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ON CONVEXLY ISOMORPHIC POSETS

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Systems of intervals, particularly of lattices, have been investigated by many authors, see, e.g., [2]–[15]. In [15], Slavík studied the relation between two lattices having isomorphic lattices of intervals. A more general situation is investigated in [14], where couples of posets, which have isomorphic posets of intervals, are dealt with. In the cited paper of V. Slavík, it is proved that the conditions "to have isomorphic lattices of intervals" and "to have isomorphic lattices of convex sublattices" are, for lattices, equivalent. In connection with this fact a natural question arises, which posets have isomorphic posets of convex subsets and whether the conditions "to have isomorphic posets of intervals" and "to have isomorphic posets of convex subsets," concerning posets, are dependent. It is easy to see that the answer to the latter question is negative. For instance, the $2n$-element crown and the $2n$-element antichain have isomorphic posets of convex subsets, while their posets of intervals are not isomorphic. On the other hand, if $\mathbb{A}$ is any partially ordered set and $\mathbb{A}^\delta$ is its dual, then $\mathbb{A} \times \mathbb{A}$ and $\mathbb{A} \times \mathbb{A}^\delta$ have isomorphic posets of intervals, but they have not isomorphic posets of convex subsets, in general.

The main result of the present paper is Theorem 4.1, where all partially ordered sets having the lattice of all convex subsets isomorphic to that of a given poset $\mathbb{A}$ are described. Further, couples of posets are investigated which have isomorphic both posets of intervals and posets of convex subsets.

Some partial results concerning couples of posets with isomorphic lattices of all convex subsets are derived in [1].

1.

Let $\mathbb{A} = (A, \leqslant)$ be any partially ordered set. A subset X of A is called convex, if $x_1 \leqslant a \leqslant x_2$, $x_1, x_2 \in X$, $a \in A$ imply $a \in X$. Let $\operatorname{Conv} \mathbb{A}$ denote the system of all convex subsets of $\mathbb{A}$. The system $\operatorname{Conv} \mathbb{A}$, ordered by the set-inclusion, is a complete lattice. The least element is the empty set, the greatest one is A. The greatest lower bound of a non-empty subsystem of the system $\operatorname{Conv} \mathbb{A}$ is its intersection. If $X \subseteq A$, the symbol $[X]$ will be used for denoting the least convex subset of $\mathbb{A}$ containing X. Instead of $[\{x_1, \ldots, x_n\}]$ we will write more briefly $[x_1, \ldots, x_n]$.

By an interval of $\mathbb{A}$ we mean a set $\langle x, y\rangle = \{a \in A\colon\ x \leqslant a \leqslant y\}$, where $x, y \in A$, $x \leqslant y$. The system of all intervals of $\mathbb{A}$ will be denoted by $\operatorname{Int} \mathbb{A}$. If $x, y \in A$, $x < y$, $\langle x, y\rangle = \{x, y\}$, we will write $x \prec y$. If x, y are incomparable elements of A, then evidently $[x, y] = \{x, y\}$ and if, e.g., $x < y$, then $[x, y] = \langle x, y\rangle$. It is also obvious that if X is a non-empty subset of A, then $[X]$ is the union of all intervals $\langle x, y\rangle$ with $x, y \in X$, $x \leqslant y$.

Two partially ordered sets will be called convexly (interval) isomorphic if they have isomorphic posets of all convex subsets (of all intervals).

If we consider another partial order on A (besides the given one $\leqslant$) denoted, e.g., by $\leqslant_1$ or $\leqslant'$, we will use the notation $\langle x, y\rangle_1$, $x \prec_1 y$, $[X]_1$ or $\langle x, y\rangle'$, $x \prec' y$, $[X]'$, instead of $\langle x, y\rangle$, $x \prec y$, $[X]$.

The set of all minimal and maximal elements of $\mathbb{A}$ is denoted by $\operatorname{Min} \mathbb{A}$ and $\operatorname{Max} \mathbb{A}$, respectively.

2.

In this section we describe three constructions which enable us to obtain new partially ordered sets, convexly isomorphic to a given one.

Let us begin with two lemmas.

2.1. Lemma. *Let $\mathbb{A} = (A, \leqslant)$ be a partially ordered set, $a, b \in A$, $a \neq b$. The set $\{a, b\}$ is convex if and only if $a \parallel b$, $a \prec b$ or $b \prec a$.*

The proof is straightforward.

2.2. Lemma. *Let $\leqslant, \leqslant_1$ be two partial orders on a set A. Then $\operatorname{Conv}(A, \leqslant) = \operatorname{Conv}(A, \leqslant_1)$ holds if and only if the following two conditions are satisfied:*

(α) *for any $a, b, c \in A$, $a < b < c$ implies $a <_1 b <_1 c$ or $a >_1 b >_1 c$;*

(β) *for any $a, b, c \in A$, $a <_1 b <_1 c$ implies $a < b < c$ or $a > b > c$.*

P r o o f. If (α) and (β) hold, then evidently $\operatorname{Conv}(A, \leqslant) = \operatorname{Conv}\ (A, \leqslant_1)$. Conversely, let $\operatorname{Conv}(A, \leqslant) = \operatorname{Conv}(A, \leqslant_1)$. To show, e.g., that ($\alpha$) holds, take $a, b, c \in A$, $a < b < c$. Suppose that a, c are incomparable in $(A, \leqslant_1)$. Then $\{a, c\} \in \operatorname{Conv}(A, \leqslant_1) = \operatorname{Conv}(A, \leqslant)$, a contradiction. Let, e.g., $a <_1 c$. Since $a, c \in \langle a, c\rangle_1 \in \operatorname{Conv}(A, \leqslant_1) = \operatorname{Conv}(A, \leqslant)$, $a < b < c$, we have $b \in \langle a, c\rangle_1$. Hence $a <_1 b <_1 c$. $\square$

In what follows, $\mathbb{A} = (A, \leqslant)$ will be any fixed partially ordered set.

