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RETRACT IRREDUCIBILITY OF MONOUNARY ALGEBRAS

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Introduction

Retract irreducibility of connected monounary algebras was investigated in [4]–[6]. In this paper all monounary algebras (not only connected) which are retract irreducible are described (Theorem 4.5).

It turns out that

1) if a monounary algebra (A, f) is retract irreducible, then $\operatorname{card} A \leqslant \aleph_0$;
2) the number of non-isomorphic types of retract irreducible monounary algebras is equal to $\aleph_0$.

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a retract of (A, f) if there is a mapping h of A onto M such that h is an endomorphism of (A, f) and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M. Further, let $R(A, f)$ be the system of all monounary algebras (B, g) such that (B, g) is isomorphic to (M, f) for some retract M of (A, f).

A monounary algebra $\mathcal{A}$ will be said to be retract irreducible if, whenever $\mathcal{A} \in R\Bigl(\prod\limits_{i\in I} \mathcal{A}_i\Bigr)$ for some monounary algebras $\mathcal{A}_i$, $i \in I$, then there exists $j \in I$ such that $\mathcal{A} \in R(\mathcal{A}_j)$. If the condition is not satisfied, then $\mathcal{A}$ will be called retract reducible.

Analogous relations between retracts and direct product decompositions of partially ordered sets were studied by D. Duffus and I. Rival [1].

In some proofs, the results and methods of M. Novotný [8], [9] concerning homomorphisms of monounary algebras are used. Homomorphisms of monounary algebras were investigated also in [2], [3] and [7]. The notion of the degree $s_f(x)$ of an element x of a monounary algebra (A, f) was introduced in [8] (cf. also [7] and [3]).

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Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ the set of all integers and let $\mathbb{P}$ be the set of all primes. For $n \in \mathbb{N}$ let $\mathbb{Z}_n$ be the set of all integers modulo n. We shall consider the following monounary algebras:

$\underline{\mathbb{N}} = (\mathbb{N}, f)$, where $f(i) = i + 1$ for each $i \in \mathbb{N}$;

$\underline{\mathbb{Z}} = (\mathbb{Z}, f)$, where $f(i) = i + 1$ for each $i \in \mathbb{Z}$;

$\underline{n} = (\mathbb{Z}_n, f)$, where $f(i) = i + 1 \pmod{n}$ for each $i \in \mathbb{Z}_n$.

As is often used, we will write also $B = (B, f)$ (without distinguishing between the set and the algebra).

By a large cycle we understand a cycle with at least two elements.

If we say that an algebra A contains no cycle, then we mean that no subalgebra of A is a cycle. Saying that A contains a cycle we mean that some subalgebra of A is a cycle.

If $B = \prod\limits_{i \in I} B_i$ is a direct product of algebras, $x \in B$ and $j \in I$, then there is a natural projection $\eta_j \colon B \to B_j$. We denote

$$x(j) = \eta_j(x).$$

Analogously, if C is a subalgebra of B, $j \in I$, then we denote

$$C(j) = \eta_j(C).$$

The following assertions can be easily proved and we sometimes use them without quotation.

Let B_j for $j \in J$ be monounary algebras, $B = \prod\limits_{j \in J} B_j$ and let K be a connected component of B.

(A) If $j \in J$, then $K(j)$ is a connected component of B_j.

(B) If there is $j \in J$ such that $K(j)$ contains no cycle, then K contains no cycle.

(C) Suppose that for each $j \in J$, $K(j)$ contains a cycle C_j.

  a) If l.c.m.$\{\operatorname{card} C_j \colon j \in J\} = n \in \mathbb{N}$, then K contains a cycle C with $\operatorname{card} C = n$.

  b) If l.c.m.$\{\operatorname{card} C_j \colon j \in J\}$ does not exist, then K contains no cycle.

(D) If K contains a cycle C with $\operatorname{card} C = n$, then, for each $j \in J$, $K(j)$ contains a cycle C_j with $\operatorname{card} C_j$ dividing n.

Let us prove the following auxiliary result:

(E) Let B be a monounary algebra containing no cycle and let C be a cycle with k elements; $k \in \mathbb{N}$.

  Then the algebra $B \times C$ is isomorphic to a disjoint sum of k copies of B.

P r o o f. Without loss of generality, assume that $C = \mathbb{Z}_k$. Let $x \in B$, $i, j \in C$, $i \neq j$. Suppose that (x, i) belongs to the same connected component of $B \times C$ as (x, j). Then there are $m, n \in \mathbb{N}$ with

$$\begin{gathered} f^n((x,i)) = f^m((x,j)), \\ (f^n(x), i+n) = (f^m(x), j+m), \\ f^n(x) = f^m(x), \quad i+n \equiv j+m \pmod{k}. \end{gathered}$$

Since $x \in B$ and B contains no cycle, we obtain $n = m$, thus $i = j$. Hence

(1) if $x \in B$, $i, j \in C$, $i \neq j$, then (x, i), (x, j) belong to distinct components of $B \times C$.

Similarly,

(2) if $x, y \in B$, $i, j \in C$, then (x, i), (y, j) belong to the same connected component of $B \times C$ if and only if there are $m, n \in \mathbb{N}$ such that $f^m(x) = f^n(y)$ and $i + m \equiv j + n \pmod{k}$.

Suppose that $(x, i) \in B \times C$. Let D and E be the connected components of $B \times C$ or of B, respectively, such that $(x, i) \in D$ and $x \in E$. For $(z, l) \in D$ define

$$\psi((z,l)) = z.$$

Then $z \in E$ and ψ is a mapping of D into E. Further, ψ is injective by (1). Let $q \in E$. There are $m, n \in \mathbb{N}$ with $f^m(x) = f^n(q)$. Then there is $j \in C$ with $j \equiv i + m - n \pmod{k}$. Hence $(q, j) \in D$, which implies that ψ is bijective. Obviously, ψ is a homomorphism, thus

$$\psi\colon\ D \to E$$

is an isomorphism, which yields the required assertion. $\square$

Remark. In what follows, if we speak about sets (e.g., $X, Y, \ldots$) or elements (e.g., $x, y, \ldots$) and if no relations between these sets or elements are explicitly stated, then we always assume that the sets under consideration are disjoint, the elements under consideration are distinct and do not belong to the sets mentioned.

In Sections 1–3 we suppose that A is a monounary algebra.

In this section assume that A contains a subalgebra which is a large cycle.

In Lemma 1.k (for $k = 2, \dots, 10$) we suppose that the assumption expressed in Lemmas 1.j, where $j = 1, \dots, k-1$ is not valid.

1.1. Lemma. *Suppose that there is a connected component D of A such that D contains a large cycle C with $C \neq D$. Then A is retract reducible.*

P r o o f. Put $n = \operatorname{card} C$ and let $\{D_j\colon j \in J\}$ be the system of all connected components D_j of A which contain cycles C_j such that $\operatorname{card} C_j = n$ and $C_j \neq D_j$. Let j be an element of J. Then [6], Theorem and [5], 1.10 imply that D_j is retract reducible and that there are connected monounary algebras $E_{j0}, E_{j1}, \dots, E_{jn}$ which satisfy the following conditions:

(1) $E_{j0} \cong \underline{n}$;
(2) $E_{j1}, \dots, E_{jn}$ contain one-element cycles $\{e_{j1}\}, \dots, \{e_{jn}\}$;
(3) $D_j \in R\Big(\prod\limits_{i=0}^{n} E_{ji}\Big)$, i.e., there is a subalgebra $\overline{D_j}$ of $\prod\limits_{i=0}^{n} E_{ji}$, an isomorphism $\nu_j\colon \overline{D}_j \to D_j$ and a retraction homomorphism φ_j of $\prod\limits_{i=0}^{n} E_{ji}$ onto $\overline{D}_j$.

If $i \in \{1, \dots, n\}$, then let B_i be the disjoint sum of the algebras E_{ji} for each $j \in J$. Denote $A' = A - \bigcup\limits_{j \in J} D_j$. Further, let B_0 be the disjoint sum of the following algebras:

1) E_{j0} for each $j \in J$,
2) A'.

