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VALUATED BUTLER GROUPS OF SPECIAL TYPE

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In the theory of torsion-free abelian groups of finite rank a host of most interesting results are known for the class of Butler groups. Butler groups B admit several equivalent characterizations of which we mention the following, more relevant ones (our notations follow those of Fuchs [4]):

(a) B is an epimorphic image of a finite rank completely decomposable torsion-free group;

(b) B is a pure subgroup of a finite rank completely decomposable torsion-free group;

(c) there exists a partition $\Pi = \Pi_1 \cup \ldots \cup \Pi_n$ of the set Π of all primes such that for each j, the localization B_{Π_j} of B at Π_j is a completely decomposable group;

(d) B has finite typeset, and for each type $\mathbf{t}$, the subgroup $B^*(\mathbf{t}) = \langle a \in B \mid \mathbf{t}(a) > \mathbf{t}\rangle$ is of finite index in its purification $B^*(\mathbf{t})*$, furthermore, $B(\mathbf{t})/B^*(\mathbf{t})*$ is a completely decomposable homogeneous group (of type $\mathbf{t}$). (Here $B(\mathbf{t}) = \{a \in B \mid \mathbf{t}(a) \geqslant \mathbf{t}\}$.)

The equivalence of conditions (a), (b) and (d) was established by Butler [3], while characterization (c) is due to Bican [2].

Arnold and Richman [1] considered finitely generated valuated groups and proved the equivalence of (a) and (b) for them. It turns out that (c) is not equivalent to them; in fact, these groups need not be completely decomposable even in the local case. Also, (d) fails in general for valuated Butler groups discussed by Arnold and Richman.

Our main purpose here is to investigate those finitely generated torsion-free valuated groups which are closer to the free valuated groups in the sense that there are no gaps in the valuations of elements. We will show that for these valuated groups the analogs of (a)–(d) are equivalent.

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1. Preliminaries

We start by recalling the definition of valuated groups. Let p be a prime number. By the *p-valuation* v_p of a group G is meant a function from G to the class of ordinals with the symbol ∞ adjoined (which is regarded to be larger than any ordinal) such that, for all $a, b \in G$,

(i) $v_p(a) = \infty$ if $a = 0$;

(ii) $v_p(pa) > v_p(a)$ provided that $v_p(a)$ is an ordinal;

(iii) $v_p(na) = v_p(a)$ whenever the integer n is relatively prime to p;

(iv) $v_p(a + b) \geqslant \min(v_p(a), v_p(b))$.

The *valuation* v of a group G is the collection of p-valuations v_p of G, one for each prime p. Thus $v(a)$ for $a \in G$ is the sequence $v_p(a)$ of p-values of a for $p = 2, 3, 5, \ldots$

We introduce the *value-matrix* $V(a)$ of a in G which will list more information about a: $V(a)$ is an $\omega \times \omega$ matrix whose i, j-entry is $v_p(p^j a)$ where p is the i-th prime and j is a non-negative integer. Using the value-matrix $V(a)$, all the values $v_p(na)$ for integers $n > 0$ can be computed. Value-matrices are partially ordered by the pointwise ordering—this is obviously a lattice-order.

A morphism φ between two valuated groups $G \to G'$ is a valuated homomorphism, i.e. a group homomorphism that does not decrease values at any prime: $v_p(a) \leqslant v_p(\varphi a)$ for all $a \in G$ and all primes p. Two valuated groups A, B are called *isometric* if there is a value-preserving isomorphism between them; we then write $A \cong B$. In particular, two valuated infinite cyclic groups are isometric if and only if they can be generated by elements with the same value-matrix. The groups A and B are *quasi-isometric* if there are subgroups A' and B' of finite indices in A and B, respectively, such that A' and B' are isometric, $A' \cong B'$.

If G is a valuated group, then its *localization* G_p at a set P of primes is the group whose elements are those of G and whose valuation w is given as follows: for $a \in G$ and a prime p, we set $w_p(a) = v_p(a)$ or ∞ according as $p \in P$ or not.