1st construction

Let $P = \{(x_i, y_i) \colon i \in I\} \subseteq \{(x, y) \in A \times A \colon x \prec y,\ x \in \operatorname{Min} \mathbb{A},\ y \in \operatorname{Max} \mathbb{A}\}$ and define a relation $\leqslant_1$ on A by

$$x \leqslant_1 y \Longleftrightarrow x \leqslant y \text{ and } (x, y) \notin P.$$

2.3. Lemma. *The above defined relation $\leqslant_1$ is a partial order and for any $x, y, z \in A$, the relation $x < y < z$ holds if and only if $x <_1 y <_1 z$.*

P r o o f. It is easy to see that $\leqslant_1$ is a partial order and $x <_1 y <_1 z$ implies $x < y < z$. Conversely, let $x < y < z$. Since y is neither a minimal nor a maximal element in $\mathbb{A}$, we have $(x, y) \notin P$, $(y, z) \notin P$. Consequently, $x <_1 y <_1 z$. $\square$

2.4. Corollary. *If $\leqslant_1$ is the above defined partial order, then* $\operatorname{Conv}(A, \leqslant) =$ *Conv* $(A, \leqslant_1)$.

Example. Let $\mathbb{A}$ be as in Fig. 1. If we take, e.g., $P = \{(a, y), (b, y)\}$, we obtain $\mathbb{A}_1 = (A, \leqslant_1)$ represented by Fig. 2.

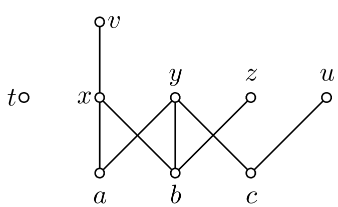

Fig. 1

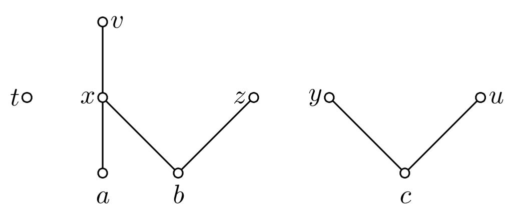

Fig. 2

2nd construction

Let $A = C \cup D$ and let c, d be incomparable whenever $c \in C$, $d \in D$. Let us define a relation $\leqslant_1$ on A by

$$x \leqslant_1 y \Longleftrightarrow \begin{array}{l} \text{either } x, y \in C \text{ and } x \leqslant y \\ \text{or } x, y \in D \text{ and } x \geqslant y. \end{array}$$

2.5. Lemma. *The above defined relation $\leqslant_1$ is a partial order and we have* $\operatorname{Conv}(A, \leqslant_1) = \operatorname{Conv}(A, \leqslant)$.

P r o o f. It is easy to verify that $\leqslant_1$ is a partial order. Let $x < y < z$. Then either $x, y, z \in C$ or $x, y, z \in D$. In the first case we have $x <_1 y <_1 z$, in the latter $x >_1 y >_1 z$. Conversely, let $x <_1 y <_1 z$. Then either $x, y \in C$, $x < y$ and simultaneously $y, z \in C$, $y < z$, or $x, y \in D$, $x > y$ and simultaneously $y, z \in D$, $y > z$. Hence either $x < y < z$ or $x > y > z$. $\square$

Remark. The assumption $c \parallel d$ for any $c \in C$, $d \in D$ can be reformulated in such a way that every maximal connected subset of $\mathbb{A}$ is contained just in one of the sets C, D.

Example. Let $\mathbb{A}$ be as in Fig. 2 and let, e.g., $C = \{a, b, x, z, v\}$, $D = \{c, y, u, t\}$. Then the second construction gives $\mathbb{A}_1$ as in Fig. 3.

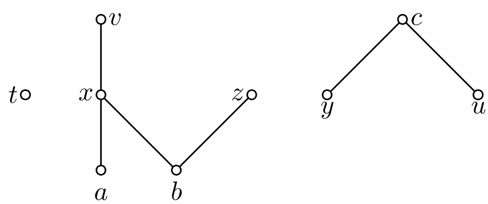

Fig. 3

3rd construction

Let $Q = \{(u_j, v_j)\colon\ j \in J\} \subseteq \{(u, v) \in A \times A\colon\ u \parallel v,\ u \in \operatorname{Min}\mathbb{A},\ v \in \operatorname{Max}\mathbb{A}\}$. Consider the following condition:

(α) $(u, v) \in Q \Longrightarrow (v, w) \in Q$ does not hold for any $w \in A$.