Then B_0 does not contain a subalgebra isomorphic to D, hence $A \notin R(B_0)$. Moreover, if $i \in \{1, \dots, n\}$, then all cycles of B_i have cardinality 1, thus $A \notin R(B_i)$. Therefore we have

(4) $A \notin R(B_i)$ for each $i \in \{0, \dots, n\}$.

Take a fixed $k \in J$ and consider the following subsets T_0, T of the direct product $\prod\limits_{i=0}^{n} B_i$. We put

$$T_0 = A' \times \prod_{i=1}^{n} \{e_{ki}\},$$
$$T = T_0 \cup \bigcup_{j \in J} \overline{D}_j.$$

Then $D_{j_1} \cap D_{j_2} = \emptyset$ if j_1, j_2 are distinct elements of J. Moreover, T is a subalgebra of $\prod_{i=0}^{n} B_i$. Define a mapping $\nu\colon T \to A$ as follows: If $x \in T$, then put

$$\nu(x) = \begin{cases} x(0) & \text{if } x \in T_0, \\ \nu_j(x) & \text{if } j \in J \text{ and } x \in \overline{D}_j. \end{cases}$$

In view of (3), ν is an isomorphism,

(5) $T \cong A$.

Now let us show that T is a retract of $\prod_{i=0}^{n} B_i$. Define a mapping φ of $\prod_{i=0}^{n} B_i$ onto T as follows.

Let K be a connected component of $\prod_{i=0}^{n} B_i$. Then by (A), $K(i)$ is a connected component of B_i for $i \in \{1, \ldots, n\}$. If $K(0) \subseteq A'$, then put $\varphi(x) = (x(0), e_{k1}, e_{k2}, \ldots, e_{kn})$ for each $x \in K$. Suppose that $K(0) \nsubseteq A'$. Then there is a uniquely determined $j \in J$ such that $K(0) = E_{j0}$. We distinguish two cases:

a) If $K(l) \subseteq E_{kl}$ for each $l \in \{1, \ldots, n\}$, then put $\varphi(x) = \varphi_j(x)$ for each $x \in K$.

b) Let there exist $l \in \{1, \ldots, n\}$ with $K(l) \nsubseteq E_{jl}$. Then

$K(0)$ contains a cycle with n elements,

$K(i)$ contains a cycle with 1 element for each $i \in \{1, \ldots, n\}$.

According to (C), K is a connected algebra with an n-element cycle. Since $\overline{D}_j \cong D_j$, $\overline{D_j}$ contains a subalgebra isomorphic to $\overline{n}$. Thus K can be homomorphically mapped into $\overline{D_j}$; take an arbitrary homomorphism φ of K into $\overline{D}_j$.

Using (3), $\varphi(x) = x$ for each $x \in T$ and φ is a homomorphism of $\prod_{i=1}^{n} B_i$ onto T, i.e., it is a retraction homomorphism. Therefore the assertion that A is retract reducible is obtained from (4) and (5). $\square$

1.2. Lemma. *Suppose that there is a connected component D of A such that D is a cycle with $p \cdot q$ elements, where $p, q \in \mathbb{N} - \{1\}$,* g.c.d. $(p, q) = 1$. *Then A is retract reducible.*

Proof. Let $\{D_j\colon j \in J\}$ be the system of all connected components of A which are cycles with $p \cdot q$ elements, $A' = A - \bigcup_{j \in J} D_j$. For $j \in J$ let D_{j1} be a cycle with p elements and D_{j2} a cycle with q elements. Then put

$$B_1 = \bigcup_{j \in J} D_{j1} \cup A',$$

$$B_2 = \bigcup_{j \in J} D_{j2} \cup A'.$$

Neither B_1 nor B_2 contains a cycle with cardinality $p \cdot q$, hence $A \notin R(B_1)$, $A \notin R(B_2)$. Consider the following subset T of $B_1 \times B_2$:

$$T = \bigcup_{j \in J} (D_{j1} \times D_{j2}) \cup \{(a, a)\colon\ a \in A'\}.$$

Then T is a subalgebra of $B_1 \times B_2$ and it is isomorphic to A (by multiplying cycles $\underline{p}$ and $\underline{q}$ we obtain a cycle isomorphic to $\underline{p \cdot q}$). We have to show that T is a retract of $B_1 \times B_2$.

(a) Let $j \in J$. Put $\varphi_{jj} = id_{D_{j1} \times D_{j2}}$. Further, if $k \in J - \{j\}$, then there exists an isomorphism φ_{jk} of $D_{j1} \times D_{k2}$ onto $D_{j1} \times D_{j2}$.

(b) If $x \in D_{j1} \times A'$, $j \in J$, then put $\varphi'(x) = (x(2), x(2))$. Then φ' is a homomorphism of $D_{j1} \times A'$ onto $\{(a, a)\colon\ a \in A'\}$.

(c) If $c \in A' \times B_2$, then put $\varphi''(x) = (x(1), (x(1))$. The mapping φ'' is a homomorphism of $A' \times B_2$ onto $\{(a, a)\colon\ a \in A'\}$.

For $x \in B_1 \times B_2$ now put

$$\varphi(x) = \begin{cases} \varphi_{jk}(x) & \text{if } x \in D_{j1} \times D_{k2}, \quad j, k \in J, \\ \varphi'(x) & \text{if } x \in D_{j1} \times A', \quad j \in J, \\ \varphi''(x) & \text{if } x \in A' \times B_2. \end{cases}$$

Then φ is a retraction endomorphism of $B_1 \times B_2$ onto T, therefore A is retract reducible. □

1.3. Lemma. *Suppose that A contains at least two connected components which are large cycles with the same cardinality. Then A is retract reducible.*

Proof. Let $n \in \mathbb{N} - \{1\}$ and let $\{D_j\colon\ j \in J\}$ be the system of all connected components of A such that D_j is a cycle with n elements, $\operatorname{card} J > 1$. Then $A' = A - \bigcup_{j \in J} D_j$ contains no cycle with cardinality n. We can suppose that $0 \notin J$; put $D_0 = \underline{n}$. Further let $\{c_j\}$, $j \in J$, be a system of mutually distinct one-element algebras. Let us define new algebras (as disjoint sums)

$$B_0 = D_0 \cup A',$$
$$B_j = D_j \cup \{c_j\} \text{ for each } j \in J.$$

Then

(1) $A \notin R(B_j)$ for each $j \in J \cup \{0\}$, since A is not isomorphic to a subalgebra of B_j for $j \in J \cup \{0\}$.

If we multiply $\underline{n}\times \prod\limits_{j\in J} D_j$, i.e., we multiply cycles with the same cardinality n, then the product consists of at least $2^{\operatorname{card} J}$ cycles with cardinality n according to (C)a). Thus there exists a subalgebra T_0 of $\prod\limits_{j\in J\cup\{0\}} D_j$ which is isomorphic to $\bigcup\limits_{j\in J} D_j$. Further,

$$T_1 = A' \times \prod_{j\in J}\{c_j\} \cong A'.$$

Then $T = T_0 \cup T_1$ is a subalgebra of $\prod\limits_{j\in J\cup\{0\}} B_j$ such that

(2) $T \cong A$.

It suffices to prove that T is a retract of $\prod\limits_{j\in J\cup\{0\}} B_j$. Consider a connected component C of $\prod\limits_{j\in J\cup\{0\}} B_j$. Then either

(3.1) $C(0) = D_0$

or

(3.2) $C(0) \subseteq A'$.

If (3.1) is valid, then $C \subseteq D_0 \times \prod\limits_{j\in J} D_j$, thus C is an n-element cycle. If $C \subseteq T_0$, we can map C identically. If $C \nsubseteq T_0$, then C can be homomorphically embedded into T_0.

Let (3.2) hold. Then C contains no cycle of cardinality n and $C \cap T_0 = \emptyset$. If $x \in C$, let $y \in T_1$ be such that $y(0) = x(0)$. Then the mapping $x \to y$ is a homomorphism of C into T_1 and, if $x \in T_1$, then x is mapped identically.

Hence there exists a homomorphism $\varphi\colon \prod\limits_{j\in J\cup\{0\}} B_j \to T$ such that $\varphi(x) = x$ for each $x \in T$, i.e.,

(4) $T \in R\Bigl(\prod\limits_{j\in J\cup\{0\}} B_j\Bigr)$.