The valuated direct sum of two valuated groups A, B is their group direct sum $A \oplus B$ equipped with the valuation

$$v(a, b) = \min(v(a), v(b)) \qquad \text{for} \quad a \in A, b \in B.$$

If C is a pure subgroup of the valuated group A, then the factor group A/C becomes a valuated group by defining

$$v(a + C) = \sup\{v(a + c) \mid c \in C\}.$$

By a *free valuated group* we mean a group F which is free as a valuated group, i.e. the group F has a set X of generators such that every function f from X into a valuated group G with $v(x) \leqslant v(fx)$ $(x \in X)$ extends uniquely to a valuated homomorphism of F into G. In the finitely generated case, we have $F \cong F_1 \oplus \ldots \oplus F_n$ such that each F_i is a valuated infinite cyclic group and, for each prime p, $v_p(pa_i) = v_p(a_i) + 1$ and $v_p(a_1 + \ldots + a_n) = \min\limits_i v_p(a_i)$ for $a_i \in F_i$.

One should be careful in distinguishing between the terms "free valuated group" and "valuated free group". The latter indicates a free group (i.e. free as an abelian group) furnished with a valuation. If the group is cyclic, say generated by a, then in both cases $v_p(a)$ is arbitrary, but in the free valuated group

$$v_p(p^j a) = v_p(a) + j$$

holds for all primes p and integers $j \geqslant 0$, while for valuated free groups we only have

$$v_p(p^{j+1} a) \geqslant v_p(p^j a) + 1.$$

Let B be a finitely generated valuated free group. We call it *of Butler type* if there is a free valuated finitely generated group $F = F_1 \oplus \ldots \oplus F_n$ (each F_i is a valuated infinite cyclic group) such that B is isometric to the valuated factor group F/G for some subgroup G of F. Thus if $\varphi\colon F \to B$ is the canonical map and $F_i = \langle x_i \rangle$, then the elements $a_i = \varphi x_i$ generate B and valuation is given by $v(a) = \sup\{\min\limits_i v(n_i x_i)\}$ where the sup is taken for all possible representations of $a \in B$ as $a = \sum n_i a_i$. Evidently, $B = \sum \varphi F_i$ which is a valuated generation, i.e. the valuation on B is induced by the valuations on the φF_i in the indicated way. It is convenient to assume that $V(a_i) = V(x_i)$ for each i—this does not mean any loss of generality, since we can change the value-matrix of the x_i accordingly.

Arnold and Richman [1] investigated the class of finitely generated valuated Butler groups and proved that this class coincides with the class of subgroups of finite direct sums of valuated cyclic groups. Thus the analog of Butler's equivalent characterizations (a) and (b) carries over to the valuated case. However, Bican's characterization of Butler groups does not work in the valuated case, since there are valuated local Butler groups which fail to be direct sums of valuated cyclic groups.

Here our main concern will be a subclass of valuated Butler groups for which Bican's characterization holds true.

A finitely generated valuated free group of Butler type will be called *special* if its valuation satisfies a condition stronger than (ii), viz.

(ii′) $v_p(pa) = v_p(a) + 1$ if $v_p(a)$ is an ordinal.

It follows easily that

Lemma 1.1. *Subgroups and finite direct sums of finitely generated valuated special Butler groups are again such groups. In particular, subgroups of free valuated groups are special Butler groups.*

Since every valuated group is an epimorphic image of a free valuated group, it is clear that epic images of free valuated groups, and hence epic images of special Butler groups need not be special Butler groups. Not even in the finitely generated case, as is shown by the following example.

Example 1.2. Let $A = \langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle a_3\rangle \oplus \langle a_4\rangle$ be a free valuated p-local group for some prime p where $v_p(a_1) = v_p(a_2) = \omega, v_p(a_3) = v_p(a_4) = 0$, and let $B = A/C$ be the valuated factor group of A modulo the subgroup $C = \langle a_1 + a_2 - pa_3 - pa_4\rangle$. Then writing $b_i = a_i + C$, in B we have $v_p(b_3 + b_4) = 0$, but $v_p(pb_3 + pb_4) = \omega$.

Moreover, it can happen for epic images of finitely generated free valuated groups that the value-matrix of an element has infinitely many gaps—this phenomenon is shown by the next example.

