, $p \leqslant q$ the algebra A_q contains a cycle of length k and for each $s \in P$, $r \leqslant s$ the algebra A_s contains a cycle of length l. Choose $t \in P$ such that $r \leqslant t$ and $p \leqslant t$. We obtain that A_t has cycles of lengths k, l. The algebra A_t is of type τ. We get that if $k \neq l$, then k does not divide l. If $k = l$, then $\overline{B} = \overline{C}$. □

Proposition 4. *Let $\mathscr{K} \subseteq \mathscr{T}$ and $n \in \mathbb{N}$. If every element of $\mathscr{K}$ has less than n components, then $\mathscr{K}$ is a direct limit class.*

P r o o f. Let (1) be valid and $A_p \in [\mathscr{K}]$ for all $p \in P$. We have $\overline{A} \in \mathscr{T}$ by the previous theorem.

Let $\{\overline{B}_i\}_{i \in I}$ be a component partition of $\overline{A}$. Put

$$m = \begin{cases} |I| & \text{if } I \text{ is finite,} \\ n & \text{otherwise.} \end{cases}$$

Let $i(1), \ldots, i(m)$ be different elements of I.

Assume that $j \in \{1, \ldots, m\}$ and $k(j) \in \mathbb{N}$ is such that $\overline{B}_{i(j)} \cong C_{k(j)}$. We use Lemma 18 and choose $p(j) \in P$ which has the following property: if $q \in P$ is such that $p(j) \leqslant q$, then the algebra A_q has a cycle of length $k(j)$.

Now let $s \in P$ be such that $p(1) \leqslant s, \ldots, p(m) \leqslant s$. The algebra A_s contains cycles of lengths $k(1), \ldots, k(m)$. Numbers $k(1), \ldots, k(m)$ are different and thus $\overline{A}$ is a subalgebra of A_s and $m < n$.

Assume that the algebra A_s has a component B such that $B \ncong C_{k(j)}$ for $j = 1, \ldots, m$. Then B is a cycle of length k and $k(j)$ does not divide k for $j = 1, \ldots, m$. So, the algebra A_s cannot be homomorphically embedded into $\overline{A}$ according to Lemma 4, which is a contradiction with Lemma 5. Thus $\overline{A} \cong A_s$ and $\overline{A} \in [\mathscr{K}]$.

□

5. One-element direct limit classes

In this section we will describe all monounary algebras A such that $\underrightarrow{\mathbf{L}}'[A] = [A]$; in this case we will speak about one-element direct limit class.

Let A be a monounary algebra.

The notion of degree $s(x)$ of an element $x \in A$ was introduced by M. Novotný [9] as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put

$$A^{(0)} = \{x \in A\colon\ f^{-1}(x) = \emptyset\}.$$

Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Let λ be an ordinal, $\lambda \neq 0$. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \Big\{x \in A - \bigcup_{\alpha < \lambda} A^{(\alpha)}\colon\ f^{-1}(x) \subseteq \bigcup_{\alpha < \lambda} A^{(\alpha)}\Big\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s(x) = \infty$, in the latter we set $s(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

Let B be a subalgebra of A. Assume that there exists a homomorphism φ of A onto B such that $\varphi(b) = b$ for each $b \in B$. Then B is said to be a retract of A and φ is called a retract mapping corresponding to B.

Retracts of monounary algebras were thoroughly studied by D. Jakubíková-Studenovská [5], [6]. In view of [5], Theorem 1.3 we have

Lemma 19. *Let $A \in \mathscr{U}$ and let B be a subalgebra of A. Then B is a retract of A if and only if the following conditions are satisfied:*

(a) *If $y \in A$ is such that $f(y) \in B$, then there is $z \in B$ such that $s(y) \leqslant s(z)$ and $f(y) = f(z)$.*

(b) *For any component D of A with $D \cap B = \emptyset$, the following conditions are satisfied:*

  (b1) *If D contains a cycle with d elements, then there is a component D' of A with $D' \cap B \neq \emptyset$ and there is $n \in \mathbb{N}$ such that n divides d and D' has a cycle with n elements.*

  (b2) *If D contains no cycle and $x \in D$, then there is $y \in B$ such that $s(f^k(x)) \leqslant s(f^k(y))$ for every $k \in \mathbb{N}_0$.*

Lemma 20. *Let $A \in \mathscr{U}$. If A contains a cycle, then there exists a retract T of A such that $T \in \mathscr{T}$.*

P r o o f. Follows from the previous statement. □

Lemma 21. *Let $A \in \mathscr{U}$ and let B be a retract of A. Then $B \in \underrightarrow{\mathbf{L}}'[A]$.*

P r o o f. Let φ be a retract mapping corresponding to B. Let P be the set of all positive integers with the natural linear order. Assume that for each $p \in P$ there is an isomorphism ψ_p of A onto A_p. Put $\varphi_{pq}(\psi_p(a)) = \psi_q(\varphi(a))$ for all $a \in A$ and $p, q \in P$ such that $p < q$. Then $\{A_p\}_{p \in P}$ is a direct family and the direct limit of this family is an algebra isomorphic to B.