Then A is retract reducible according to (1), (2) and (4). $\square$

1.4. Lemma. *Suppose that A contains connected components C_1 and C_2 which are cycles such that C_1 has cardinality k, C_2 has cardinality n and $1 < k < n$, k divides n. Then A is retract reducible.*

Proof. Take B_1 such that B_1 is a disjoint union of two cycles C_1', C_1'' and of $A' = A - C_1 - C_2$, where $\operatorname{card} C_1' = \operatorname{card} C_1'' = k$. Further, let B_2 be a disjoint union of two cycles, C_2 and $\{c\}$. We can suppose that the assumption of 1.3 fails to hold. Then A is not isomorphic to any subalgebra of B_1. Obviously,

(1) $A \notin R(B_1)$, $A \notin R(B_2)$,

because A' contains no cycle isomorphic to C_2.

Since k divides n, we get that $C_1'' \times C_2$ consists of k cycles with n elements, thus there is a subalgebra T_0 of $C_1'' \times C_2$ such that $T_0 \cong C_2$. Denote

$$T = T_0 \cup ((A' \cup C_1') \times \{c\}).$$

Then

(2) $T \cong A$.

Let D be a connected component of $B_1 \times B_2$. One of the following conditions is valid:

(3.1) $D \subseteq C_1'' \times C_2$,
(3.2) $D \subseteq C_1'' \times \{c\}$,
(3.3) $D \subseteq (C_1' \cup A') \times C_2$,
(3.4) $D \subseteq (C_1' \cup A') \times \{c\}$.

Let (3.1) hold. Then D is a cycle with n elements. Since $T_0 \subseteq C_1'' \times C_2$ is a cycle with n elements, too, thus either $D = T_0$ or $D \cap T_0 = \emptyset$. If $D \cap T_0 = \emptyset$, then (3.1) yields

$$D \cap T \subseteq (C_1'' \times C_2) \cap ((A_1' \cup C_1') \times \{c\}) = \emptyset.$$

If $D = T_0$, put $\varphi \restriction D = \mathrm{id}_D$. If $D \cap T = \emptyset$, then the relation $D \cong C_2$ (D and C_2 are n-element cycles) implies that there is an isomorphism ψ_D of D onto T_0; in this case put $\varphi \restriction D = \psi_D$.

Let (3.2) be valid. Then $D = C_1'' \times \{c\} \cong C_1' \times \{c\}$ is a cycle with k elements, $D \cap T_0 = \emptyset$. Analogously as above, $D \cap T = \emptyset$. There is an isomorphism $\eta_D \colon D \to C_1' \times \{c\} \subseteq T$; put $\varphi \restriction D = \eta_D$.

Suppose that (3.3) holds. Then

$$\begin{aligned} D \cap T &\subseteq [(C_1' \cup A') \times C_2] \cap [(C_1'' \times C_2) \cup ((A' \cup C_1') \times \{c\})] \\ &= [(C_1' \cup A') \times C_2] \cap (C_1'' \times C_2) = [(C_1' \cup A') \cap C_1''] \times C_2 = \emptyset. \end{aligned}$$

If $x \in D$, denote

$$\varphi(x) = (x(1), c).$$

Obviously, φ is a homomorphism of D into $(A' \cup C_1') \times \{c\} \subseteq T$.

Now let (3.4) be valid. Then $D \subseteq T$ and we set $\varphi \restriction D = \mathrm{id}_D$.

The mapping φ described for each connected component D of $B_1 \times B_2$ is a homomorphism of $B_1 \times B_2$ into T and such that $\varphi \restriction T = \mathrm{id}_T$. Thus T is a retract of $B_1 \times B_2$. In view of (1) and (2), A is retract reducible. $\square$

1.5. Lemma. *Suppose that A contains no connected component with a one-element cycle and that there is a connected component of A which contains no cycles. Then A is retract reducible.*

P r o o f. Let D be the union of all elements of A which belong to a cycle, $A' = A - D$. The assumption of 1.1 is not valid, thus $D \neq \emptyset$ is a subalgebra of A and $A' \neq \emptyset$ is a subalgebra of A, too. Take some new element c. Let B_2 be a disjoint union of a one-element cycle $\{c\}$ and of A', $B_1 = D$. Then A is not isomorphic to any subalgebra of B_1, A is not isomorphic to any subalgebra of B_2, thus

(1) $A \notin R(B_1)$, $A \notin R(B_2)$.

By (E), $B_1 \times A'$ consists of some copies of A'; take one of them and denote it by T_1. If we denote $T_2 = B_1 \times \{c\}$, then $D \cong T_2$. We have $A = D \cup A'$, $D \cong T_2$, $A' \cong T_1$, $T_1 \cap T_2 = \emptyset$, thus

(2) $A \cong T_1 \cup T_2$.

Obviously, $T_1 \cup T_2$ is a retract of $B_1 \times B_2$. Therefore A is retract reducible. □

1.6. Lemma. *Suppose that some of the following conditions is satisfied:*

(a) *there is a connected component of A which contains no cycles,*
(b) *the set A' of all elements of A which belong to connected components with one element cycles has at least two elements.*

Then A is retract reducible.

P r o o f. Let D be the set of all elements which belong to a large cycle, $A'' = A - (A' \cup D)$. Then A'' is the set of all elements of the connected components of A which do not contain a cycle. Let c be an arbitrary new element. Further, let B_1 be a disjoint sum of D and of a cycle $\{c\}$ and let B_2 be a disjoint sum of $A' \cup A''$ and of a cycle $\{c\}$. Consider the following subsets of $B_1 \times B_2$:

$$T_1 = D \times \{c\}, \quad T_2 = \{c\} \times (A' \cup A''), \quad T = T_1 \cup T_2.$$

Obviously,

(1) $A \cong T$.

Further, since (a) or (b) holds, we get that A is not a subalgebra of B_1, thus

(2) $A \notin R(B_1)$.

The algebra B_2 contains no large cycles, thus

(3) $A \notin R(B_2)$.

Let C be a connected component of $B_1 \times B_2$. By (A), the set $C(1)$ is a connected component of B_1 and $C(2)$ is a connected component of B_2. There are the following possibilities:

(4.1) $C(1) \subseteq D$, $\quad C(2) = \{c\}$,
(4.2) $C(1) = \{c\}$, $C(2) \subseteq A' \cup A''$,
(4.3) $C(1) \subseteq D$, $\quad C(2) \subseteq A' \cup A''$,
(4.4) $C(1) = \{c\}$, $C(2) = \{c\}$.

If (4.1) or (4.2) is valid, then $C \subseteq T$ and we can map C identically. If (4.3) or (4.4) holds, then C can be homomorphically mapped onto a one-element cycle, thus C can be homomorphically mapped into A', hence into T_2 as well. Therefore T is a retract of $B_1 \times B_2$ and (1)–(3) imply that A is retract reducible. □

1.7. Lemma. *Suppose that A contains a one-element cycle and that there are at least two connected components of A which are large cycles. Then A is retract reducible.*

Proof. The assumptions of 1.6 and 1.1 are not valid, thus A is a union of a one-element cycle $\{c\}$ and of some (at least two) large cycles. Let $\{D_j \colon j \in J\}$ be the system of all large cycles of A. For $j \in J$ put $B_j = D_j \cup \{c\}$. Further, let

$$T_j = \Big\{x \in \prod_{i \in J} B_i \colon\ x(j) \in B_j, x(i) = c \text{ for each } i \in J, i \neq j\Big\},$$

$$T_0 = \Big\{x \in \prod_{i \in J} B_i \colon\ x(i) = c \text{ for each } i \in J\Big\},$$

$$T = T_0 \cup \bigcup_{j \in J} T_j.$$

Obviously,

(1) $A \notin R(B_j)$ for each $j \in J$,
(2) $A \cong T$.

Further, analogously as above, each connected component C of $\prod_{j \in J} B_j$ either satisfies $C \subseteq T$ or can be homomorphically mapped onto a one-element cycle, i.e., onto T_0. Hence T is a retract of $\prod_{j \in J} B_j$ and A is retract reducible. □

1.8. Lemma. *Suppose that A consists of two cycles, one having p^n elements (p being a prime, $n \in \mathbb{N}$), the second having 1 element. Then A is retract irreducible.*

P r o o f. By way of contradiction assume that A is retract reducible. Then there are B_j, $j \in J$ and $T \subseteq \prod_{j\in J} B_j$ such that

(1) $T \cong A$,

(2) there is a retraction homomorphism of $\prod_{j\in J} B_j$ onto T,

(3) $A \notin R(B_j)$ for each $j \in J$.