Example 1.3. Let $A = \langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle a_3\rangle \oplus \langle a_4\rangle$ be a free valuated group where $v(a_1) = v(a_2) = (\omega, \omega, \ldots, \omega, \ldots)$ and $v(a_3) = v(a_4) = (0, 0, \ldots 0, \ldots)$, and let $B = A/C$ be the valuated group where $C = \langle a_1 + a_2 - qa_3 - qa_4, a_1 - a_2 - ra_3 - ra_4\rangle$ with relatively prime integers q, r. Then for each prime p we can find integers s, t such that $qt - rs = p$. Writing $b_i = a_i + C$ as before, we have $v_p(b_3 + b_4) = 0$, but $v_p(pb_3 + pb_4) = v_p((t - s)b_1 + (t + s)b_2) \geqslant \omega$. Thus the value-matrix $V(b_3 + b_4)$ in B contains a gap in each row.

It is evident that if B is a special valuated Butler group, then there is a valuated epimorphism $\varphi\colon F = \langle x_1\rangle \oplus \ldots \oplus \langle x_n\rangle \to B$ where F is a finitely generated free valuated group. We will write $B = \sum_1^k \langle a_i\rangle$ and understand that B is equipped with the valuation induced by F. There is no loss of generality in assuming that φ acts as an isometry on the cyclic summands $\langle x_i\rangle$. Moreover, we may suppose that in B the images of the $\langle x_i\rangle$ are pure subgroups; in fact, if $\varphi x_i = a_i$ is divisible by a prime p in B, say $a_i = pa_i'$, then we can simply replace x_i by x_i' with $x_i = px_i'$ as generator and assign to x_i' the value $v(x_i') = v(a_i')$.

2. The homogeneous case

We shall call a valuated finitely generated free group $F = F_1 \oplus \ldots \oplus F_n$ *homogeneous* if the $F_i = \langle a_i\rangle$ are isometric valuated cyclic groups. Though we are exclusively interested in the case when F is a free valuated group, the proof applies to a more general situation.

Lemma 2.1. *A pure cyclic subgroup $\langle b\rangle$ of a homogeneous valuated finitely generated free group $F = \langle a_1\rangle \oplus \ldots \oplus \langle a_n\rangle$ is a valuated summand, i.e.*

$$F = \langle b\rangle \oplus \langle b_2\rangle \oplus \ldots \oplus \langle b_n\rangle$$

holds for suitable $b_i \in F$.

P r o o f. *Case 1.* If $b = \pm a_i$ for some i, then the claim is evident.

Case 2. Next let $b = a_2 - a_1$ or $b = a_2 + a_1$. We claim that $F = \langle a_1\rangle \oplus \langle a_2 \pm a_1\rangle \oplus \ldots \oplus \langle a_n\rangle$ is a valuated direct sum. Write $x \in F$ in the form $x = m_1a_1 + m_2a_2 + \ldots + m_na_n$ where, for each prime p, $v_p(x) = \min\{v_p(m_ia_i) \mid i = 1, \ldots, n\}$. Then $x = (m_2 \mp m_1)a_1 + m_2(a_2 \pm a_1) + \ldots + m_na_n$ where $v_p(m_2a_2) = v_p(m_2(a_2 \pm a_1))$ by homogeneity. We have to ascertain that $v_p(x)$ can also be computed as the minimum of the p-values of the terms in the second sum; this is evident in all cases except when $v_p(m_1a_1)$ or $v_p((m_2 \mp m_1)a_1)$ is smaller than $\min\{v_p(m_ia_i) \mid i = 2, \ldots, n\}$. But in this exceptional case $v_p(m_1a_1) = v_p((m_2 \mp m_1)a_1)$, since $v_p(m_1a_1) < v_p(m_2a_2) = v_p(m_2a_1)$.

