Czechoslovak Mathematical Journal

Szymon Dolecki; Tsugunori Nogura

Two-fold theorem on Fréchetness of products

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 2, 421–429

Persistent URL: `http://dml.cz/dmlcz/127498`

Czechoslovak Mathematical Journal, 49 (124) (1999), 421–429

TWO-FOLD THEOREM ON FRÉCHETNESS OF PRODUCTS

S. Dolecki, Dijon, and T. Nogura, Matsuyama

(Received November 3, 1996)

Abstract. A refined common generalization of known theorems (Arhangel'skii, Michael, Popov and Rančin) on the Fréchetness of products is proved. A new characterization, in terms of products, of strongly Fréchet topologies is provided.

Keywords: α_3, α_4, β_3, β_4 spaces, Φ-space, product space, sequential space, sequentially subtransverse, strongly Fréchet, transverse

MSC 2000: 54A20, 54B10, 54D50, 54D55, 54G15

Introduction and formulation of main results

Recall that a topological space X is called *Fréchet* if its every point x is *Fréchet*, i.e., if $A \subset X$ and $x \in \operatorname{cl} A$, then there exists a sequence $(x_n)_n$ in A converging to x; a topological space X is called *strongly Fréchet* if its every point x is *strongly Fréchet*, i.e., if $(A_n)_n$ is a decreasing sequence of subsets of X and $x \in \operatorname{cl} A_n$ for every n, then there exists a sequence $(x_n)_n$ converging to x and such that $x_n \in A_n$.

It is well-known that a product of two Fréchet topologies need not be Fréchet (e.g., [3], [13]); in fact, if X is a first countable space that contains a nontrivial sequence, then $X \times Y$ is Fréchet if and only if Y is strongly Fréchet [5], and there exist Fréchet not strongly Fréchet spaces.

In order to better situate our main result (Theorem 4), we reformulate the sufficiency part of the above result of Michael in terms of the properties α of Arhangel'skii. Recall that a topological space is strongly Fréchet if and only if it is Fréchet and α_4

The first author's work has been partly supported by the Ministère des Affaires Etrangères and by the Ehime University in Matsuyama The idea of the two-fold theorem arose after the lecture of the paper [12] of Popov and Rančin. The authors are grateful to Professor A. V. Arhangel'skii for having indicated this reference to them as well as for providing the example used in Proposition 10.

[1, Theorem 5.23]. The property α_4 [1] of a topological space means that for every convergent stationary bisequence

$$x_{n,k} \xrightarrow[k]{} x_n = x \tag{1}$$

there exist sequences (n_p) and (k_p) such that n_p tends to ∞ and $(x_{n_p,k_p})_p$ tends to x. This property can be rephrased as follows: there exists a compact metrizable subset C of $\{x\} \cup \{x_{n,k}\colon\ n, k \in \omega\}$ such that

$$|\{n\colon\ C \cap \{x_{n,k}\colon\ k \in \omega\} \neq \emptyset\}| = \omega. \tag{2}$$

Now the sufficiency part of the above theorem of Michael can be stated in terms that are suitable for the development of the paper.

0. Theorem. [5] *If X is first countable and Y is a Fréchet α_4-space, then $X \times Y$ is Fréchet.*

The theorem of Arhangel'skii on the Fréchetness of products [1] uses the property α_3: for every stationary bisequence converging to x, there exist sequences (n_p) and (k_p) such that $(x_{n_p,k_p})_p$ converges to x and

$$|\{n\colon\ |\{k_p\colon\ n_p = n\}| = \omega\}| = \omega.$$

We reformulate this property in the following way: for every stationary bisequence converging to x, there exists a compact metrizable subset C of $\{x\} \cup \{x_{n,k}\colon\ k \in \omega\}$ such that

$$|\{n\colon\ |C \cap \{x_{n,k}\colon\ k \in \omega\}| = \omega\}| = \omega. \tag{3}$$

First countable spaces are of course α_3.