Let us define a relation $\leqslant_1$ on A by

$$x \leqslant_1 y \Longleftrightarrow x \leqslant y \text{ or } (x, y) \in Q.$$

The following can be proved easily:

2.6. Lemma. *Let the set Q satisfy (α). Then the above defined relation $\leqslant_1$ is a partial order and for any $x, y, z \in A$, the relation $x < y < z$ holds if and only if $x <_1 y <_1 z$.*

2.7. Corollary. *If the set Q satisfies (α), then* $\operatorname{Conv}(A, \leqslant_1) = \operatorname{Conv}(A, \leqslant)$.

Example. Let $\mathbb{A}$ be as in Fig. 3 and let, e.g., $Q = \{(a, z), (a, t), (y, v)\}$. Then the third construction gives $\mathbb{A}_1$ as in Fig. 4.

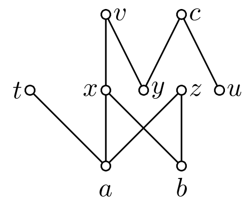

Fig. 4

If $\mathbb{A}' = (A, \leqslant')$ is a partially ordered set such that there exists a finite sequence of posets $\mathbb{A}_0 = (A, \leqslant_0)$, $\mathbb{A}_1 = (A, \leqslant_1), \ldots, \mathbb{A}_n = (A, \leqslant_n)$ $(n \geqslant 0)$ satisfying $\mathbb{A}_0 = \mathbb{A}$, $\mathbb{A}_n = \mathbb{A}'$ and for each $i \in \{1, \ldots, n\}$ $\mathbb{A}_i$ arises from $\mathbb{A}_{i-1}$ by using some of the above mentioned constructions, then evidently $\operatorname{Conv} \mathbb{A}' = \operatorname{Conv} \mathbb{A}$. For example, posets represented by Figures 1–4 have the same system of convex subsets.

3.

In this section we will prove that if $\mathbb{A} = (A, \leqslant)$, $\mathbb{B} = (B, \leqslant^*)$ are convexly isomorphic posets, then there exist partially ordered sets $\mathbb{A}_0 = (A, \leqslant_0)$, $\mathbb{A}_1 = (A, \leqslant_1)$, $\mathbb{A}_2 = (A, \leqslant_2)$, $\mathbb{A}_3 = (A, \leqslant_3)$ such that $\mathbb{A}_i$ arises from $\mathbb{A}_{i-1}$ using the i-th construction $(i = 1, 2, 3)$, $\mathbb{A}_0 = \mathbb{A}$ and $\mathbb{A}_3$ is isomorphic to $\mathbb{B}$.

So, let us suppose that f is an isomorphism of $\operatorname{Conv} \mathbb{A}$ onto $\operatorname{Conv} \mathbb{B}$. Define $f'\colon A \to B$ by

$$f'(a) = b \iff f([a]) = [b]^*.$$

Since f assigns atoms of $\operatorname{Conv} \mathbb{B}$ to atoms of $\operatorname{Conv} \mathbb{A}$, f' is a bijective map. Evidently, $f([a]) = [f'(a)]^*$ for every $a \in A$. First, we will derive some properties of the map f'.

3.1. Lemma. *Let $x, y \in A$. Then $f([x, y]) = [f'(x), f'(y)]^*$.*

Proof. We have $f([x, y]) = f([x] \vee [y]) = f([x]) \vee f([y]) = [f'(x)]^* \vee [f'(y)]^* = [f'(x), f'(y)]^*$. □

3.2. Lemma. *Let $x, y \in A$, $x < y$, but $x \not\prec y$. Then $f'(x)$, $f'(y)$ are comparable elements of $\mathbb{B}$.*

Proof. The set $[x, y] = \langle x, y \rangle$ contains at least three atoms in the lattice $\operatorname{Conv} \mathbb{A}$, so $[f'(x), f'(y)]^*$ also contains at least three atoms in the lattice $\operatorname{Conv} \mathbb{B}$, hence $[f'(x), f'(y)]^* \neq \{f'(x), f'(y)\}$. Consequently $f'(x)$, $f'(y)$ are comparable in $\mathbb{B}$. □

3.3. Lemma. *Let $x, y, z \in A$, $x < y < z$. Then either $f'(x) <^* f'(y) <^* f'(z)$ or $f'(x) >^* f'(y) >^* f'(z)$.*

P r o o f. Since $x < y < z$, we have either $f'(x) <^* f'(z)$ or $f'(x) >^* f'(z)$, by 3.2. Suppose the first possibility occurs. Because of $[y] \subset [x, z] = \langle x, z\rangle$, we have $[f'(y)]^* = f([y]) \subset f([x, z]) = [f'(x), f'(z)]^* = \langle f'(x), f'(z)\rangle^*$. Consequently $f'(x) <^* f'(y) <^* f'(z)$. In the case $f'(x) >^* f'(z)$ we obtain analogously $f'(x) >^* f'(y) >^* f'(z)$. $\square$

As a consequence of 3.2 and 3.3 we have:

3.4. Corollary. *If $x, y \in A$, $x < y$ and $f'(x), f'(y)$ are incomparable, then $x \prec y$, $x \in \operatorname{Min} \mathbb{A}$, $y \in \operatorname{Max} \mathbb{A}$.*

Taking into account that f^{-1} is also an isomorphism of $\operatorname{Conv} \mathbb{B}$ onto $\operatorname{Conv} \mathbb{A}$ and $(f^{-1})' = (f')^{-1}$, we obtain:

3.5. Corollary. *If $f'(x) <^* f'(y) <^* f'(z)$, then either $x < y < z$ or $x > y > z$. If $f'(x) <^* f'(y)$, $x \parallel y$, then $f'(x) \prec^* f'(y)$, $f'(x) \in \operatorname{Min} \mathbb{B}$, $f'(y) \in \operatorname{Max} \mathbb{B}$.*

Using 3.5 we obtain immediately:

3.6. Lemma. *Let $x_1, x_2, z \in A$, $x_1 < z$, $x_2 < z$.*
(1) If $f'(x_1) <^* f'(z)$, then $f'(x_2) \not>^* f'(z)$.
(2) If $f'(x_1) >^* f'(z)$, then $f'(x_2) \not<^* f'(z)$.