Corollary 5. *Let $\mathscr{K}$ be a direct limit class. Let $A \in \mathscr{K}$. If B is a retract of A, then $B \in [\mathscr{K}]$.*

P r o o f. We have $\underrightarrow{\mathbf{L}}'[A] \subseteq \underrightarrow{\mathbf{L}}'[\mathscr{K}] = [\mathscr{K}]$. Thus Lemma 21 yields this assertion. □

Corollary 6. *Let $\mathscr{K}$ be a direct limit class of monounary algebras. If $\mathscr{K}$ contains an algebra with a cycle, then $\mathscr{K}$ contains an algebra of type τ.*

P r o o f. The class $[\mathscr{K}]$ possesses an algebra of type τ according to Lemma 20 and Corollary 1. So, the claim follows from $[\mathscr{T}] = \mathscr{T}$. □

Lemma 22. *Let $A \in \mathscr{U}$. Then there exists an algebra $B \in \underrightarrow{\mathbf{L}}'[A]$ such that each component of B is isomorphic to C_k for some $k \in \mathbb{N}_0$.*

P r o o f. Let P be the set of all positive integers with the natural linear order. Assume that for each $p \in P$ there is an isomorphism ψ_p of A onto A_p. If $a \in A$, we will write $\psi_p(a) = a_p$. If $p \in P$, then we define $\varphi_{p,p+1}$ by setting

$$\varphi_{p,p+1}(a_p) = f(a_{p+1})$$

for each $a \in A$. So we have defined a direct family of monounary algebras. Let $\{A_p\}_{p \in P} \longrightarrow \overline{A}$. We will show that the operation f on $\overline{A}$ is an injective and surjective mapping. Then the proof will be ready.

Assume that $u, v \in \overline{A}$ and $f(u) = f(v)$. Choose $p, q \in P$ and $a, b \in A$ such that $a_p \in u$, $b_q \in v$. Then $\overline{f(a_p)} = \overline{f(b_q)}$ and therefore there exists $s \in P$ such that $p \leqslant s$, $q \leqslant s$ and $\varphi_{ps}(f(a_p)) = \varphi_{qs}(f(b_q))$. Thus $f^{s+1-p}(a_s) = f^{s-p}(f(a_s)) = f^{s-q}(f(b_s)) = f^{s+1-q}(b_s)$. This yields $f^{s+1-p}(a) = f^{s+1-q}(b)$ because ψ_s is an isomorphism. We get $\varphi_{p,s+1}(a_p) = f^{s+1-p}(a_{s+1}) = f^{s+1-q}(b_{s+1}) = \varphi_{q,s+1}(b_q)$. This means $u = v$.

Further, let $p \in P$ and $a \in A$. Then $\overline{a_p} = \overline{\varphi_{p,p+1}(a_p)} = \overline{f(a_{p+1})} = f(\overline{a_{p+1}})$. □

Corollary 7. *Let $\mathscr{K}$ be a direct limit class of monounary algebras. If $\mathscr{K} \neq \emptyset$, then there exists an algebra $B \in \mathscr{K}$ such that each component of B is isomorphic to C_k for some $k \in \mathbb{N}_0$.*

P r o o f. Lemma 22 yields this assertion. □

Theorem 1. *Let $A \in \mathscr{U}$. The following conditions are equivalent:*

(i) *$\{A\}$ is a direct limit class,*

(ii) $\underrightarrow{\mathbf{L}}'[A] = [A]$,

(iii) *either $A \cong C_0$ or $A \in \mathscr{T}$.*

P r o o f. The equivalence (i) and (ii) follows from the definition. Now let (iii) hold. The set $\{C_0\}$ is a direct limit class in view of Proposition 1. If A is an algebra of type τ, then $\{A\}$ is a direct limit class by Lemma 17.

Conversely, assume that $A \ncong C_0$ and A is not of type τ. Let $\mathscr{K}$ be a direct limit class and $A \in \mathscr{K}$.

If A has a cycle, then $\mathscr{K}$ contains an algebra of type τ by Corollary 2. Thus $\mathscr{K}$ has more than one element.

If A has no cycle, then $\mathscr{K}$ contains an algebra B such that each component of B is isomorphic to C_k for some $k \in \mathbb{N}_0$ according to Corollary 3. If $A \ncong B$, then $\mathscr{K}$ possesses more than one element. If $A \cong B$, then A is not connected and each component of A is isomorphic to C_0. Thus there exists a retract C of A such that $C \cong C_0$ and $C \in [\mathscr{K}]$ by Corollary 1. Therefore $\mathscr{K}$ has more than one element. □

Theorem 2. *Let $\mathscr{K}$ be a direct limit class of monounary algebras. Then there exists an algebra $A \in \mathscr{K}$ such that $\{A\}$ is a direct limit class.*

P r o o f. We will prove that $\mathscr{K} \cap (\mathscr{T} \cup [C_0]) \neq \emptyset$.

If $\mathscr{K}$ contains an algebra with a cycle, then $\mathscr{K} \cap \mathscr{T} \neq \emptyset$ according to Corollary 2.

Let $\mathscr{K}$ contain no algebra with a cycle. Then Corollary 3 implies that $\mathscr{K}$ contains an algebra B which has all components isomorphic to C_0. Thus B has a retract C isomorphic to C_0. We have $C \in [\mathscr{K}]$ by Corollary 1 and $C \in [C_0]$. Conclude $\mathscr{K} \cap [C_0] \neq \emptyset$. □

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