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ON σ-DISCRETE BOREL MAPPINGS VIA QUASI-METRICS

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Abstract. Let X and Y be metrizable spaces. We show that, for a mapping $f\colon X\to Y$, there exists a quasi-metric ϱ on X inducing the topology of X such that f regarded as a mapping from $(X,\max\{\varrho,\varrho^{-1}\})$ to Y is continuous if and only if f in the original topology of X is a σ-discrete map of Borel class 1. Further, we prove that, for every σ-discrete mapping $f\colon X\to Y$ of Borel class $\alpha+1$, there exists a compatible quasi-metric ϱ on X such that $f\colon(X,\max\{\varrho,\varrho^{-1}\})\to Y$ is of Borel class α. We also investigate a more general situation when the range of the mapping under consideration is not necessarily metrizable. In passing, we obtain some results related to the behaviour of absolutely Borel sets and absolutely analytic spaces with respect to compatible quasi-metrics.

Keywords: quasi-metric, continuous map, Borel map, σ-discrete map, σ-discretely decomposable family, absolutely Borel set, absolutely analytic space

MSC 2000: 54E35, 54H05, 28A05, 26A21

1. Introduction

A *quasi-metric* on a set X is a function ϱ from $X\times X$ to the non-negative real numbers such that $\varrho(x,y)\leqslant\varrho(x,z)+\varrho(z,y)$ for all $x,y,z\in X$, and $\varrho(x,y)=0$ if and only if $x=y$. The conjugate ϱ^{-1} of the quasi-metric ϱ is the function defined by $\varrho^{-1}(x,y)=\varrho(y,x)$ for all $x,y\in X$. Then ϱ^{-1} is also a quasi-metric and the function $\varrho^*=\max\{\varrho,\varrho^{-1}\}$ is a metric on X. The topology on X induced by the quasi-metric ϱ is the topology $\mathscr{G}(X,\varrho)$ on X having the collection of all ϱ-balls

$$B_\varrho(x,\varepsilon)=\{y\in X\colon\ \varrho(x,y)<\varepsilon\}$$

as a base for the open sets. Given a metric space (X,d), we shall say that a quasi-metric ϱ on X is *compatible* (with d or on (X,d)) if $\mathscr{G}(X,\varrho)=\mathscr{G}(X,d)$. In general,

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a compatible quasi-metric on a topological space X is a quasi-metric on the set X which induces the original topology of X.

If X is a subspace of a metric space (Y, d), we shall write (X, d) to denote the space $(X, d\restriction_{X\times X})$.

A natural problem related to the theory of quasi-metrics is the question of when a mapping f from a metric space (X, d) to a topological space Y can become continuous if we replace the topology $\mathscr{G}(X, d)$ by $\mathscr{G}(X, \varrho^*)$ for some suitably chosen quasi-metric ϱ on X that is compatible with d. In the present paper we shall prove that, for a mapping f from a metric space (X, d) to a topological space Y, there exists a compatible with d quasi-metric ϱ on X such that $f\colon (X, \varrho^*) \to Y$ is continuous if and only if f has in (X, d) a σ-discrete base (in the sense of Hansell) consisting of F_σ subsets of (X, d). This answer to the above-mentioned question restricts our attention to σ-discrete Borel mappings and leads us to a more general problem whether the Borel class of a σ-discrete Borel mapping between metric spaces X and Y can be lowered if we replace the original topology of X by the topology $\mathscr{G}(X, \varrho^*)$ where ϱ is a compatible quasi-metric on X. We shall show that a mapping f from a metric space (X, d) to a metrizable space Y is a σ-discrete Borel mapping of class $\alpha+1$ if and only if there exists a compatible with d quasi-metric ϱ on X such that $f\colon (X, \varrho^*) \to Y$ is a σ-discrete Borel mapping of class α. Since, in classical descriptive set theory, it is often useful to replace the original topology of a Polish space by a Polish zero-dimensional topology, we shall turn our attention to the problem whether the quasi-metric ϱ lowering the Borel class of a σ-discrete Borel mapping can be chosen in such a way that the covering dimension $\dim(X, \varrho^*) = 0$ and that ϱ^* is complete if (X, d) is complete. Indeed, we shall prove that, for every compatible quasi-metric ϱ on a metric space (X, d), there exists a compatible quasi-metric $\tilde{\varrho}$ on the completion $(\tilde{X}, \tilde{d})$ of (X, d) such that $\mathscr{G}(X, \varrho^*) \subseteq \mathscr{G}(X, \tilde{\varrho}^*)$, $\dim(\tilde{X}, \tilde{\varrho}^*) = 0$, $\tilde{\varrho}^*$ is complete and, in addition, if (X, d) is an absolute $\mathscr{F}_{\alpha+1}$-set (resp. $\mathscr{G}_{\alpha+1}$-set), then $(\tilde{X}, \tilde{\varrho}^*)$ is an absolute $\mathscr{G}_\alpha$-set (resp. $\mathscr{F}_\alpha$-set). Our results on Borel mappings generalize those obtained in [11] for second-countable Y's.