3.7. Corollary. *Let $x_1, x_2, z \in A$, $x_1 < z$, $x_2 < z$, $f'(x_1) <^* f'(z)$ $(f'(x_1) >^* f'(z))$. Then either $f'(x_2) <^* f'(z)(f'(x_2) >^* f'(z))$ or $x_2 \prec z$, $x_2 \in \operatorname{Min} \mathbb{A}$, $z \in \operatorname{Max} \mathbb{A}$, $f'(x_2) \parallel^* f'(z)$.*

The following can be proved dually.

3.8. Corollary. *Let $x_1, x_2, z \in A$, $x_1 > z$, $x_2 > z$, $f'(x_1) >^* f'(z)$ $(f'(x_1) <^* f'(z))$. Then either $f'(x_2) >^* f'(z)$ $(f'(x_2) <^* f'(z))$ or $z \prec x_2$, $z \in \operatorname{Min} \mathbb{A}$, $x_2 \in \operatorname{Max} \mathbb{A}$, $f'(x_2) \parallel^* f'(z)$.*

Using 3.7, 3.8 and 3.3 we obtain:

3.9. Corollary. *Let $x_0, x_1, \ldots, x_n \in A$ $(n \geqslant 0)$ be such that $a = x_0 \leqslant x_1$, $x_1 \geqslant x_2, \ldots, x_{n-1} \leqslant x_n = b$, $f'(x_{i-1}) \nparallel^* f'(x_i)$ for each $i \in \{1, \ldots, n\}$. Then f' either preserves all the above relations or reverses each of them.*

Now let us denote by P the set $\{(x,y) \in A \times A\colon\ x < y,\ f'(x) \parallel^* f'(y)\}$. By 3.4, we have $P \subseteq \{(x,y) \in A \times A\colon\ x \prec y,\ x \in \operatorname{Min} \mathbb{A},\ y \in \operatorname{Max} \mathbb{A}\}$, hence in view of 2.3, the relation $\leqslant_1$ defined by

$$x \leqslant_1 y \Longleftrightarrow x \leqslant y \text{ and } (x,y) \notin P$$

is a partial order and for any $x, y, z \in A$, $x < y < z$ is equivalent to $x <_1 y <_1 z$. Evidently, if $x <_1 y$, then $f'(x), f'(y)$ are comparable.

Introduce a relation $\sim$ on A as follows: $a \sim b$ means that there exists a finite sequence of elements $x_0, \ldots, x_n \in A$ $(n \geqslant 0)$ such that $a = x_0 \leqslant x_1,\ x_1 \geqslant x_2, \ldots, x_{n-1} \leqslant x_n = b$ and $f'(x_{i-1}) \nparallel^* f'(x_i)$ for every $i \in \{1, \ldots, n\}$. Clearly the relation $\sim$ is an equivalence relation. If $a \in A$, let $\tilde{a}$ mean the equivalence class containing a.

3.10. Lemma. *For each $a \in A$, the restriction of f' to $\tilde{a}$ is an isotone or antitone map, with respect to the above defined order $\leqslant_1$.*

P r o o f. If $\tilde{a} = \{a\}$, the statement is evident. Suppose $\operatorname{card} \tilde{a} > 1$. Then there exists a couple of elements $x, y \in \tilde{a}$ such that $x < y$ and $f'(x), f'(y)$ are comparable. Let, e.g., $f'(x) <^* f'(y)$. We are going to show that for any $u, v \in \tilde{a}$, $u <_1 v$ implies $f'(u) <^* f'(v)$. Hence let $u, v \in \tilde{a}$, $u <_1 v$. Then $u < v$ and $f'(u), f'(v)$ are comparable. Since $x \sim v$, there exists a sequence $x_0, \ldots, x_n \in A$ such that $x = x_0 \leqslant x_1,\ x_1 \geqslant x_2, \ldots, x_{n-1} \leqslant x_n = v$ and $f'(x_{i-1}) \nparallel^* f'(x_i)$ for each $i \in \{1, \ldots, n\}$. Considering the sequence $y, x = x_0, x_1, \ldots, x_n = v, u$ and taking into account that $f'(x) <^* f'(y)$, 3.9 yields $f'(u) <^* f'(v)$.