Case 3. Let $b = k_1a_1 + k_2a_2 + \ldots + k_na_n$ where $(k_1, k_2, \ldots, k_n) = 1$ with $k_i \in \mathbb{Z}$. We induct on the integer $k = |k_1| + |k_2| + \ldots + |k_n|$. The starting case $k = 1$ is covered by Case 1 above, so assume $k > 1$. Then at least two of the k_i are different from 0, say, $|k_1| \geqslant |k_2| > 0$. Then either $|k_1 + k_2| < |k_1|$ or $|k_1 - k_2| < |k_1|$, and thus $|k_1 \pm k_2| + |k_2| + \ldots + |k_n| < k$ for one of the two signs. Applying the induction hypothesis to the homogeneous valuated direct sum $F = \langle a_1\rangle \oplus \langle a_2 \pm a_1\rangle \oplus \ldots \oplus \langle a_n\rangle$ and to $b = (k_1 \pm k_2)a_1 + k_2(a_2 \pm a_1) + \ldots + k_na_n$, the claim follows at once. $\square$

Note that the complement of $\langle b\rangle$ in F is likewise a direct sum of valuated cyclic groups.

The following is an analog of a well-known result by R. Baer (see e.g. [4, 86.8]).

Theorem 2.2. *Pure subgroups of homogeneous valuated finitely generated free groups are valuated summands. They are themselves free valuated groups.*

P r o o f. Let G be a pure subgroup of the homogeneous valuated finitely generated free group F. If $b \in G$ generates a pure subgroup in G, then by (2.1) we have $F = \langle b\rangle \oplus C$ for some C which is again a homogeneous valuated finitely generated free group. We obtain $G = \langle b\rangle \oplus (C \cap G)$ where the second summand is a pure subgroup of C. An obvious induction on the rank of G leads to the conclusion that G is a summand of F. The proof shows that G itself is a free valuated group. $\square$

We can now derive the following consequence for special valuated Butler groups.

Corollary 2.3. *Let $B = \sum_1^k \langle a_i\rangle$ be a valuated finitely generated free group which is a special Butler group with each $\langle a_i\rangle$ pure in B. If $v(a_1) = v(a_2) = \ldots = v(a_k)$, then B is a free valuated group.*

P r o o f. B is a valuated epic image of a homogeneous valuated free group F. Owing to (2.2), the kernel K is a summand, so B is again a free valuated group as it is isomorphic to a summand of F. $\square$

3. Auxiliary lemmas

The following two lemmas are crucial for the proof of Theorem 5.1 infra. The first generalizes our (2.3) above.

Lemma 3.1. *Let $B = \sum_1^k \langle a_i \rangle$ be a finitely generated valuated free group which is a special Butler group with each $\langle a_i \rangle$ pure in B. If $v(a_1) \leqslant v(a_2) \leqslant \ldots \leqslant v(a_k)$, then $\langle a_k \rangle$ is a valuated summand of B, i.e. $B = \langle a_k \rangle \oplus C$ for some valuated Butler group C.*

Proof. Since $\langle a_k \rangle$ is pure in the free group B, group-theoretically $\langle a_k \rangle$ is a summand of B, therefore there is a free subgroup C of B such that $B = \langle a_k \rangle \oplus C$ is a group-theoretical direct sum. It remains to prove that this is a valuated direct sum.

In order to do this, we first show that if $b \in B$ satisfies

$$v_p(b) \geqslant v_p(a_k) + j \tag{3}$$

for some prime p and integer $j \geqslant 0$, then b is divisible by p^j in B. Indeed, B is a valuated epic image of a free valuated group $F = \langle x_1 \rangle \oplus \ldots \oplus \langle x_k \rangle$ where $v(x_i) = v(a_i)$ for each i. The element b is the image of certain linear combinations $m_1x_1 + m_2x_2 + \ldots + m_kx_k$ and its p-value is the supremum of all p-values $v_p(m_1x_1 + m_2x_2 + \ldots + m_kx_k) = \min v_p(m_ia_i)$. Clearly, (3) can occur only if in one of the linear combinations all the coefficients are divisible by p^j.