By (1), $T = \{c\} \cup \{a, f(a), f^2(a), \dots, f^{k-1}(a)\}$, where $f(c) = c$, $k = p^n$, $f^k(a) = a$ and $f^l(a) \neq a$ for $1 < l < k$. Then, for $j \in J$,

$$f(c(j)) = (f(c))(j) = c(j),$$

i.e., B_j contains a one-element subalgebra. Further, if $j \in J$, then

$$f^k(a(j)) = (f^k(a))(j) = a(j),$$

hence $a(j)$ belongs to a cycle with cardinality k_j dividing k. Then

(4) $f^{k_j}(a(j)) = a(j)$.

Now suppose

(5) $k_j = k$ for some $j \in J$.

Then B_j contains a subalgebra isomorphic to A. Moreover, $A \in R(B_j)$, since a connected component of B_j containing $a(j)$ can be homomorphically embedded into the cycle $\{a(j), f(a(j)), \dots, f^k(a(j))\}$, and the other elements of B_j can be mapped onto $c(j)$. Thus (5) contradicts (3). Hence

(5′) $k_j < k$ for each $j \in J$.

If $j \in J$, we have

$$k_j = p^{n_j} \text{ for some } 0 \leqslant n_j < n,$$

thus $m = \text{l.c.m.}\{p^{n_j}\colon\ j \in J\} < p^n$. Then k_j divides m for each $j \in J$ and (4) implies

$$f^m(a(j)) = a(j).$$

Hence $f^m(a) = a$, $m < p^n$, which is a contradiction. □

1.9. Lemma. *Let the following conditions be valid:*

(a) *A consists of cycles C_i, $i \in I$, where* $\operatorname{card} C_i = p_i^{k_i}$, *$p_i \in \mathbb{P}$, $k_i \in \mathbb{N}$;*

(b) $\mathbb{P} - \{p_i\colon\ i \in I\} \neq \emptyset$.

Then A is retract reducible.

Proof. The assumptions of 1.3 and 1.4 are not satisfied, thus $p_i \neq p_k$ for each $i, k \in I$, $i \neq k$. Let $q \in \mathbb{P} - \{p_i \colon i \in I\}$. For $j \in \mathbb{N}$ denote by B_j the disjoint sum of A and of a cycle D_j with q^j elements. Since $p_i^{k_i}$ does not divide q^j, we obtain that D_j cannot be homomorphically mapped onto A, thus

(1) $A \notin R(B_j)$ for each $j \in \mathbb{N}$.

Denote

$$T = \left\{ x \in \prod_{j\in\mathbb{N}} B_j \colon \text{ there is } a \in A \text{ with } x(j) = a \text{ for each } j \in \mathbb{N} \right\}.$$

Obviously,

(2) $T \cong A$.

Let E be a connected component of $\prod_{j\in\mathbb{N}} B_j$. By (A), the set $E(j)$ is a connected component of B_j for each $j \in \mathbb{N}$. If $E \subseteq T$, then we can map E identically. Suppose that $E \nsubseteq T$. Then $E \cap T = \emptyset$. If $E(j) = D_j$ for each $j \in \mathbb{N}$, then E contains no cycle, thus E can be homomorphically mapped onto an arbitrary cycle, hence also into T. Suppose that there is $j \in \mathbb{N}$ with $E(j) \subseteq A$, i.e., there is $i \in \mathbb{N}$ such that $E(j)$ is a cycle with $p_i^{k_i}$ elements. Then E can be homomorphically mapped (by the natural projection) onto $E(j)$, i.e., onto a $p_i^{k_i}$-element cycle, thus E can be homomorphically mapped into T. Therefore we obtain

(3) T is a retract of $\prod_{j\in\mathbb{N}} B_j$

and, in view of (1) and (2), A is retract reducible. $\square$

1.9.1. Corollary (cf. also [6], Theorem). *If A is a large cycle, then A is retract reducible.*

Proof. It follows from 1.9 if $\operatorname{card} I = 1$. $\square$

1.10. Lemma. *Let* (a) *of* 1.9 *hold and suppose that the set $\{i \in I \colon k_i > 1\}$ is infinite. Then A is retract reducible.*

Proof. Let $J = \{i \in I \colon k_i > 1\}$. For $j \in J$ denote by B_j the disjoint sum of $A - C_j$ and of C'_j, where C'_j is a cycle with p_j elements. The assumption of 1.4 is not valid, thus $p_i \neq p_k$ for each $i, k \in I$, $i \neq k$, hence

(1) $A \notin R(B_j)$ for each $j \in J$.

Denote, for $i \in I - J$,

$$S_i = \left\{ x \in \prod_{j \in J} B_j \colon\ x(j) \in C_i \text{ for each } j \in J \right\},$$

and if $i \in J$, let

$$S_i = \left\{ x \in \prod_{j \in J} B_j \colon\ x(j) \in C_i \text{ for each } j \in J - \{i\}, x(i) \in C_i' \right\}.$$

First let $i \in I - J$. If K is a connected component of S_i, then $K(j) = C_i$ for each $i \in I$, i.e., $K(j)$ is a cycle with p_i elements. Then (C) implies that K is a cycle with the same number of elements. Hence S_i contains only cycles with p_i elements and there is a subalgebra T_i of S_i with $T_i \cong C_i$.

Now let $i \in J$. If K is a connected component of S_i, then $K(i) = C_i'$ and $K(j) = C_i$ for each $j \in I - J$. Since C_i' is a cycle with p_i elements and C_i is a cycle with $p_i^{k_i}$ elements, the assertion $(C)a)$ implies that K is a cycle with $p_i^{k_i}$ elements, i.e., $K \cong C_i$. Hence there is a subalgebra T_i of S_i with $T_i \cong C_i$, too.

Denote $T = \bigcup\limits_{i \in I} T_i$. As above, a connected component E of $\prod\limits_{j \in J} B_j$ satisfies either

(3.1) $E \subseteq T$

or

(3.2) $E \cap T = \emptyset$.

In the first case E can be mapped identically. Let (3.2) hold. If E contains no cycle, then there is a homomorphism of E into an arbitrary cycle, thus there exists a homomorphism of E into T. Suppose that E contains a cycle with m elements. This implies $E \cong \underline{m}$. By (D), $E(j)$ is a cycle with m_j elements, where m_j divides m (for each $j \in J$). Then

(4) $m = \text{l.c.m.}\{m_j \colon\ j \in J\}$.

First assume that the following condition is satisfied:

(5.1) there is $i \in I$ such that $p_i^{k_i}$ divides m.

Then $\underline{m}$ can be homomorphically embedded onto a cycle $\underline{p_i^{k_i}}$, thus E can be homomorphically embedded into T.

Now let (5.1) fail to hold. Then, for each $j \in J$, $m_j \notin \{p_i^{k_i} \colon\ i \in I\}$, therefore $m_j = p_j$ for each $j \in J$. Thus (4) yields

$$m = \text{l.c.m.}\{p_j \colon\ j \in J\},$$

a contradiction, because the set J is infinite.

Hence we have proved that there exists a retraction homomorphism of $\prod\limits_{j \in J} B_j$ onto T and according to (1) and (2) we obtain that A is retract reducible. $\square$

1.11. Lemma. *Suppose that the following conditions are satisfied:*

(a) *A consists of cycles C_i, $i \in I$, where* $\operatorname{card} C_i = p_i^{k_i}$, $p_i \in \mathbb{P}$, $k_i \in \mathbb{N}$ *and* $p_i \neq p_k$ *for each* $i, k \in I$, $i \neq k$;

(b) $\{p_i \colon i \in I\} = \mathbb{P}$;

(c) $\{i \in I \colon k_i > 1\}$ *is finite.*

Then A is retract irreducible.

P r o o f. By way of contradiction, assume that A is retract reducible. There are $B_j, j \in J$ and $T \subseteq \prod\limits_{j \in J} B_j$ such that

(1) $T \cong A$,

(2) there is a retraction homomorphism of $\prod\limits_{j \in J} B_j$ onto T,

(3) $A \notin R(B_j)$ for each $j \in J$.