Analogously it can be shown that in the case $f'(x) >^* f'(y)$, the map f' is antitone on $(\tilde{a}, \leqslant_1)$. $\square$

Let C be the union of all classes $\tilde{a}$ on which the map f' is isotone (with respect to $\leqslant_1$). Further let $D = A - C$. Then the following is evident:

3.11. Lemma. *For any $c \in C$, $d \in D$ we have $c \parallel_1 d$.*

We proceed by introducing $\leqslant_2$ on A as follows:

$$\begin{aligned} x \leqslant_2 y \Longleftrightarrow\ & \text{either } x, y \in C \text{ and } x \leqslant_1 y \\ & \text{or } x, y \in D \text{ and } x \geqslant_1 y. \end{aligned}$$

In view of 2.5 and 3.11, the relation $\leqslant_2$ is a partial order on A and $\operatorname{Conv}(A, \leqslant_2) = \operatorname{Conv}(A, \leqslant_1)$.

3.12. Lemma. *f' is an isotone map of $\mathbb{A}_2 = (A, \leqslant_2)$ onto $\mathbb{B}$.*

Proof. The map f' is isotone on $(C, \leqslant_1)$ and antitone on $(D, \leqslant_1)$. Let $x <_2 y$. Then either $x, y \in C$, $x <_1 y$ or $x, y \in D$, $x >_1 y$. In both cases $f'(x) <^* f'(y)$. $\square$

Finally, set $Q = \{(u, v) \in A \times A\colon\ u \parallel v, f'(u) <^* f'(v)\}$.

3.13. Lemma. *We have $Q \subseteq \{(u, v) \in A \times A\colon\ u \parallel_2 v, u \in \operatorname{Min} \mathbb{A}_2, v \in \operatorname{Max} \mathbb{A}_2\}$.*

Proof. By the definitions of $\leqslant_2$ and $\leqslant_1$, if two elements of A are comparable in $\mathbb{A}_2$, then they are comparable in $\mathbb{A}$, too. Hence if $(u, v) \in Q$, then $u \parallel_2 v$. Further let us suppose that $(u, v) \in Q$, $u \notin \operatorname{Min} \mathbb{A}_2$. Then there exists $t \in A$ with $t <_2 u$. This implies $f'(t) <^* f'(u)$, by 3.12. Realizing that $f'(u) <^* f'(v)$, we obtain $t < u < v$ or $t > u > v$ by 3.5, which is a contradiction with $u \parallel v$. The proof of $v \in \operatorname{Max} \mathbb{A}_2$ would be analogous. $\square$

The following is a consequence of 3.5.

3.14. Lemma. *If $(u, v) \in Q$, then there exists no $w \in A$ satisfying $(v, w) \in Q$.*

Let us define a relation $\leqslant_3$ on A by

$$x \leqslant_3 y \Longleftrightarrow x \leqslant_2 y \text{ or } (x, y) \in Q.$$

In view of 3.13, 3.14 and 2.6, $\mathbb{A}_3 = (A, \leqslant_3)$ is a partially ordered set and for any $x, y, z \in A$, $x <_2 y <_2 z$ is equivalent to $x <_3 y <_3 z$.

3.15. Lemma. *The map f' is an isomorphism of $\mathbb{A}_3 = (A, \leqslant_3)$ onto $\mathbb{B}$.*

Proof. If $x <_3 y$, then either $x <_2 y$ or $(x, y) \in Q$. If $x <_2 y$, then $f'(x) <^* f'(y)$ by 3.12. If $(x, y) \in Q$, then the definition of Q yields $f'(x) <^* f'(y)$.

It remains to show that $f'(x) <^* f'(y)$ implies $x <_3 y$. So let $f'(x) <^* f'(y)$ for some $x, y \in A$. If $x \parallel y$, then $(x, y) \in Q$ and this implies $x <_3 y$. Assume that x, y are comparable in $\mathbb{A}$. Since $f'(x) <^* f'(y)$, we have $y \in \tilde{x}$ and $(x, y) \notin P$, $(y, x) \notin P$. Hence x, y are comparable in $\mathbb{A}_1$ and either $x, y \in C$ or $x, y \in D$. Consequently, x, y are comparable in $\mathbb{A}_2$, too. But by virtue of 3.12, $x >_2 y$ is impossible. So $x <_2 y$, which implies $x <_3 y$. $\square$

4.

Summarizing the results of the preceding two sections, we obtain:

4.1. Theorem. *Let $\mathbb{A} = (A, \leqslant)$ be any partially ordered set. Partially ordered sets convexly isomorphic to $\mathbb{A}$ are (up to isomorphism) just those which can be obtained by applying successively three constructions:*

(1) *We construct $\mathbb{A}_1 = (A, \leqslant_1)$, where $x \leqslant_1 y$ means that $x \leqslant y$ and $(x, y) \notin P$, for a subset P of the set $\{(x, y) \in A \times A\colon\ x \prec y,\ x \in \operatorname{Min} \mathbb{A},\ y \in \operatorname{Max} \mathbb{A}\}$.*

(2) *Having $\mathbb{A}_1$ we construct $\mathbb{A}_2 = (A, \leqslant_2)$, setting $x \leqslant_2 y$ whenever either $x, y \in C$, $x \leqslant_1 y$ or $x, y \in D$, $x \geqslant_1 y$ holds, for a decomposition $A = C \cup D$ of the set A satisfying $c \parallel_1 d$ for every $c \in C$, $d \in D$.*

(3) *Taking $\mathbb{A}_2$, we construct $\mathbb{A}_3 = (A, \leqslant_3)$ in such a way that we put $x \leqslant_3 y$ if $x \leqslant_2 y$ or $(x, y) \in Q$, for a subset Q of the set $\{(x, y) \in A \times A\colon\ x \parallel_2 y,\ x \in \operatorname{Min} \mathbb{A}_2,\ y \in \operatorname{Max} \mathbb{A}_2\}$ satisfying the following condition:*