Basic facts concerning Borel sets and Borel mappings can be found in [12]. We recommend [4] to the reader to get more information about quasi-metrics. All other topological notions which we refer to can be found in [1].

2. σ-discretely decomposable families of Borel sets

In order to prove our main results on σ-discrete Borel mappings, we must have a deeper look at σ-discretely decomposable collections of Borel sets. To begin with, let us establish the appropriate terminology and notation.

Given a collection $\mathscr{A}$ of subsets of a set X, denote by $\mathscr{A}_\sigma$, $\mathscr{A}_\delta$ and $\mathscr{A}_c$ the collections of, respectively, all countable unions, all countable intersections and complements of members of $\mathscr{A}$. Put $\mathscr{F}_0(\mathscr{A}) = \mathscr{G}_0(\mathscr{A}) = \mathscr{A}$ and, for every non-zero ordinal $\alpha < \omega_1$, define

$$\mathscr{F}_\alpha(\mathscr{A}) = \begin{cases} \Big[\bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(\mathscr{A})\Big]_\delta & \text{when } \alpha \text{ is even,} \\ \Big[\bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(\mathscr{A})\Big]_\sigma & \text{when } \alpha \text{ is odd,} \end{cases}$$

and

$$\mathscr{G}_\alpha(\mathscr{A}) = \begin{cases} \Big[\bigcup_{\gamma<\alpha} \mathscr{G}_\gamma(\mathscr{A})\Big]_\sigma & \text{when } \alpha \text{ is even,} \\ \Big[\bigcup_{\gamma<\alpha} \mathscr{G}_\gamma(\mathscr{A})\Big]_\delta & \text{when } \alpha \text{ is odd.} \end{cases}$$

If X is a topological space, we put $\mathscr{F}_\alpha(X)$ and $\mathscr{G}_\alpha(X)$ to denote, respectively, the classes $\mathscr{F}_\alpha\big(\mathscr{F}(X)\big)$ and $\mathscr{G}_\alpha\big(\mathscr{G}(X)\big)$ where $\mathscr{F}(X)$ is the collection of all closed subsets of X, while $\mathscr{G}(X)$ is the topology of X. If α is even, the sets in $\mathscr{G}_\alpha(X)$ are of additive class α, while the sets in $\mathscr{F}_\alpha(X)$ are of multiplicative class α; if α is odd, the sets in $\mathscr{F}_\alpha(X)$ are of additive class α and those in $\mathscr{G}_\alpha(X)$ are of multiplicative class α. If the topology of X is induced by a quasi-metric ϱ, we shall often use the symbols $\mathscr{F}_\alpha(X,\varrho)$ and $\mathscr{G}_\alpha(X,\varrho)$ to denote the classes $\mathscr{F}_\alpha(X)$ and $\mathscr{G}_\alpha(X)$, respectively.

A metrizable space X is an *absolute $\mathscr{F}_\alpha$-set* (resp. *absolute $\mathscr{G}_\alpha$-set*) if X is of type $\mathscr{F}_\alpha$ (resp. $\mathscr{G}_\alpha$) in every metrizable space containing X as a subspace (up to a homeomorphic embedding). A metrizable space X is *absolutely analytic* if, in every metrizable space Y containing X as a subspace, the space X can be represented in the form

$$X = \bigcup_{\tau\in\mathbb{N}^\omega} \bigcap_{n=1}^{\infty} F(\tau \mid n)$$

where $F(\tau \mid n) \in \mathscr{F}(Y)$ for $\tau \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$. (cf. e. g. [6] and [15]).