For $i \in I$ we denote by D_i the cycle of T which corresponds to C_i under the isomorphism (1).

Consider $j \in J$. If $i \in I$, then by (D), $D_i(j)$ is a cycle whose cardinality divides $\operatorname{card} C_i$, i.e. $\operatorname{card} D_i(j) = p_i^{n_{ij}}$, where $0 \leqslant n_{ij} \leqslant k_i$. There are two possibilities:

(4.1) $n_{ij} = k_i$ for each $i \in I$,

(4.2) there is $i \in I$ with $n_{ij} < k_i$.

Let (4.1) hold. Then there is a subalgebra of B_j which is isomorphic to A. In view of (3) we get that

(5) there exists a cycle E_j in B_j with q_j elements such that if $i \in I$, then $p_i^{k_i}$ does not divide q_j.

If (4.2) is valid, we can put $q_j = p_i^{n_{ij}}$ and then (5) is valid, too.

According to (b), there is $I_1 \subseteq I$ with $I_1 \neq \emptyset$ such that

(6) $q_j = \prod\limits_{i \in I_1} p_i^{\alpha_i}$ for some numbers $\alpha_i \in \mathbb{N}$.

Let $i \in I_1$. Then p_i divides q_j. In view of (5) we obtain that $p_i^{k_i}$ does not divide q_j, thus $k_i > 1$. By (c), the set I_1 is then finite. Further, $\alpha_i < k_i$ for each $i \in I_1$. Hence

(7) $q_j = \prod\limits_{i \in I_1} p_i^{\alpha_i}$, $1 \leqslant \alpha_i < k_i$ for each $i \in I_1$ and I_1 is finite.

For $j \in J$ take an arbitrary element $z_j \in E_j$ and let $z \in \prod\limits_{j \in J} B_j$ be such that $z(j) = z_j$. Denote

(8) $m = \text{l.c.m.}\{q_j \colon j \in J\}$;

it exists in view of (c) and (7). Then z belongs to a cycle E with m elements. From (2) and (1) we get that

$$p_i^{k_i} \text{ divides } m \text{ for some } i \in I.$$

According to (7) and (8) we have a contradiction. Therefore A is retract irreducible. $\square$

1.12. Proposition. *Suppose that A contains a subalgebra which is a large cycle. Then A is retract irreducible if and only if A satisfies the assumption of* 1.8 *or of* 1.11.

Proof. If the assumptions of 1.8 or of 1.11 are satisfied, then A is retract irreducible. Conversely, let A be retract irreducible. By 1.1, each connected component of A containing a large cycle is a cycle and, in view of 1.2, with p^k elements, where $p \in P$, $k \in N$. From 1.3 and 1.4 it follows that if C_1 and C_2 are distinct large cycles of A, then there are distinct primes p, q and $k, l \in N$ such that $\operatorname{card} C_1 = p^k$, $\operatorname{card} C_2 = q^l$. If there is a connected component of A which contains no cycles, then A is retract reducible in view of 1.6(a), a contradiction. Suppose that each connected component of A contains a cycle. By 1.6(b), the set M of all elements of A which belong to no large cycle has at most one element. If $\operatorname{card} M = 1$, then 1.7 implies that the assumption of 1.8 is valid. Let $M = \emptyset$. Then 1.9 and 1.10 imply that the assumption of 1.11 is satisfied. $\square$

2. Small cycles

In this section assume that A contains no connected component with a large cycle and that A contains an element x_0 with $f(x_0) = x_0$.

2.1. Lemma. *Suppose that there is a connected component D of A which contains a cycle $\{d\}$ and such that $D \neq \{d\}$. Further, let A be not connected. Then A is retract reducible.*

Proof. Let $\{D_j \colon j \in J\}$ be the system of all connected components of A which contain a cycle (by the assumption, a one-element cycle). We can suppose that $1 \in J$ and $d \in D_1$. Denote $A' = A - \bigcup\limits_{j \in J} D_j$. Then either

(1.1) $\operatorname{card} J > 1$

or

(1.2) $A' \neq \emptyset$.

Let $c \notin A$ and let $\{c\}$ be a cycle. Denote by D_0 a disjoint union of $\{c\}$ and A'. Put $B_1 = D_1$. If $j \in J - \{1\}$, let B_j be the disjoint union of $\{c\}$ and D_j. Since A' contains no cycles, the assumption yields that $A \notin R(B_0)$. Obviously, $A \notin R(B_1)$. If

(1.1) is valid, then A is not isomorphic to a subalgebra of B_j for $j \in J - \{1\}$, hence $A \notin R(B_j)$. If (1.2) holds, then $A \notin R(B_j)$ for $j \in J - \{1\}$ as well. We have got

(2) $A \notin R(B_j)$ for each $j \in J \cup \{0\}$.

Consider the direct product $B = \prod\limits_{j \in J \cup \{0\}} B_j$ and denote

$$T_0 = \{x \in B\colon\ x(0) \in A', x(1) = d, x(j) = c \text{ for each } j \in J - \{1\}\},$$
$$T_1 = \{x \in B\colon\ x(1) \in D_1, x(j) = c \text{ for each } j \in J \cup \{0\}, j \neq 1\}$$

and, for $k \in J - \{1\}$,

$$T_k = \{x \in B\colon\ x(k) \in D_k, x(1) = d, x(j) = c \text{ for each } j \in (J \cup \{0\}) - \{k, 1\}\}.$$

Then $T_0 \cong A'$, $T_1 \cong D_1$, $T_k \cong D_k$ for each $k \in J - \{1\}$. Further, let $T = \bigcup\limits_{k \in J \cup \{0\}} T_k$. We obviously have

(3) $T \cong A$.

Let $\overline{d} \in T_1$ be such that $\overline{d}(1) = d$, $\overline{d}(j) = c$ for each $j \in J \cup \{0\} - \{1\}$.

Let K be a connected component of $\prod\limits_{j \in J \cup \{0\}} B_j$. Then (A) implies that $K(j)$ is a connected component of B_j for each $j \in J \cup \{0\}$, hence $K(1) = B_1$.

We are going to define a retraction homomorphism of B into T. Let us describe how to map the connected component K.

a) First suppose $K(0) \subseteq A'$. If $j \in J \cup \{0\} - \{1\}$, then $K(0) \cap T_j(0) \subseteq A' \cap \{c\} = \emptyset$, thus

(4) $K \cap T_j = \emptyset$ for each $j \in J \cup \{0\} - \{1\}$.

If $K(j) = \{c\}$ for each $J - \{1\}$, $x \in K$, then define $\varphi(x) \in T_0$ by the formula

$$(\varphi(x))(j) = \begin{cases} x(j) & \text{if } \ j \in J \cup \{0\} - \{1\}, \\ d & \text{if } \ j = 1. \end{cases}$$

Let there be $k \in J - \{1\}$ with $K(k) = D_k$. Then

$$K(k) \cap T_1(k) \subseteq D_k \cap \{c\} = \emptyset,$$

hence $K \cap T_1 = \emptyset$ and (4) implies $K \cap T = \emptyset$. In this case put $\varphi(x) = \overline{d}$ for each $x \in K$.

The mapping $\varphi \restriction K$ is a homomorphism of K into T.

b) Now suppose that $K(0) = \{c\}$ and that there are $j_1, j_2 \in J - \{1\}$, $j_1 \neq j_2$, such that

(5) $K(j_1) = D_{j_1}, \quad K(j_2) = D_{j_2}$.

We have

$$K(j_1) \cap T_k(j_1) = D_{j_1} \cap \{c\} = \emptyset \text{ for each } k \in J \cup \{0\} - \{j_1\},$$
$$K(j_2) \cap T_k(j_2) = D_{j_2} \cap \{c\} = \emptyset \text{ for each } k \in J \cup \{0\} - \{j_2\},$$

which yields

$$K \cap T_k = \emptyset \text{ for each } k \in J \cup \{0\},$$

i.e., $K \cap T = \emptyset$. Define $\varphi(x) = \overline{d}$ for each $x \in K$. Then $\varphi \restriction K$ is a homomorphism of K into T.

c) Suppose that $K(0) = \{c\}$ and that there is $j_0 \in J - \{1\}$ such that $K(j_0) = D_{j_0}$ and $K(j) = \{c\}$ for each $j \in J - \{1, j_0\}$. If $x \in K$, then define

$$(\varphi(x))(j) = \begin{cases} d & \text{if } j = 1 \\ x(j) & \text{otherwise.} \end{cases}$$

Then $\varphi(x) \in T_{j_0}$ and φ is a homomorphism of K into T.