1. Theorem. [1] *If X is a regular countably compact Fréchet space and Y is a Fréchet α_3-space, then $X \times Y$ is Fréchet.*

V. V. Popov and D. V. Rančin [12] say that a topological space X is a Φ-*space* if for every $x \in X$ and for each $A \subset X$ with $x \in \operatorname{cl} A$, there exists a sequence $(Q_n)_n$ of open sets such that $Q_n \cap A \neq \emptyset$ for each n, and

$$\lim Q_n = x,$$

that is, for every $V \in N(x)$ there exists n_V such that $Q_n \subset V$ for each $n \geqslant n_V$, where $N(x)$ is the set of neighborhoods of x in X. They prove the following

2. Theorem. *If X and Y are compact Fréchet spaces and X is a Φ-space, then $X \times Y$ is Fréchet.*

Before considering its further strengthening, we give here an intermediate generalization of Theorem 2 in order to indicate its place in the framework of this paper. We say that a space X is a *β_3-space* if for every convergent bisequence

$$x_{n,k} \underset{k}{\longrightarrow} x_n \underset{n}{\longrightarrow} x \tag{4}$$

which is free (i.e., such that $x_n \neq x$)there exists a compact metrizable subset C of $\{x\} \cup \{x_n : n \in \omega\} \cup \{x_{n,k} : n, k \in \omega\}$ such that (3) holds. This amounts to the existence, for every convergent free bisequence (4), of a subbisequence, i.e., a bisequence of the form

$$x_{n_m, k_q^m} \underset{q}{\longrightarrow} x_{n_m} \underset{m}{\longrightarrow} x$$

such that $(x_{n_m, k^m_{q_m}})$ converges to x for each sequence (q_m).

Of course, every Φ-space is a β_3-space. On the other hand, each regular locally countably compact Fréchet space is α_4. Now we are in position to formulate the announced intermediate re-enforcement of Theorem 2.

3. Theorem. *If X is a regular locally countably compact Fréchet β_3-space and Y is a Fréchet α_4-space, then $X \times Y$ is Fréchet.*

The property β_3 is clearly an analogue of the property α_3, the difference is that in the latter case the bisequence is free and in the former case the bisequence is stationary. Let us introduce, in analogy with α_4, property β_4: for every convergent free bisequence (4) there exists a compact metrizable subset C of $\{x\} \cup \{x_n : n \in \omega\} \cup \{x_{n,k} : n, k \in \omega\}$ such that (2) holds. Notice that for a sequential topology, the property β_4 amounts to Fréchetness.

It turns out that the conjunction of properties α_3 and β_4 (Theorem 1) and the conjunction of properties β_3 and α_4 (Theorem 3) play the same (perfectly asymmetric) role in the (simultaneous) proof that we give. In other words, Theorems 1 and 3 are twin theorems.

A space X is called a *q-space* if every $x \in X$ is a *q-point*, i.e., such that there exists a sequence $(Q_n)_n$ of neighborhoods of x with the property that if $x_n \in Q_n$, then the sequence $(x_n)_n$ has an accumulating point [5]. A point of a topological space is *regular* if it admits a base of closed sets; a q-point x is *q-regular* if there exists a defining sequence $(Q_n)_n$ such that x is regular in $\bigcap_n \operatorname{cl} Q_n$. Of course, each regular q-point is q-regular.

Since first countable spaces are q-regular (q-spaces), and locally countably compact spaces are q-spaces, the following main result of this paper constitutes a common strengthening of all the theorems above. We call it a two-fold theorem, because its hypothesis is the alternative of two asymmetric conditions that play the same role in the argument.

All the α and β properties of a topological space X can be rephrased (in an obvious way) as properties at x for every $x \in X$; we use some of these point variants in the sequel.

4. Two-fold theorem. *Let X and Y be Fréchet spaces and let x be q-regular. If either x is β_3 and y is α_4, or x is β_4 and y is α_3, then (x, y) is a Fréchet point.*

The first variant of Theorem 4 can actually be strengthened to become another characterization (at the beginning of this paper we formulated that of Michael) of strong Fréchetness.

5. Theorem. *A space Y is strongly Fréchet if and only if for every non discrete regular β_3 Fréchet q-space X, the product $X \times Y$ is Fréchet.*

The second variant of Theorem 4 implies

6. Corollary. *The product of countably many regular α_3 Fréchet q-spaces is a regular α_3 Fréchet q-space.*

Undefined notions and concepts can be found in [4].