   (α) *$(u, v) \in Q$, $(v, w) \in Q$ do not hold simultaneously for any $u, v, w \in A$.*

Notice that, in view of 2.3,2.5 and 2.6, each of the above three constructions gives a partially ordered set which is not only convexly isomorphic to the given one, but has even the same system of convex subsets. Hence we have

4.2. Theorem. *If partially ordered sets $\mathbb{A} = (A, \leqslant)$, $\mathbb{B} = (B, \leqslant^*)$ are convexly isomorphic, then there exists a poset $\mathbb{A}' = (A, \leqslant')$ isomorphic to $\mathbb{B}$ such that $\operatorname{Conv} \mathbb{A} = \operatorname{Conv} \mathbb{A}'$.*

We can see that for some partially ordered sets $\mathbb{A}$, the only posets convexly isomorphic to $\mathbb{A}$ (up to isomorphism) are $\mathbb{A}$ and $\mathbb{A}^\delta$. This is the case, e.g., when $\mathbb{A}$ is connected and has no minimal or no maximal elements, or when $\mathbb{A}$ is a directed poset containing more than two elements. In particular, every lattice containing more than two elements is of this sort. On the other hand, the partially ordered sets in Fig. 5 are convexly isomorphic, but they are neither isomorphic nor dually isomorphic. Moreover, the first is connected, while the other fails to be connected.

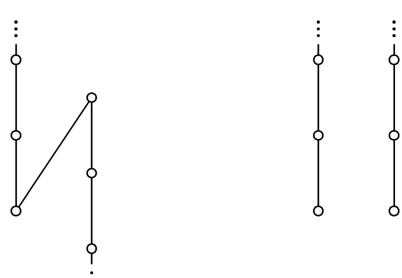

Fig. 5

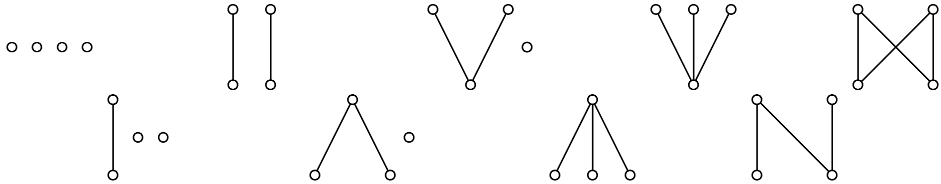

Fig. 6

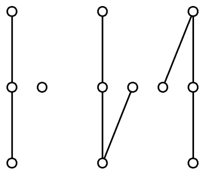

Fig. 7

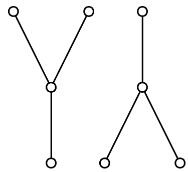

Fig. 8

Fig. 9

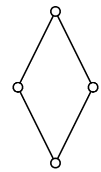

Fig. 10

In Figures 6–10, we depict all 4-element posets, divided into the classes of mutually convexly isomorphic.

5.

In this section we are interested in couples of partially ordered sets, which are both convexly and interval isomorphic.

Let $\mathbb{A} = (A, \leqslant)$ be any partially ordered set, U, V binary relations on A. Consider the following conditions concerning U, V:

(P1) $U, V \subseteq \{(x,y) \in A \times A \colon\ x \nparallel y\}$;

(P2) $x, y \in A$, $x \leqslant y \Longrightarrow$ there exists a unique couple of elements $p, q \in \langle x, y\rangle$ satisfying $pVxUqVyUp$;

(P3) $u \leqslant x, y$, $xVuUy \Longrightarrow u = \inf\,\{x,y\}$, there exists $v = \sup\,\{x,y\}$ and $yVvUx$ holds;

(P3′) $v \geqslant x, y$, $yVvUx \Longrightarrow v = \sup\,\{x,y\}$, there exists $u = \inf\,\{x,y\}$ and $xVuUy$ holds.

The following theorem is proved in [14].

5.1. Theorem. *Let $\mathbb{A}$ be any connected partially ordered set.*

(a) *If U, V are binary relations on A satisfying the conditions* (P1)–(P3′), *then the relation $\leqslant_1$ defined by*

$$x \leqslant_1 y \Longleftrightarrow \text{ there exists } u \in A, u \leqslant x, y \text{ with } xVuUy$$

is a partial order and $\mathbb{A}_1 = (A, \leqslant_1)$ *is interval isomorphic to* $\mathbb{A}$.

(b) *If for a partially ordered set* $\mathbb{A}' = (A, \leqslant')$ *there exists an isomorphism* f *of* *Int* $\mathbb{A}$ *onto Int* $\mathbb{A}'$ *and* f' *is the mapping* $A \to A$ *defined by*

$$f'(a) = b \Longleftrightarrow f(\langle a \rangle) = \langle b \rangle,$$

then the binary relations $U = \{(x, y) \in A \times A\colon\ x \leqslant y,\ f'(x) \leqslant' f'(y)\} \cup \{(x, y) \in A \times A\colon\ x \geqslant y,\ f'(x) \geqslant' f'(y)\}$, $V = \{(x, y) \in A \times A\colon\ x \leqslant y,\ f'(x) \geqslant' f'(y)\} \cup \{(x, y) \in A \times A\colon\ x \geqslant y,\ f'(x) \leqslant' f'(y)\}$ *satisfy the conditions* (P1)–(P3′) *and the partially ordered set* $\mathbb{A}_1 = (A, \leqslant_1)$ *corresponding to* U, V *in the sense of* (*a*) *is isomorphic to* $\mathbb{A}'$.