From the definition of φ it follows that if $x \in T$, then $\varphi(x) = x$. Therefore φ is a retraction homomorphism of B onto T and, in view of (2) and (3), A is retract reducible. $\square$

2.2. Lemma. *Suppose that there are at least two one-element connected components of A and that* card $A > 2$. *Then A is retract reducible.*

Proof. Let $\{D_j\colon\ j \in J\}$ be the system of all one-element connected components $\{d_j\}$ of A, $A' = A - \bigcup\limits_{j \in J} D_j$. By 2.1 we can suppose that A' contains no subalgebra which is a cycle. Let $\{c\}$ be a new cycle. We set

$$B_0 = \{c\} \cup A',$$
$$B_j = \{c\} \cup \{d_j\} \text{ for each } j \in J.$$

By the assumption,

(1) $A \notin R(B_j)$ for each $j \in J \cup \{0\}$.

Denote

$$T_0 = \left\{x \in \prod_{j \in J \cup \{0\}} B_j\colon\ x(0) \in A', x(j) = c \text{ for each } j \in J\right\}$$

and, if $k \in J$,

$$T_k = \left\{x \in \prod_{j \in J \cup \{0\}} B_j\colon\ x(k) = d_k, x(j) = c \text{ for each } j \in J \cup \{0\}, j \neq k\right\}.$$

Put

$$T = \prod_{j \in J \cup \{0\}} T_j.$$

Obviously,

(2) $T \cong A$.

Let K be a connected component of $\prod\limits_{j \in J \cup \{0\}} B_j$. If $K \subseteq T$, put $\varphi(x) = x$ for each $x \in K$; if $K \cap T = \emptyset$, $x \in K$, define $(\varphi(x))(j) = c$ for each $j \in J \cup \{0\}$. The relation

(3) $K \nsubseteq T$, $K \cap T \neq \emptyset$

implies a contradiction since if (3) is valid, then we have either

(4.1) $K \cap T_j \neq \emptyset$ for some $j \in J$, $K \nsubseteq T$,

or

(4.2) $K \cap T_0 \neq \emptyset$, $K \nsubseteq T$.

If (4.1) is valid, then

$$\emptyset \neq K(l) \cap T_j(l) = \begin{cases} K(l) \cap \{c\} & \text{if } \ l \in J \cup \{0\} - \{j\}, \\ K(l) \cap \{d_j\} & \text{if } \ l = j, \end{cases}$$

and according to (A),

$$K(l) = \begin{cases} \{c\} & \text{if } \ l \in J \cup \{0\} - \{j\}, \\ \{d_j\} & \text{if } \ l = j. \end{cases}$$

Then $K = T_j \subseteq T$, a contradiction to (4.1). Further, suppose that (4.2) holds. If $l \in J$, then $\emptyset \neq K(l) \cap T_0(l) = K(l) \cap \{c\}$, hence $K(l) = \{c\}$. Next, $\emptyset \neq K(0) \cap T_0(0) \subseteq K(0) \cap A'$ and since $K(0)$ is connected, we obtain that $K(0) \subseteq A'$. Therefore $K \subseteq T_0 \subseteq T$, which is a contradiction to (4.2).

Hence φ is a retraction endomorphism of $\prod\limits_{j \in J \cup \{0\}} B_j$ onto T. According to (1) and (2), A is retract reducible. $\square$

2.3. Lemma. *Let A consist of two one-element cycles. Then A is retract irreducible.*

P r o o f. Suppose that A is retract reducible. Then there are B_j, $j \in J$ and $T \subseteq \prod\limits_{j \in J} B_j$ such that

(1) $T \cong A$,

(2) there is a retraction homomorphism of $\prod\limits_{j \in J} B_j$ onto T,

(3) $A \notin R(B_j)$ for each $j \in J$.

According to (1), $T = \{c, d\}$, where $\{c\}$ and $\{d\}$ are distinct cycles. If $j \in J$, then $\{c(j)\}$ and $\{d(j)\}$ are cycles of B_j. From (3) it follows that B_j can contain at most one one-element cycle, thus $c(j) = d(j)$. But, since $c \neq d$, there exists $i \in J$ with $c(i) \neq d(i)$, a contradiction. $\square$

2.4.1. Remark. If A does not satisfy the assumptions of 2.1–2.3, then either

(a) $A = \{c\} \cup A'$, where $\{c\}$ is a cycle of A and A' contains no connected component with a cycle,

or

(b) A is connected.

2.4.2. Lemma. *Let* (a) *of* 2.4.1 *hold. If A' is retract reducible, then A is retract reducible.*

Proof. Suppose that A' is retract reducible. Then there are B'_j, $j \in J$ and $T' \subseteq \prod_{j \in J} B'_j$ such that

(1′) $T' \cong A'$,

(2′) there is a retraction homomorphism φ' of $\prod_{j \in J} B'_j$ onto T',

(3′) $A' \notin R(B'_j)$ for each $j \in J$.

We can suppose that $c \notin B'_j$ for each $j \in J$. If $j \in J$, then put

$$B_j = \{c\} \cup B'_j.$$

Let $\overline{c} \in \prod_{j \in J} B_j$ be such that $\overline{c}(j) = c$ for each $j \in J$ and let

$$T = T' \cup \{\overline{c}\}.$$

Then (1′) implies

(1) $T \cong A$

and (3′) yields

(3) $A \notin R(B_j)$ for each $j \in J$.

Let $x \in \prod_{j \in J} B_j$. Define

$$\varphi(x) = \begin{cases} \varphi'(x) & \text{if } x \in \prod_{j \in J} B'_j, \\ \overline{c} & \text{otherwise.} \end{cases}$$

Then $\varphi(x) = x$ for each $x \in T$ and φ is a homomorphism of $\prod_{j \in J} B_j$ onto T according to (2′), i.e.,

(2) there is a retraction homomorphism φ of $\prod_{j \in J} B_j$ onto T.

Hence (1)–(3) imply that A is retract reducible. $\square$

3. Monounary algebras without cycles

Throughout this section suppose that no subalgebra of A is a cycle.

3.1. Lemma. *Assume that there is $x \in A$ with $s_f(x) = \infty$. Then A is retract reducible.*

Proof. Let $\{D_j : j \in J\}$ be the system of all connected components of A such that, for each $j \in J$, there is $x_j \in D_j$ with $s_f(x_j) = \infty$. Let $j \in J$. According to [6], Theorem and [5], 2.8, there exist a system of distinct primes $\{p_i : i \in \mathbb{Z}\}$ and a system of algebras $\{B_{ji} : i \in \mathbb{Z}\}$ with the following properties:

(1) if $i \in Z$, then B_{ji} is connected and contains a cycle with p_i elements,

(2) there is a subalgebra T_j of $\prod_{i \in \mathbb{Z}} B_{ji}$ such that $T_j \cong D_j$,

(3) there is a retraction homomorphism φ_j of $\prod_{i \in \mathbb{Z}} B_{ji}$ onto T_j.

Denote $A' = A - \bigcup_{j \in J} D_j$. If $i \in \mathbb{Z}$, then put

$$B_i = \bigcup_{j \in J} B_{ji} \cup A'.$$

We can suppose that $B_{j_1 i_1} \cap B_{j_2 i_2} = \emptyset$ for each distinct $(i_1, j_1), (i_2, j_2) \in \mathbb{Z} \times J$. The algebra A is not isomorphic to any subalgebra of B_i for $i \in \mathbb{Z}$, thus

(4) $A \notin R(B_i)$ for each $i \in \mathbb{Z}$.

Further, denote

$$T_0 = \left\{ x \in \prod_{i \in \mathbb{Z}} B_i : (\exists a \in A')(\forall i \in \mathbb{Z})(x(i) = a) \right\},$$

$$T = \bigcup_{j \in J \cup \{0\}} T_j.$$

Then (2) implies

(5) $T \cong A$.