It remains to prove that for every integer n and for every $c \in C$, $v_p(na_k + c) = \min(v_p(na_k), v_p(c))$ holds. Otherwise, we would have inequality $>$ which can occur only if $v_p(na_k) = v_p(c) < v_p(na_k + c)$. If p^j denotes the largest p-power dividing n, then the preceding paragraph implies that $na_k + c$ is divisible by p^{j+1}. In view of the direct decomposition of B, both na_k and c have to be divisible by p^{j+1}. But p^j was chosen to be the highest power of p dividing na_k. □

The proof of the following lemma follows closely Bican's argument [2].

Lemma 3.2. *Let again $B = \sum_1^k \langle a_i \rangle$ be a finitely generated valuated free group which is a special Butler group with each $\langle a_i \rangle$ pure in B. There is a finite partition $\Pi = \Pi_1 \cup \ldots \cup \Pi_n$ of the set Π of all primes such that, for each index j, the localization $B_{\Pi_j} = B_j$ is a special valuated Butler group with a valuated cyclic summand.*

Proof. For each prime p, there is a permutation ϱ of the indices $\{1, 2, \ldots, k\}$ such that the p-values satisfy $v_p(a_{\varrho(1)}) \leqslant v_p(a_{\varrho(2)}) \leqslant \ldots \leqslant v_p(a_{\varrho(k)})$. For each permutation ϱ define

$$\Pi_\varrho = \{p \in \Pi \mid v_p(a_{\varrho(1)}) \leqslant v_p(a_{\varrho(2)}) \leqslant \ldots \leqslant v_p(a_{\varrho(k)})\}.$$

The union of all these Π_ϱ is evidently Π, and if we cancel repeated occurrences of primes, then we obtain a finite partition of Π into at most $k!$ non-empty subsets.

The localizations B_j of B are obtained by localizing B at each Π_ϱ. Therefore the generators in each localization satisfy the hypotheses of (3.1), and hence (3.1) implies our claim. □

4. Types

We introduce an equivalence relation between values of elements. Let $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$ and $\beta = (\beta_1, \ldots, \beta_n, \ldots)$ be two sequences of ordinals and symbols ∞ (representing values of elements). α and β will be considered *equivalent* if they have the same entries for almost all n, and whenever they are different at some n, then both α_n and β_n are ordinals and $\alpha_n = \beta_n + m$ for some positive or negative integer m. The equivalence classes will be called *types*.

Thus linearly dependent elements in special Butler groups have the same type. The set of types is partially ordered, inheriting the partial order of the value sequences $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$.

In case all the elements have the same type, we can show that special Butler groups are close to be free valuated groups.

Lemma 4.1. *Let the valuated finitely generated free group B be special Butler. If all the elements of B have the same type, then B contains a free valuated group of finite index.*

Proof. Let $a_1, \ldots, a_k$ be generators of B such that each of them generates a pure subgroup of B. They have the same type, which means that their values are the same at almost all primes. If p is a prime at which some values are different, then choose powers $p^{n_1}, \ldots, p^{n_k}$ of p such that the elements $p^{n_1}a_1, \ldots, p^{n_k}a_k$ all have the same p-value, say β. The subgroup generated by $p^{n_i}a_i$ must be pure in the subgroup B_1 of B generated by these elements. Since the index of B_1 in B is a power of p, only the p-purity is questionable. But $p^{n_i-1}a_i$ can not belong to B_1, because its p-value is $\beta - 1$, while B_1 is generated by elements of p-value β.

We conclude that B contains a subgroup B_1 of finite index whose generators generate pure subgroups such that the values of the generators differ at a smaller number of primes. Repeating the above argument, in a finite number of steps we obtain a finite index subgroup C of B whose generators generate isometric pure subgroups in C. (2.3) implies that C is a free valuated group. □

Next we verify an analog of Baer's ubiquitous lemma:

Lemma 4.2. *Let A be a finitely generated valuated free group of special Butler type, and B a pure subgroup such that A/B is a free valuated cyclic group. If all the elements of $A \setminus B$ have the same type* $\mathbf{t}$ *as the type of A/B, then B is a valuated summand of A.*