It is well known that, for $\alpha > 1$ (resp. $\alpha > 0$), a metric space (X,d) is an absolute $\mathscr{F}_\alpha$-set (resp. absolute $\mathscr{G}_\alpha$-set) if and only if (X,d) is of type $\mathscr{F}_\alpha$ (resp. $\mathscr{G}_\alpha$) in its completion or, equivalently, in a completely metrizable space containing (X,d) as a subspace. Similarly, a metric space (X,d) is absolutely analytic if and only if it can be obtained by applying the Suslin operation to closed sets of the completion of (X,d).

The proposition given below was established in [11]. It is an immediate consequence of [10, Thm. 4] and the fact that, if ϱ is a compatible quasi-metric on a metric space (X,d), then the bitopological space $\big(X,\mathscr{G}(X,\varrho),\mathscr{G}(X,\varrho^{-1})\big)$ is pairwise perfectly normal (cf. [13]).

Proposition 1. *If ϱ is a compatible quasi-metric on a metric space (X,d), then the spaces (X,d) and (X,ϱ^*) are of the same weight; furthermore*

$$\mathscr{G}(X,d) \subseteq \mathscr{G}(X,\varrho^*) \subseteq \mathscr{F}_1(X,d).$$

Theorem 2. *If ϱ is a compatible quasi-metric on a completely metrizable space X, then the space (X,ϱ^*) is completely metrizable.*

P r o o f. Let d be a complete compatible metric on X. The function $\varrho_0 = \max\{d,\varrho\}$ is a compatible quasi-metric on (X,d) such that $\mathscr{G}(X,\varrho^*) = \mathscr{G}(X,\varrho_0^*)$. For $n \in \mathbb{N}$, let $\mathscr{U}_n$ be the collection of all those sets $U \in \mathscr{G}(X,\varrho_0^*)$ for which there is $x_U \in X$ such that $\mathrm{cl}_{(X,d)}\, U \subseteq B_{\varrho_0^*}(x_U, 2^{-n})$. Since $\mathscr{G}(X,\varrho_0^*) = \mathscr{G}(X,\varrho_0^{-1})$, it follows from the pairwise regularity of $(X,\varrho_0,\varrho_0^{-1})$ that $\mathscr{U}_n$ is a cover of X. In order to show that (X,ϱ^*) is completely metrizable, it suffices to prove that $\langle \mathscr{U}_n \rangle_{n\in\mathbb{N}}$ is a complete sequence of open covers of (X,ϱ_0^*). To this end, consider an arbitrary filter $\mathscr{F}$ of closed sets of (X,ϱ_0^*) such that, for each $n \in \mathbb{N}$, there are $F_n \in \mathscr{F}$ and $U_n \in \mathscr{U}_n$ with $F_n \subseteq U_n$. As $d \leqslant \varrho_0^*$, we have $d(x,y) \leqslant 2^{-n+1}$ for any $x,y \in \mathrm{cl}_{(X,d)}\, F_n$; therefore, $\emptyset \neq \bigcap\limits_{n=1}^{\infty} \mathrm{cl}_{(X,d)}\, F_n$ because the metric d is complete. Let $x_0 \in \bigcap\limits_{n=1}^{\infty} \mathrm{cl}_{(X,d)}\, F_n$ and suppose, if possible, that there exists $F \in \mathscr{F}$ such that $x_0 \notin F$. Take $n \in \mathbb{N}$ such that $B_{\varrho_0^*}(x_0, 2^{-n}) \cap F = \emptyset$. Since $\mathrm{cl}_{(X,d)}\, F_{n+1} \subseteq B_{\varrho_0^*}(x_0, 2^{-n})$, we have $F_{n+1} \cap F = \emptyset$, which is absurd. The contradiction obtained shows that $x_0 \in \bigcap\{F\colon F \in \mathscr{F}\}$, which completes the proof. $\square$

Definitions. Let $\mathscr{A}$ and $\mathscr{D}$ be collections of subsets of a metric space (X,d) and let $\varepsilon > 0$. Then:

(i) $\mathscr{A}$ is *ε-discrete in* (X,d) if, for every pair of sets A, $B \in \mathscr{A}$ such that $A \neq B$, we have $d(x,y) > \varepsilon$ whenever $x \in A$ and $y \in B$;

(ii) $\mathscr{A}$ is *metrically discrete in* (X,d) if there exists $\varepsilon > 0$ such that $\mathscr{A}$ is ε-discrete in (X,d);

(iii) $\mathscr{A}$ is *σ-metrically discrete in* (X,d) if it is the countable union of metrically discrete in (X,d) collections of sets;

(iv) $\mathscr{A}$ is *σ-discretely decomposable in* (X,d) *into members of* $\mathscr{D}$ if there exists a sequence of discrete in (X,d) collections $\mathscr{D}_n \subseteq \mathscr{D}$ such that each member of $\mathscr{A}$ is the union of some members of $\bigcup\limits_{n=1}^{\infty} \mathscr{D}_n$.