Discussion

In [2] S. Dolecki and S. Sitou call a point x *transverse* if for every free sequence $(x_n)_n$ converging to x there exists a sequence of open sets $(Q_n)_n$ converging to x with $x_n \in Q_n$. A point x is called *subtransverse* if for every free sequence (x_n) converging to x there exists a subsequence (x_{n_k}) and a sequence of open sets (Q_k) converging to x with $x_{n_k} \in Q_k$. A space is called *(sub)transverse* if every point is (sub)transverse. A point x is called *sequentially transverse* if for every free bisequence (4) (convergent to x) there exists $f\colon \omega \to \omega$ such that $lim_n x_{n,k}$ provided that $k_n \geqslant f(n)$ for all n. A point x is called *sequentially subtransverse* if every free bisequence converging to x admits a sequentially transverse subbisequence, that is if x is β_3.

7. Proposition. *A space is a Φ-space if and only if it is sequential and subtransverse.*

It is proved in [12] and in [2] that Lašnev spaces are subtransverse. Popov and Rančin in [12] essentially proved the following

8. Proposition. *Let x be a non-isolated point in X and let $C = \{y\} \cup \{y_n : n \in \omega\}$ be a free convergent sequence. If $X \times C$ is subtransverse, then x is a first countable point in X.*

We compare here the properties listed above. However, since in some cases we succeed in obtaining subtler results than needed in this task of comparison, we introduce here a general framework that enables us to better render these subtleties and to provide a broader view of the topic.

Along with properties α_4 and α_3, A. V. Arhangel'skii considered in [1] properties α_2 and α_1 (the definitions that have been adopted in literature differ slightly from the original ones); P. J. Nyikos introduced in [10] property $\alpha_{1.5}$. Analogous properties were studied by J. Novák in [9] under the names α, β, γ, δ, and so on.

Our reformulation of property α_3 in terms of compact metrizable subsets of stationary bisequences admit analogues relative to the other α-properties. Formula (3) means that the set of indices n such that C is a frequent set with respect to the indices $(n,k)_k$ is frequent (we call it *frequent/frequent* with the understanding that the first qualification concerns n's and the second k's). Similarly α_2 corresponds to *eventual/frequent*, $\alpha_{1.5}$ to *frequent/eventual* and α_1 to *eventual/eventual*. The same properties of free bisequences can be called β_2, $\beta_{1.5}$ and β_1. In these terms sequential subtransversity is β_1. It is known (e.g., [10], [11]) that the bigger is i, the weaker is α_i; the same holds for β_i's.

To start our comparative review, we observe that there exists a sequentially subtransverse Fréchet space which is not sequentially transverse. Actually, more is true. Some modifications of the argument of [11, Proposition 2.4] of Nyikos enable us to prove that the Cantor tree fulfils the following

9. Proposition. *There exists a compact Fréchet space which is α_3 and β_3 but neither α_2 nor β_2.*

Once having noticed that the Cantor tree is a sequential topological space, we see that the above proposition implies that it is Fréchet recovering [9, Proposition 2.2].

10. Proposition. *There exists a sequentially compact Fréchet space (hence a q-space) which is α_3 and sequentially subtransverse (i.e., β_3) but not subtransverse.*

The following proposition shows that the Fréchetness of a point $(x,y) \in X \times Y$ does not imply that one of x or y is a q-point.

11. Proposition. *There is an α_3 and sequentially subtransverse (i.e., β_3) Fréchet space X which is not a q-space, and such that X^2 is Fréchet. Moreover, X is not subtransverse.*

The following result generalizes the fact [1] that every regular locally countably compact Fréchet space is α_4.

12. Proposition. *Each regular q-point in a Fréchet space is α_4.*

The following proposition shows that Theorem 4 generalizes Theorem 1 even if we restrict ourselves to the case of Y being α_3.

13. Proposition. *There exists a Fréchet regular q-space which is neither locally countably compact nor first countable.*

Proofs

Considering subsets of the product of two sets as multifunctions (i.e., relations), we use the following notation: for $A \subset X \times Y$, $x \in X$ and $y \in Y$, let $Ax = \{y\colon (x,y) \in A\}$, $A^{-}y = \{x\colon (x,y) \in A\}$; for $V \subset X$ and $W \subset Y$, let $AV = \bigcup_{x\in V} Ax$ and $A^{-}W = \{x\colon Ax \cap W \neq \emptyset\}$.