We will prove, using 5.1 and 4.1, that if $\mathbb{A}$ is a connected partially ordered set, then $\mathbb{A}$ and $\mathbb{A}^\delta$ are the only posets (up to isomorphism) which are both convexly and interval isomorphic to $\mathbb{A}$.

5.2. Lemma. *Let* $\mathbb{A} = (A, \leqslant)$ *be a connected partially ordered set and let there exist a couple of elements* $u, v \in A$ *such that* $u \prec v$, $u \in \operatorname{Min} \mathbb{A}$, $v \in \operatorname{Max} \mathbb{A}$. *If* U, V *are binary relations on* A *satisfying* (P1)–(P3′), *then either* $U = \{(x, y) \in A \times A\colon\ x \nparallel y\}$, $V = \{(z, z)\colon\ z \in A\}$ *or* $U = \{(z, z)\colon\ z \in A\}$, $V = \{(x, y) \in A \times A\colon\ x \nparallel y\}$.

Proof. We have either uUv or uVv, by (P2). Let us suppose, e.g., that the first alternative occurs. We will show that $U = \{(x, y) \in A \times A\colon\ x \nparallel y\}$ and then evidently $V = \{(z, z)\colon\ z \in A\}$, again by (P2). Let us prove, by induction on n, that if for some $x_0, \ldots, x_n \in A$ ($n \in \mathbb{N}$) we have $x_0 = v$, $x_{i-1} \nparallel x_i$ for each $i \in \{1, \ldots, n\}$, then $x_{i-1} U x_i$ holds also for each $i \in \{1, \ldots, n\}$. Let $n = 1$. Then $v = x_0 \geqslant x_1$, since $v \in \operatorname{Max} \mathbb{A}$. In view of (P2) there exists $z \in \langle x_1, x_0 \rangle$ satisfying $x_1 U z V x_0$. Using (P3′) we obtain that there exists $w = \inf\,\{u, z\}$. As $u \in \operatorname{Min} \mathbb{A}$, we have $u \leqslant z$ and $x_0 = \sup\{u, z\} = z$. We have $x_1 U x_0$. Assume that the assertion is true for $n = k$. Let us have a sequence $v = x_0, x_1, \ldots, x_{k+1}$ such that every two adjoining elements are comparable. The induction hypothesis yields $x_0 U x_1 U x_2 \ldots x_{k-1} U x_k$. It remains to show $x_k U x_{k+1}$. Without loss of generality we can suppose that $x_0 > x_1$, $x_1 < x_2$, $x_2 > x_3, \ldots$ (namely if $x < y < z$, $xUyUz$, then xUz by 2.5 of [14]). Let, e.g., k be even (in the case of odd k the argument would be analogous). Then $x_{k-2} > x_{k-1}$, $x_{k-1} < x_k$, $x_k > x_{k+1}$. By (P2) there exists $z \in \langle x_{k+1}, x_k \rangle$ satisfying $x_{k+1} U z V x_k$. In view of (P3′) $x_k = \sup\,\{x_{k-1}, z\}$, there exists $w = \inf\,\{x_{k-1}, z\}$ and $x_{k-1} V w U z$. Using the induction hypothesis for the sequence $x_0, x_1, \ldots, x_{k-2}, w$, we obtain $x_{k-2} U w$. The convexity of U-classes (cf. 2.7 of [14]) yields $x_{k-1} U w$ which, together with x_{k-1}, Vw, gives $x_{k-1} = w$ by (P2). But then $x_{k-1} \leqslant z$ and $x_k = z$. We have proved $x_{k+1} U x_k$. Now if we take any $(x, y) \in A \times A$, $x \nparallel y$ and use the connectivity of $\mathbb{A}$, we obtain xUy. In the case of uVv we proceed analogously concluding that $V = \{(x,y) \in A \times A\colon\ x \nparallel y\}$. $\square$

If $U = \{(x,y) \in A \times A\colon\ x \nparallel y\}$, $V = \{(z,z)\colon\ z \in A\}$, then $\mathbb{A}_1 = (A, \leqslant_1)$, in the sense of 5.1(a), is $\mathbb{A}$. If $U = \{(z,z)\colon\ z \in A\}$, $V = \{(x,y) \in A \times A\colon\ x \nparallel y\}$, then $\mathbb{A}_1 = \mathbb{A}^\delta$. Hence we have:

5.3. Corollary. *If $\mathbb{A}$ is a connected partially ordered set which contains elements u, v satisfying $u \prec v$, $u \in \operatorname{Min}\mathbb{A}$, $v \in \operatorname{Max}\mathbb{A}$, then $\mathbb{A}$ and $\mathbb{A}^\delta$ are the only posets (up to isomorphism) interval isomorphic to $\mathbb{A}$.*