Definitions. A binary relation V on a topological space X is called a *neighbournet* on X if, for each point $x \in X$, the set $V(x) = \{y \in X\colon (x,y) \in V\}$ is a neighbourhood of x. A neighbournet on X is called *unsymmetric* if, for all $x,y \in X$, we have $V(x) = V(y)$ whenever $x \in V(y)$ and $y \in V(x)$ (cf. [8, p. 88] or [4, pp. 4–5]).

Lemma 3. *Let ϱ be a compatible quasi-metric on a metric space (X, d) and let $\alpha < \omega_1$. If a collection $\mathscr{A} \subseteq \mathscr{F}_\alpha(X, \varrho^*)$, resp. $\mathscr{A} \subseteq \mathscr{G}_\alpha(X, \varrho^*)$, is ε-discrete in (X, ϱ^*), then $\mathscr{A}$ is σ-discretely decomposable in (X, d) into members of $\mathscr{G}_{\alpha+1}(X, d)$, resp. of $\mathscr{F}_{\alpha+1}(X, d)$.*

Proof. Let $U = \{(x, y) \in X \times X\colon\ \varrho(x, y) < \frac{\varepsilon}{4}\}$. By Theorem 4.4 of [8], there exists an unsymmetric neighbournet V on (X, d) such that $V \subseteq U^2$. Then $V \cap V^{-1}$ is an equivalence relation on X; accordingly, in view of Theorem 4.8 of [8], the partition $\{V \cap V^{-1}(x)\colon\ x \in X\}$ of X has a refinement $\mathscr{D} = \bigcup_{n=1}^{\infty} \mathscr{D}_n$ such that each collection $\mathscr{D}_n$ consists of closed subsets of (X, d) and is discrete in (X, d). Let

$$\mathscr{E}_n = \{A \cap D\colon\ A \in \mathscr{A}\ \&\ D \in \mathscr{D}_n\}$$

for $n \in \mathbb{N}$. It follows from Proposition 1 that if $\mathscr{A} \subseteq \mathscr{F}_\alpha(X, \varrho^*)$, resp. $\mathscr{A} \subseteq \mathscr{G}_\alpha(X, \varrho^*)$, then $\mathscr{E}_n \subseteq \mathscr{G}_{\alpha+1}(X, d)$, resp. $\mathscr{E}_n \subseteq \mathscr{F}_{\alpha+1}(X, d)$, for $n \in \mathbb{N}$. We shall show that the collections $\mathscr{E}_n$ are discrete in (X, d). To this end, it suffices to check that each $D \in \mathscr{D}_n$ meets at most one member of $\mathscr{A}$.

Suppose, if possible, that there exist $A, B \in \mathscr{A}$ and $D \in \mathscr{D}_n$, such that $A \neq B$ and $A \cap D \neq \emptyset \neq B \cap D$. Let $y_0 \in A \cap D$ and $z_0 \in B \cap D$. There exists $x_0 \in X$ such that $D \subseteq V \cap V^{-1}(x_0)$. Put $W = \{(x, y) \in X \times X\colon\ \varrho(x, y) < \frac{\varepsilon}{2}\}$. Obviously, $V \cap V^{-1}(x_0) \subseteq W \cap W^{-1}(x_0)$, which implies that $\varrho(x_0, y_0) < \frac{\varepsilon}{2}$, $\varrho(x_0, z_0) < \frac{\varepsilon}{2}$, $\varrho(y_0, x_0) < \frac{\varepsilon}{2}$ and $\varrho(z_0, x_0) < \frac{\varepsilon}{2}$. All the inequalities taken together give that $\varrho^*(y_0, x_0) < \frac{\varepsilon}{2}$ and $\varrho^*(x_0, z_0) < \frac{\varepsilon}{2}$; in consequence, $\varrho^*(y_0, z_0) < \varepsilon$, but this is impossible because the collection $\mathscr{A}$ is ε-discrete in (X, ϱ^*). The contradiction obtained proves that each collection $\mathscr{E}_n$ is discrete in (X, d). Since $\mathscr{D}$ is a cover of X, each member of $\mathscr{A}$ can be expressed as the union of some members of $\bigcup_{n=1}^{\infty} \mathscr{E}_n$, which concludes the proof. □

We shall make use of the following theorem which was proved in [11]:

Theorem 4. *If $\mathscr{H}$ is a σ-discrete closed network for a metric space (X, d), then there exists a compatible quasi-metric ϱ on (X, d) such that $\mathscr{H}$ is a subbase for (X, ϱ^*).*

Recall that a quasi-metric ϱ on X is *bicomplete* if the metric ϱ^* is complete (cf. e. g. [14]).