P r o o f o f T h e o r e m 4. Let $(x,y) \in \operatorname{cl} A$ and let $(Q_m)_m$ be a defining sequence of a q-point x. Then for every neighborhood V of x closed in $\bigcap_m \operatorname{cl} Q_m$ and each m, one has $y \in \operatorname{cl} A(V \cap Q_m)$. As Y is strongly Fréchet, there exists a sequence (y_m^V) converging to y and such that $y_m^V \in A(V \cap Q_m)$. Let $x_m^V \in V \cap Q_m$ be such that $y_m^V \in Ax_m^V$. By the definition of $(Q_m)_m$ and the Fréchetness of X, there exists a subsequence $(x^V_{m(V,k)})_k$ of $(x_m^V)_m$ that converges to an element x^V of V. If $x^V = x$, then the proof is complete, so that we can assume that $x^V \neq x$.

Of course, the corresponding subsequence $(y^V_{m(V,k)})_k$ of $(y_m^V)_m$ converges to y. Since $x \in \operatorname{cl}\{x^V\colon V \in N(x)\}$, by the Fréchetness of X, there exists a sequence $(x_n)_n = (x_n^{V_n})_n$ converging to x. Let $x_{n,k} = x^{V_n}_{m(V_n,k)}$ and $y_{n,k} = y^{V_n}_{m(V_n,k)}$. Of course, $(x_{n,k})$ is a free bisequence in X and $(y_{n,k})$ is a stationary bisequence in Y.

If X is β_3, then choose a compact metric space C of the form

$$C = \{x\} \cup \{x_{n_m}\colon m \in \omega\} \cup \{x_{n_m,k_q^m}\colon m,q \in \omega\},$$

where $(n_m)_m$ tends to ∞ and $(k_q^m)_q$ tends to ∞ for every m. Take the subsequence of $(y_{n,k})$ corresponding to the same indices. By the strong Fréchetness of Y we can choose a subsequence $(y_{n_{m_t},k_{q_t}^{m_t}})$ converging to y with (m_t) converging to ∞. Now the sequence $(x_{n_{m_t},k_{q_t}^{m_t}}, y_{n_{m_t},k_{q_t}^{m_t}})$ converges in A to (x,y).

If Y is α_3, then there exists a compact metrizable set of the form $C = \{x\} \cup \{x_{n_m,k_q^m}\colon m,q \in \omega\}$, where $(n_m)_m$ tends to ∞ and $(k_q^m)_q$ tends to ∞ for every m.

Now, by the Fréchetness of X, choose a subsequence $(x_{n_{m_t},k_{q_t}^{m_t}})$ converging to x with (m_t) converging to ∞. Then the sequence $(x_{n_{m_t},k_{q_t}^{m_t}}, y_{n_{m_t},k_{q_t}^{m_t}})$ in A converges to (x, y). The proof is complete. $\square$

Proof of Corollary 6. The product of countably many q-spaces is a q-space. By Theorem 3, $\prod\limits_{n\leqslant m} X_n$ is Fréchet for every $m \in \omega$. Now the conclusion follows from [9, Theorem 2.4] and [8, Theorem 3.1] which respectively say that α_3-property is countable productive and if all finite products are α_3 and Fréchet, then the whole product is Fréchet. $\square$

Proof of Proposition 7. Let x be a Φ-point and let $(x_n)_n$ be a free sequence converging to x. Then we may suppose that $x \in \operatorname{cl}\{x_n\colon n \in \omega\} \setminus \{x_n\colon n \in \omega\}$. Since x is a Φ-point, there exists a sequence $(Q_n)_n$ of open sets converging to x and such that $Q_k \cap \{x_n\colon n \in \omega\} \neq \emptyset$. If $x_{n_k} \in Q_k$, then (n_k) tends to ∞. Consequently x is subtransverse.