Let K be a connected component of $\prod\limits_{i\in\mathbb{Z}} B_i$. Suppose that K contains a cycle with m elements. Then $K(i)$ contains a cycle for each $i \in \mathbb{Z}$, too. Since $K(i) \subseteq B_i$, we get that $K(i)$ contains a cycle with p_i elements, hence

(6) p_i divides m for each $i \in \mathbb{Z}$.

We have $p_i \neq p_k$ for each $i, k \in \mathbb{Z}$, $i \neq k$, thus (6) yields a contradiction. Therefore K contains no cycle. Consider the following conditions:

(7.1) there is $j \in J$ such that $K \subseteq \prod\limits_{i\in\mathbb{Z}} B_{ji}$,

(7.2) $K \subseteq T_0$,

(7.3) neither (7.1) nor (7.2) is valid.

If (7.1) holds and $x \in K$, then put $\varphi(x) = \varphi_j(x)$. If (7.2) holds, define $\varphi(x) = x$ for each $x \in K$. Let (7.3) hold. First suppose

(8.1) $K(i) \nsubseteq A'$ for each $i \in \mathbb{Z}$.

Since (7.1) is not valid, there are $k, l \in \mathbb{Z}$, $k \neq l$ and $j_1, j_2 \in J$, $j_1 \neq j_2$ such that $K(k) = B_{j_1 k}$, $K(l) = B_{j_2 l}$. Then

$$K(k) \cap T_0(k) \subseteq B_{j_1 k} \cap A' = \emptyset,$$

therefore

(9.1) $K \cap T_0 = \emptyset$.

Further,

$$K(l) \cap T_{j_1}(l) \subseteq B_{j_2 l} \cap B_{j_1 l} = \emptyset,$$

hence

(9.2) $K \cap T_{j_1} = \emptyset$.

If $j \in J - \{j_1\}$, then

$$K(k) \cap T_j(k) \subseteq B_{j_1 k} \cap B_{jk} = \emptyset$$

and

(9.3) $K \cap T_j = \emptyset$.

Thus (9.1)–(9.3) imply

(9) $K \cap T = \emptyset$.

Since K contains no cycle, it can be homomorphically mapped into $\underline{\mathbb{Z}}$, hence it can be homomorphically mapped into an arbitrary T_j, $j \in J$, thus K can be homomorphically mapped into T, too.

Now suppose

(8.2) there are $i, l \in \mathbb{Z}$ with $K(i) \subseteq A'$ and $K(l) = B_{jl}$ for some $j \in J$.

Analogously as in the case dealing with the condition (8.1), we get $K \cap T = \emptyset$ and K can be homomorphically mapped into T.

Finally, suppose

(8.3) $K(i) \subseteq A'$ for each $i \in \mathbb{Z}$.

If $x \in K$, then put $\varphi(x) = y$, where

$$y(i) = x(0) \text{ for each } i \in \mathbb{Z}.$$

Then $y \in T_0$ and φ is a homomorphism of K into T.

Thus φ is a retraction homomorphism of $\prod_{i \in \mathbb{Z}} B_i$ onto T. Therefore (4) and (5) imply that A is retract reducible. $\square$

Suppose that X is a connected monounary algebra which satisfies the following condition:

($*$) $s_f(x) \neq \infty$ for each $x \in X$ and there are distinct elements $a, b \in X$ with $f(a) = f(b)$.

In [5], 3.4, to each such X a uniquely defined positive integer m was assigned. We slightly modify the notation from [5] and write $m(X)$ instead of m. Let us recall the definition.

According to ($*$), the set $L = \{x \in A\colon\ f^{-1}(x) = \emptyset\}$ is nonempty and

$$\{k \in \mathbb{N}\colon\ \operatorname{card} f^{-1}(f^k(x)) > 1\} \neq \emptyset \text{ for each } x \in L.$$

If $x \in L$, then denote

$$k(x) = \min\, \{k \in \mathbb{N}\colon\ \operatorname{card} f^{-1}(f^k(x)) > 1\}$$

and put

$$m(X) = \min\, \{k(x)\colon\ x \in L\}.$$

3.2. Lemma. *Suppose that $s_f(x) \neq \infty$ for each $x \in A$ and that there are distinct elements $a, b \in A$ with $f(a) = f(b)$. Then A is retract reducible.*

Proof. Let $m = \min\{m(D)\colon\ D$ is a connected component of A and D satisfies $(*)\}$. Further, let $\{D_j\colon\ j \in J\}$ be the system of all connected components of A which satisfy ($*$) and such that $m(D_j) = m$ for each $j \in J$. Let $j \in J$. According to [5], 3.4–3.7, D_j is retract reducible and there exist a set I_j and algebras B_{ji} for each $i \in I_j \cup \{0\}$ with the following properties:

(1) B_{j0} satisfies ($*$) and $m(B_{j0}) > m$,

(2) if $i \in I_j$, then B_{ji} is isomorphic to $(\{0, 1, \dots, m\}, f)$, where $f(0) = 1$, $f(1) = 2, \dots, f(m-1) = f(m) = m$,

(3) there is a subalgebra T_j of $\prod_{i \in I_j \cup \{0\}} B_{ji}$ such that $T_j \cong D_j$,

(4) there is a retraction homomorphism φ_j of $\prod_{i \in I_j \cup \{0\}} B_{ji}$ onto T_j.

Denote $A' = A - \bigcup_{j \in J} D_j$, $I = \bigcup_{j \in J} I_j$. We can suppose that the sets I_j for $j \in J$ are mutually disjoint. If $j \in J$, $i \in I - I_j$, then let B_{ji} be a one-element cycle $\{c\}$. If $i \in I \cup \{0\}$, then put

$$B_i = \bigcup_{j \in J} B_{ji} \cup A'$$

and let

$$B = \prod_{i \in I \cup \{0\}} B_i.$$

Obviously, B_0 contains no cycles. We have $m(X) > m$ for each connected component X of A' in view of the choice of m, thus (1) implies that $m(X) > m$ for each connected component X of B_0, hence

(5) $A \notin R(B_0)$.

Similarly, (2) implies that if $i \in I$, then no connected component of B_i satisfies $(*)$, thus A is not isomorphic to any subalgebra of B_i and

(6) $A \notin R(B_i)$ for each $i \in I$.

Further, denote

$$T_0' = \left\{ x \in \prod_{i \in I \cup \{0\}} B_i \colon\ (\exists a \in A')(\forall i \in I \cup \{0\})(x(i) = a) \right\}.$$

If $j \in J$, then we put

$$\begin{gathered} T_j' = \{x \in B \colon (\exists t \in T_j)(x(i) = t(i) \text{ for each } i \in I_j \cup \{0\}, \\ x(i) = c \text{ for each } i \in I - I_j)\}; \end{gathered}$$

obviously,

$$T_j' \cong T_j.$$

Let

$$T = \bigcup_{j \in J \cup \{0\}} T_j'.$$

According to (3) and in view of the relation $A' \cong T_0'$ we obtain

(7) $T \cong A$.

We are going to define a retraction homomorphism of B onto T. We can suppose that $I \cup \{0\}$ is well-ordered. Let $x \in B$. If there is $i \in I \cup \{0\}$ such that $x(i) \in A'$ and

$$i_0 = \min \{i \in I \cup \{0\} \colon x(i) \in A'\},$$

then define $\varphi(x) = y \in T'_0$, where

$$y(i_0) = x(i_0).$$

Now let $x(i) \notin A'$ for each $i \in I \cup \{0\}$. Then $x(0) \in B_{k0}$ for some $k \in J$. There is $y \in T'_k$ with

$$y(0) = x(0);$$

put $\varphi(x) = y$.

It can be verified that φ is a retraction homomorphism of B onto T. Then A is retract reducible according to (5), (6) and (7). □

3.3. Lemma. *Let A not satisfy the assumptions of* 3.1 *and* 3.2. *If A is not connected, then A is retract reducible.*

Proof. Suppose that A is not connected. The assumption implies that A consists of connected components D_j, $j \in J$ such that if $j \in J$, then $D_j \cong \underline{\mathbb{N}}$. Let $0 \notin J$ and $D_0 \cong \underline{\mathbb{N}}$. Denote by B_0 a disjoint sum of D_0 and of a one-element cycle $\{c\}$. Put $B_j = D_j$ for $j \in J$. Obviously,

(1) $A \notin R(B_j)$ for each $j \in J \cup \{0\}$.