P r o o f. Let $a + B$ ($a \in A$) be a generator of A/B. If we can find an element a' in the coset $a + B$ such that $v(a') = v(a + B)$, then $A = B \oplus \langle a' \rangle$ will be a valuated direct sum, and we are done. In the search of such an a', note that $v(a) \leqslant v(a + B)$ where by hypothesis, $v(a)$ and $v(a+B)$ are equivalent. Evidently, there exist a finite number of elements $a_1, \ldots, a_k \in a+B$ such that $v(a_1) \vee \ldots \vee v(a_k) = v(a+B)$. Thus it will be sufficient to show that if $x, y \in a+B$ have equivalent valuations, then there is a $z \in a + B$ satisfying $v(z) \geqslant v(x) \vee v(y)$. By equivalence, there are relatively prime positive integers m, n such that $v(mx) = v(ny)$. If m', n' are integers with $mm' + nn' = 1$, then $z = mm'x + nn'y \in a + B$ satisfies $v(z) \geqslant v(mx) \geqslant v(x) \vee v(y)$. $\square$

The proof of the preceding lemma establishes the following statement as well: Suppose A is a finitely generated valuated free group of special Butler type and B a pure subgroup of A. If the value sequence $v(a + B)$ is equivalent to $v(a + b)$ for every $b \in B$, then there is a $b' \in B$ such that $v(a + b') = v(a + B)$.

We exhibit an example of an indecomposable special Butler group of rank 2.

Example 4.3. Let $F = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$ be the free valuated group where $v_2(a) = v_3(b) = \infty$ and $v_5(c) = 1$, while all the other values of a, b, c at primes are 0. Define the group B as the valuated epic image of F modulo the (pure) subgroup $\langle a + b - c \rangle$. In this way, B becomes a special Butler group. We claim that B is indecomposable as a valuated group. Assume on the contrary that B is the direct sum of two valuated cyclic groups, say, $\langle u \rangle$ and $\langle w \rangle$. There are integers q, r, s, t with $qt - rs = 1$ (or -1) such that $u = qa + rb$, $w = sa + tb$ (for the sake of simplicity, the images of a, b, c in B are denoted by the same symbols). Thus $a = tu - rw$, $b = -su + qw$ in B. Noting that $v_2(a) = \infty$, it is clear that if t, r are not zero, then both $v_2(u) = \infty$ and $v_2(w) = \infty$, which is impossible. Say, $r = 0$, i.e. q and t are ± 1 and $\langle u \rangle = \langle a \rangle$, $\langle w \rangle = \langle b \rangle$. But then $v_5(c) = \min(v_5(a), v_5(b)) = 0$ is a contradiction.

5. The main result

Let C be a subgroup of a finitely generated valuated free group B, such that B/C is torsion. We say that B is a *close extension* of C if $p^k b = c \in C$ ($b \in B$) with $p^{k-1} b \notin C$ implies $v_p(c) = v_p(b) + k$ (i.e. the valuation in C determines the values of all the elements of B; the values of elements in $B \setminus C$ are the largest values consistent with the valuation in C). It is clear that a close extension of a special valuated Butler group is again Butler of special type.

We are now ready for the main result which gives equivalent characterizations of valuated Butler groups of special type.

Theorem 5.1. *For a finitely generated valuated free group B the following conditions are equivalent:*

(a) *B is a special Butler group;*

(b) *B is isometric to a pure subgroup of a free valuated group;*

(c) *there exists a partition $\Pi = \Pi_1 \cup \ldots \cup \Pi_n$ of the set Π of all primes such that for each j, the localization B_{Π_j} is a free valuated group (where the values of the generators form a totally ordered set);*

(d) *B has a finite typeset and for each type $\mathbf{t}$, $B(\mathbf{t})$ is a close extension of a finite index subgroup $B'(\mathbf{t})$ such that $B'(\mathbf{t}) = B_{\mathbf{t}} \oplus B^*(\mathbf{t})$ where $B_{\mathbf{t}}$ is a homogeneous free valuated group (of type $\mathbf{t}$).*

P r o o f. (b) $\Longrightarrow$ (a) A pure subgroup B of a free valuated group F is of Butler type by Arnold-Richman [1]. Since F satisfies condition (ii$'$), so does B, thus B is special Butler.