Further, consider a connected partially ordered set $\mathbb{A}$ which does not contain couples of elements u, v with $u \prec v$, $u \in \operatorname{Min}\mathbb{A}$, $v \in \operatorname{Max}\mathbb{A}$. Let $\mathbb{B} = (B, \leqslant^*)$ be a partially ordered set convexly isomorphic to $\mathbb{A}$. Then $\mathbb{B}$ is isomorphic to a partially ordered set $\mathbb{A}' = (A, \leqslant')$ which is obtained from $\mathbb{A}$ by using the 3rd construction for a subset Q of the set $\{(x,y) \in A \times A\colon\ x \parallel y,\ x \in \operatorname{Min}\mathbb{A},\ y \in \operatorname{Max}\mathbb{A}\}$ satisfying the condition (α), by 4.1. If $Q = \emptyset$, then $\mathbb{A}' = \mathbb{A}$. If $(x,y) \in Q$, then $x \prec' y$, $x \in \operatorname{Min}\mathbb{A}'$, $y \in \operatorname{Max}\mathbb{A}'$, so that, by 5.3 only $\mathbb{A}'$ and ${\mathbb{A}'}^\delta$ are interval isomorphic to $\mathbb{A}'$. We have proved:

5.4. Corollary. *If $\mathbb{A}$ is any connected partially ordered set, then $\mathbb{A}$ and $\mathbb{A}^\delta$ are (up to isomorphism) the only posets which are both interval and convexly isomorphic to $\mathbb{A}$.*

Now consider a disconnected partially ordered set $\mathbb{A}$. If we turn upside down some of its maximal connected subsets, we obtain a partially ordered set convexly isomorphic to $\mathbb{A}$. But there exist also other couples of convexly isomorphic posets, as 5.6 shows. We will use the following evident assertion:

5.5. Lemma. *Let $\mathbb{A}$ be the cardinal sum of some partially ordered sets $\mathbb{A}_i$, $i \in I$. Then the lattice $\operatorname{Conv}\mathbb{A}$ is isomorphic to the direct product of the lattices $\operatorname{Conv}\mathbb{A}_i$.*

5.6. Example. Let $\mathbb{C}$ and $\mathbb{D}$ be as in Figs. 11 and 12, respectively.

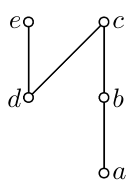

Fig. 11

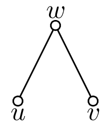

Fig. 12

Let $\mathbb{A}_1 = \mathbb{C} \times \mathbb{D}$, $\mathbb{B}_1 = \mathbb{C} \times \mathbb{D}^\delta$. For any $i \in \mathbb{N}$ let $\mathbb{C}_i$ be an isomorphic copy of $\mathbb{C}$, with elements denoted by a_i, b_i, c_i, d_i, e_i, and analogously for $\mathbb{D}_i$. Now let $\mathbb{K}$ be the cardinal sum of all $\mathbb{C}_i \times \mathbb{D}_i$ ($i \in \mathbb{N}$) and let $\mathbb{L}$ be the cardinal sum of all $\mathbb{C}_i \times \mathbb{D}_i^\delta$. Let $\mathbb{A}_2$ be obtained from $\mathbb{K}$ by using the 3rd construction, taking $Q = \{((a_i, v_i), (e_{i+1}, w_{i+1}))\colon\ i \in \mathbb{N}\}$. Finally, let $\mathbb{A}_3$ be obtained from $\mathbb{L}$ by using the 3rd construction, taking $Q = \{((a_i, w_i), (e_{i+1}, u_{i+1}))\colon\ i \in \mathbb{N}\}$. Then if $\mathbb{A}$ is the cardinal sum of $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ and $\mathbb{B}$ is the cardinal sum of $\mathbb{B}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, then, evidently, $\mathbb{A}$ and $\mathbb{B}$ are interval isomorphic, because $\mathbb{A}_1$ and $\mathbb{B}_1$ are interval isomorphic. But $\mathbb{A}$ and $\mathbb{B}$ are convexly isomorphic, too. Namely, as $\operatorname{Conv} \mathbb{A}_2 \cong \operatorname{Conv} \mathbb{K} \cong \prod_{i\in\mathbb{N}} \operatorname{Conv}(\mathbb{C}_i \times \mathbb{D}_i)$ and analogously $\operatorname{Conv} \mathbb{A}_3 \cong \prod_{i\in\mathbb{N}} \operatorname{Conv}(\mathbb{C}_i \times \mathbb{D}_i^\delta)$, we infer that $\operatorname{Conv} \mathbb{A} \cong \operatorname{Conv}(\mathbb{C}\times\mathbb{D}) \times \prod_{i\in\mathbb{N}} \operatorname{Conv}(\mathbb{C}_i \times \mathbb{D}_i) \times \prod_{i\in\mathbb{N}} \operatorname{Conv}(\mathbb{C}_i \times \mathbb{D}_i^\delta)$ and $\operatorname{Conv} \mathbb{B} \cong \operatorname{Conv}(\mathbb{C} \times \mathbb{D}^\delta) \times \prod_{i\in\mathbb{N}} \operatorname{Conv}(\mathbb{C}_i \times \mathbb{D}_i) \times \prod_{i\in\mathbb{N}} \operatorname{Conv}(\mathbb{C}_i \times \mathbb{D}_i^\delta)$. Hence $\operatorname{Conv} \mathbb{A}$ is isomorphic to $\operatorname{Conv} \mathbb{B}$.

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