Then $\prod\limits_{j \in J \cup \{0\}} B_j$ consists of connected components isomorphic to $\underline{\mathbb{N}}$ and there is a subalgebra T of $\prod\limits_{j \in J \cup \{0\}} B_j$ such that $T \cong A$. This yields that $A \in R\Big(\prod\limits_{j \in J \cup \{0\}} B_j\Big)$ and that A is retract reducible. □

3.4. Proposition. *Suppose that no subalgebra of A is a cycle. Then A is retract irreducible if and only if $A \cong \underline{\mathbb{N}}$.*

Proof. If $A \cong \underline{\mathbb{N}}$, then [6], Thm. (b) and [5], (R1) imply that A is retract irreducible. Conversely, let A be retract irreducible. By 3.1, we get

(1) $s_f(x) \neq \infty$ for each $x \in A$.

Then 3.2 yields

(2) $(\forall a, b \in A)(f(a) = f(b) \Rightarrow a = b)$.

According to 3.3, A is connected, thus (1) and (2) imply that $A \cong \underline{\mathbb{N}}$. □

4. Main result

4.1. Notation. Let $n \in \mathbb{N}$. We will denote by S_n the following algebra (S_n, f) (all elements written here are mutually distinct):

$$\begin{aligned} S_n &= \{a_i^n \colon\ i \in \mathbb{N}\} \cup \{b_1^n, \dots, b_n^n\} \cup \{c_{jk}^n \colon\ j \in \mathbb{N}, k \in \{1, \dots, j\}\}, \\ f(a_i^n) &= a_{i+1}^n \text{ for each } i \in \mathbb{N}, \\ f(b_i^n) &= b_{i+1}^n \text{ for each } i \in \{1, \dots, n-1\}, \quad f(b_n^n) = a_{n+1}^n, \\ f(c_{jk}^n) &= c_{j,k+1}^n \text{ for each } j \in \mathbb{N}, k \in \{1, \dots, j-1\}, \\ f(c_{jj}^n) &= b_1^n \text{ for each } j \in \mathbb{N}. \end{aligned}$$

(Cf. Fig. 1 for the case $n = 2$).

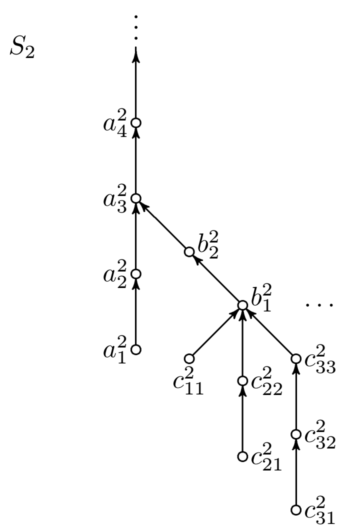

Fig. 1

4.2. Lemma. *If $n \in \mathbb{N}$, then $\underline{\mathbb{N}} \notin R(S_n)$.*

P r o o f. The assertion is a corollary of [5], 3.1. $\square$

4.3. Lemma. *There is a connected component K of $\prod\limits_{n\in\mathbb{N}} S_n$ such that $\underline{\mathbb{N}} \in R(K)$.*

P r o o f. If $k \in \mathbb{N}$, then denote by $\overline{a}_k$ the element of $\prod\limits_{n\in\mathbb{N}} S_n$ such that $\overline{a}_k(n) = a_k^n$ for each $k \in \mathbb{N}$.

Let K be a connected component of $\prod_{n\in\mathbb{N}} S_n$ such that $\overline{a_1} \in K$. We are going to show that

(1) for each $k \in \mathbb{N}$, $f^{-(k+1)}(f^k(\overline{a_1})) = \emptyset$;

then [5], 3.1 implies that $\underline{\mathbb{N}} \in R(K)$. By way of contradiction, let $k \in \mathbb{N}$ be such that $f^{-(k+1)}(f^k(\overline{a_1}) \neq \emptyset$, $y \in f^{-(k+1)}(f^k(\overline{a_1}))$. We have

$$f^{k+1}(y) = f^k(\overline{a_1}) = \overline{a_{k+1}},$$

thus

$$f^{k+1}(y(k+1)) = (f^{k+1}(y))(k+1) = a_{k+1}^{k+1},$$
$$y(k+1) \in f^{-(k+1)}(a_{k+1}^{k+1}) = \emptyset,$$

a contradiction. □

4.4. Lemma. *Suppose that A is a monounary algebra with two connected components, where one is a one-element cycle and the other is isomorphic to $\underline{\mathbb{N}}$. Then A is retract reducible.*

P r o o f. Let $\{c\}$ be a cycle and denote by B_n (for $n \in \mathbb{N}$) a disjoint sum of $\{c\}$ and of S_n. Then 4.2 implies

(1) $A \notin R(S_n)$ for each $n \in \mathbb{N}$.

Let $\overline{c} \in \prod_{n\in\mathbb{N}} B_n$ be such that $\overline{c}(n) = c$ for each $n \in \mathbb{N}$. By 4.3 there is a connected component K of $\prod_{n\in\mathbb{N}} S_n$ with $\underline{\mathbb{N}} \in R(K)$, i.e., there exist $T_0 \subseteq K$ with $T_0 \cong \mathbb{N}$ and a retraction homomorphism φ_0 of K onto T_0. Denote $T = T_0 \cup \{\overline{c}\}$. Hence

(2) $T \cong A$.

If C is a connected component of $\prod_{n\in\mathbb{N}} S_n$, then either $C \subseteq K$ or $C \cap K = \emptyset$. In the first case put $\varphi(x) = \varphi_0(x)$ for each $x \in C$; in the latter case, if $x \in C$, put $\varphi(x) = \overline{c}$. Then φ is a retraction homomorphism of $\prod_{n\in\mathbb{N}} B_n$ onto T and, according to (1) and (2), A is retract reducible. □

4.5. Theorem. *Let A be a monounary algebra. Then A is retract irreducible if and only if one of the following conditions is satisfied:*

(a) *A is a disjoint union of two cycles, one of them having p^k elements ($p \in \mathbb{P}$, $n \in \mathbb{N}$), the other having* 1 *element;*

(b) *A satisfies the assumption of* 1.11*;*

(c) *A contains a cycle $\{c\}$, is connected and $f(a) = f(b)$ for $a, b \in A - \{c\}$ implies $a = b$;*

(d) *A is a disjoint union of two 1-element cycles;*

(e) $A \cong \underline{\mathbb{N}}$.

P r o o f. If A satisfies (a) or (b), then A is retract irreducible in view of 1.8 and 1.11. If A satisfies (c) or (e), then [5] (R), (R1) and [6], Theorem imply that A is retract irreducible. If A satisfies (d), then A is retract irreducible by 2.3.

Now suppose that A is retract irreducible.

a) Let A contain a subalgebra which is a large cycle. Then 1.12 yields that either (a) or (b) is valid.

Assume that A contains no subalgebra which is a large cycle.

b) Let there be $x_0 \in A$ with $f(x_0) = x_0$. If A is connected, then (c) holds according to [4]. Suppose that A is not connected. By 2.1, each connected component with a one-element cycle is a cycle and 2.2 implies that either

(1.1) A consists of two 1-element cycles

or

(1.2) A contains only one 1-element cycle $\{c\}$.

The condition (1.1) is (d). Let (1.2) be valid. Then 2.4.2 yields that $A - \{c\}$ is retract irreducible. By 3.4, $A - \{c\} \cong \underline{N}$. But then 4.4 implies that A is retract reducible, which is a contradiction.

Assume that A contains no subalgebra which is a cycle. We obtain according to 3.4 that $A \cong \underline{N}$, i.e., (e) is satisfied. □

4.6. Corollary. a) *If A is a retract irreducible monounary algebra, then* $\operatorname{card} A \leqslant \aleph_0$.

b) *The number of non-isomorphic types of retract irreducible monounary algebras is equal to $\aleph_0$.*

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