(a) $\Longrightarrow$ (c) Next assume that B is a special Butler group. To derive (c), we use induction on the rank r of B. If $r = 1$, then there is nothing to prove. So suppose B is of rank $r > 1$, and (c) holds for valuated free groups of rank $< r$. In view of (3.2), there is a finite partition $\Pi = \Pi_1 \cup \ldots \cup \Pi_n$ of the set of primes Π, such that for each j, the localization B_j of B at Π_j has a valuated decomposition $B_j = \langle a_h \rangle_{\Pi_j} \oplus C_j$ for some generator a_h and a suitable subgroup C_j which is again special Butler, its rank is $r - 1$. Using the induction hypothesis, we can find a finite partition of $\Pi_j = \Pi_{j1} \cup \ldots \cup \Pi_{jn}$ such that the localizations of B at each Π_{jm} will be free valuated groups. The set of all Π_{jm} (taken for all j) will be a partition of the primes with the indicated property.

(c) $\Longrightarrow$ (b) Assume that there is a partition $\Pi = \Pi_1 \cup \ldots \cup \Pi_n$ of the set of primes such that the localization B_j of B at each Π_j is a free valuated group. Set $B^* = B_1 \oplus \ldots \oplus B_n$ (valuated direct sum), and let δ denote the diagonal map $B \to B^*$. It is clear that B is isometric to its image $\delta(B)$ in B^* which is manifestly a pure subgroup of B^*. Hence B satisfies (b).

(a) + (b) $\Longrightarrow$ (d) As the typeset of a free valuated group is evidently finite, (b) implies that the same must be true for B. If B is an epic image of a free valuated group F, then without loss of generality the subgroup $B(\mathbf{t})$ can be assumed to be an epic image of a summand of F (which is the direct sum of those cyclic summands whose types are $\geqslant \mathbf{t}$). Hence the proof is reduced to the case when all the cyclic summands of F are of type $\geqslant \mathbf{t}$ and $B = B(\mathbf{t})$. Since every element of B, not in

$B^*(\mathbf{t})$ must have type $\mathbf{t}$, there is a subgroup B' of B such that $B' \oplus B^*(\mathbf{t})$ has finite index in B and all the elements of B' have type $\mathbf{t}$. In view of (4.1), there is a free valuated subgroup $B_{\mathbf{t}}$ of finite index in B', and evidently, $B(\mathbf{t})$ is a close extension of $B'(\mathbf{t}) = B_{\mathbf{t}} \oplus B^*(\mathbf{t})$.

(d) $\Longrightarrow$ (a) For each $\mathbf{t}$ in the typeset of B, pick a free valuated subgroup $B_{\mathbf{t}}$ as stated, and let $C = \sum B_{\mathbf{t}}$ be their sum in B. This C is evidently a valuated Butler subgroup of finite index in B. Hence B is likewise a valuated Butler group. The types in B can be listed as $\mathbf{t}_1, \ldots, \mathbf{t}_k$ in such a way that never $\mathbf{t}_i < \mathbf{t}_j$ if $i < j$. We prove inductively that $C_j = \sum_{i \leqslant j} B_{\mathbf{t}_j}$ is special Butler. For $j = 1$, this is obvious, since $B_{\mathbf{t}_1}$ is free valuated. If the intersection $C_j \cap B_{\mathbf{t}_{j+1}}$ is pure in $B_{\mathbf{t}_{j+1}}$, then by (2.1) it is a summand of $B_{\mathbf{t}_{j+1}}$, so C_{j+1} is the direct sum of C_j and a complement to the intersection, thus it is again special Butler. If the intersection is not pure in $B_{\mathbf{t}_{j+1}}$, then C_{j+1} will be a close extension of a special Butler group, and so again special Butler. Finally, in order to show that B itself is special, it suffices to point out that by induction it follows that each $B(\mathbf{t})$ is a close extension of the corresponding $\sum B_{\mathbf{t}_j}$.

This completes the proof of the theorem. $\square$

Imitating the argument used in the proof of Theorem 3.4 by Arnold-Richman [1], we can even show that in (b) the condition of purity can be dropped, so (b) can be replaced by

(b′) B is isometric to a subgroup of a free valuated group.

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