Conversely, let x be a sequential subtransverse point and let $x \in \operatorname{cl} A \setminus A$. Then there exists a sequence $(x_n)_n \subset \operatorname{cl} A \setminus A$ converging to x. By subtransversity, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a sequence $(Q_k)_k$ of open sets converging to x such that $x_{n_k} \in Q_k$ for each k. Since $x_k \in \operatorname{cl} A$, one has $Q_k \cap A \neq \emptyset$. $\square$

Proof of Proposition 9. Consider the Cantor tree $T \cup C \cup \{\infty\}$ as presented in [11]: C is the Cantor set, T is the set of finite restrictions of the representations of the elements of C; the elements of T are isolated and the neighborhood filter of $c \in C$ corresponds to the cofinite filter on the branch of c in T. The point ∞ is the Alexandroff compactifying point.

In view of the results of [11], it remains to prove that the space is β_3 but not β_4. To see β_3 we consider the only non trivial case of bisequences of the form

$$t_{n,k} \xrightarrow[k]{} c_n \xrightarrow[n]{} \infty, \tag{6}$$

where $t_{n,k}$'s belong to T and c_n's to C. Then there exists an isotone subsequence (c_{n_p}) of (c_n) converging to some $c \in C$ in the Euclidean topology. Now there exists $f\colon \omega \to \omega$ such that if $k_p \geqslant f(p)$, then $(t_{n_p,k_p})_p$ converges to c in the Euclidean topology (induced from the plane into which the Cantor tree is embedded) and is out of the branch of c. Therefore $(t_{n_p,k_p})_p$ converges to ∞ in the topology considered.

To prove that the space is not β_2, take a countable dense (in the Euclidean topology) subset $\{c_n\colon n \in \omega\}$ of C and consider (6) with $(t_{n,k})$ on the branch of c_n for each n. If β_2 held, there would exist a compact metrizable eventual/frequent subset K of (6). Let (t_{n,k_n}) be a sequence in K (hence converging to ∞) such that the ordinates $y(t_{n,k_n})$ tend to 0. Let C_m be the set of all those c_n's for which $\frac{1}{m}$-wedges of their branches miss the sequence (t_{n,k_n}). By the Baire category theorem, there exists a Euclidean-open subset I of C and $\delta > 0$ such that $y(t_{n,k_n}) \leqslant \delta$ for each $c_n \in I$, contradicting the fact that $y(t_{n,k_n})$ tends to 0. $\square$

Proof of Proposition 10. Let $\kappa \geqslant \omega_1$ be a cardinal. Let X be a Σ-product of κ copies of the two point space $\{0,1\}$, with a base point $(0,0,0,\ldots)$, i.e.,

$$X = \{x \in \{0,1\}^\kappa \colon\ |\{\alpha < \kappa \colon\ x(\alpha) \neq 0\}| \leqslant \omega\}.$$

Then X is an α_3 Fréchet space [1, Theorem 6.16]. Actually we can show that every countable subset of X is metrizable, so that X is β_3.

It is well-known that X is sequentially compact. Since X^2 is homeomorphic to X, and X is not first countable, Proposition 8 implies that X is not subtransverse. $\square$

Proof of Proposition 11. Let X be a Σ-product of uncountably many copies of the space of reals with a base point $(0,0,0,\ldots)$. Then X is α_3, β_3 and Fréchet as in Proposition 10. We show that X is not a q-space. First note that X is not countably compact, so X contains an infinite closed discrete subset.

Let $(U_n)_n$ be a sequence of neighborhoods of 0. We show that there exist points $x_n \in U_n$ such that $\{x_n \colon\ n \in \omega\}$ is discrete. Each U_n includes

$$V_n = \Big(\prod_{\alpha \in A_n} T_\alpha^n \times \prod_{\alpha \in \kappa \setminus A_n} R_\alpha \Big) \cap X,$$

where $R_\alpha = R$ and each A_n is finite. Without loss of generality we may assume $A_n \subset A_{n+1}$, $T_\alpha^{n+1} \subset T_\alpha^n$ for each $\alpha \in A_n$ and $\bigcap_{n\in\omega} T_\alpha^n = \{0\}$ for every $\alpha \in \bigcup_{n\in\omega} A_n$. Then $\bigcap_{n\in\omega} V_n$ is homeomorphic to X. Hence X is not a q-space. $\square$

Proof of Proposition 12. Let $x_{n,k} \underset{k}{\longrightarrow} x_n = x$ be a convergent stationary bisequence and let $(Q_n)_n$ be a defining sequence of a q-point x. For every closed neighborhood U of x, consider a sequence $x_{n,k(U,n)} \in U \cap Q_n$. It follows from the assumption that $(x_{n,k(U,n)})_n$ has an accumulation point x^U and by Fréchetness there exists a sequence $(n(U,p))_p$ converging to ∞ and such that $x_{n(U,p),k(U,n(U,p))} \underset{p}{\longrightarrow} x^U$. Of course, $x^U \in U$. If $x^U = x$,then the proof is complete; if not, then $x \in \operatorname{cl}\{x^U \colon \operatorname{cl} U = U \in N(x)\} \setminus \{x^U \colon \operatorname{cl} U = U \in N(x)\}$. Therefore, by Fréchetness, there exists a free sequence $x_m = x^{U_m}$ that converges to x. Let $x_{n(m,p),l(m,p)} = x_{n(U_m,p),k(U_m,n(U_m,p))}$ and let $f(m)$ be such that for each $p > f(m)$, one has $n(m,p) > m$. Since

$$x_{n(m,p),l(m,p)} \underset{p}{\longrightarrow} x_m \underset{m}{\longrightarrow} x$$

is a free bisequence, by Fréchetness there exist sequences (m_q) tending to ∞ and (p_q) with $p_q > f(m_q)$ such that $x_{n(m_q,p_q),l(m_q,p_q)}$converges to x with $(n(m_q,p_q))_q$ converging to ∞. □

P r o o f o f P r o p o s i t i o n 1 3. Let X be a compact Fréchet space that admits a point x that is not first countable. Let Y be a first countable space that admits a point y which is not locally countably compact. Then the product $X \times Y$ is Fréchet by Theorem 0, but it is neither first countable nor locally countably compact. As a product of two q-spaces, $X \times Y$ is a q-space.

If we take the quotient of the topological sum of X and Y by $\{x, y\}$, then we obtain also a space which is not locally countably compact and not first countable. This is a q-space, because $X \oplus Y$ is a q-space and the corresponding quotient map is actually almost open. □

References

[1] *A. V. Arhangel'skii*: The frequency spectrum of a topological space and the product operation. Trans. Moscow Math. Soc. (1981), 164–200.
[2] *S. Dolecki and S. Sitou*: Precise bounds for sequential order of products of some Fréchet topologies. Topology Appl.. To appear.
[3] *E. K. van Douwen*: The product of a Fréchet space and a metrizable space. Topology Appl. *47* (1992), 163–164.
[4] *R. Engelking*: General Topology. PWN Warszawa (1977).
[5] *E. Michael*: A quintuple quotient quest. General Topology Appl. *2* (1972), 91–138.
[6] *E. Michael*: A note on closed maps and compact sets. Israel J. Math. *2* (1964), 174–176.
[7] *T. Nogura*: Fréchetness of inverse limits and products. Topology and Appl. *20* (1985), 59–66.
[8] *T. Nogura*: The product of $\langle\alpha_i\rangle$-spaces. Topology and Appl. *21* (1985), 251–259.
[9] *J. Novák*: On convergence group. Czechoslovak Math. J. *20* (1970), 357–374.
[10] *P. Nyikos*: The Cantor tree and the Fréchet-Urysohn property. Annals N.Y. Acad. Sc. *552* (1989), 109–123.
[11] *P. Nyikos*: Subsets of ${}^{\omega}\omega$ and the Fréchet-Urysohn and α_i-properties. Topology Appl. *48* (1992), 91–116.
[12] *V. V Popov and D. V. Rančin*: On certain strengthening of the property of Fréchet-Urysohn. Vest. Moscow Univ. *2* (1978), 75–80.
[13] *Y. Tanaka*: Products of sequential spaces. Proc. Amer. Math. Soc. *54* (1976), 371–375.

Authors' addresses: Département de Mathématiques, Université de Bourgogne, B.P. 47870, 21078 Dijon, France; Department of Mathematical Science, Faculty of Science, Ehime University Matsuyama, Japan.