Lemma 5. *Let X be a subspace of a complete metric space (Y, d). If $\mathscr{A} = \{A_s\colon s \in S\}$ is a discrete in X collection of F_σ-subsets of X, then there exists a compatible bicomplete quasi-metric ϱ on (Y, d) such that $\dim(Y, \varrho^*) = 0$ and $\mathscr{A} \subseteq \mathscr{G}(X, \varrho^*)$.*

P r o o f. Using similar arguments as in the proof of Lemma 2 of [6], we can represent each set A_s in the form $A_s = \bigcup_{n=1}^{\infty} A_{n,s}$ where $A_{n,s}$ are closed sets in (X,d) and each collection $\{A_{n,s}\colon s\in S\}$ is metrically discrete in (X,d). Then each collection $\mathscr{A}_n = \{\mathrm{cl}_{(Y,d)} A_{n,s}\colon s\in S\}$ is discrete in (Y,d). Choose a closed network $\mathscr{D} = \bigcup_{m=1}^{\infty} \mathscr{D}_m$ for (Y,d) such that the collections $\mathscr{D}_m$ are discrete in (Y,d). For $m,n\in\mathbb{N}$, define $\mathscr{H}_{m,n} = \{D\cap \mathrm{cl}_{(Y,d)} A_{n,s}\colon s\in S,\ D\in\mathscr{D}_m \ \&\ D\cap A_{n,s}\neq\emptyset\}\cup\{D\in\mathscr{D}_m\colon D\cap\mathrm{cl}_{(Y,d)} A_{n,s}=\emptyset$ for each $s\in S\}$. Then $\mathscr{H}_{m,n}$ are discrete collections of closed subsets of (Y,d). It is easily seen that the collection $\mathscr{H} = \bigcup_{m,n=1}^{\infty}\mathscr{H}_{m,n}$ serves as a network for (Y,d). In view of Theorem 4, there exists a compatible quasi-metric ϱ on (Y,d) such that $\mathscr{H}$ is a subbase for (Y,ϱ^*). By Theorem 2, the space (Y,ϱ^*) is completely metrizable; therefore, according to Theorem 2.1 of [14], we may assume that ϱ is bicomplete. By Proposition 1, the collection $\mathscr{H}$ consists of clopen subsets of (Y,ϱ^*) and is σ-discrete in (Y,ϱ^*). It follows from [1, 7.3.2 & 7.3.6] that $\dim(Y,\varrho^*)=0$. Since $\mathscr{D}$ is a network for (Y,d), each set $\mathrm{cl}_{(Y,d)} A_{n,s}$ is expressible as the union of some members of $\mathscr{H}$; therefore $\mathscr{A}\subseteq\mathscr{G}(X,\varrho^*)$. □

In what follows, we shall regard 0 as a limit ordinal.

Theorem 6. *Let X be a subspace of a complete metric space (Y,d) and let $\alpha<\omega_1$. If we are given a compatible quasi-metric ϱ on X, then there exists a compatible bicomplete quasi-metric ϱ_0 on (Y,d) such that $\dim(Y,\varrho_0^*)=0$ and $\mathscr{G}(X,\varrho^*)\subseteq\mathscr{G}(X,\varrho_0^*)$; furthermore:*

(6.1) *if $(X,d)\in\mathscr{F}_{\alpha+1}(Y,d)$, resp. $(X,d)\in\mathscr{G}_{\alpha+1}(Y,d)$, then $(X,\varrho_0^*)\in\mathscr{G}_\alpha(Y,\varrho_0^*)$, resp. $(X,\varrho_0^*)\in\mathscr{F}_\alpha(Y,\varrho_0^*)$;*

(6.2) *if $\alpha\neq 0$ is a limit ordinal and $(X,d)\in\mathscr{F}_\alpha(Y,d)$, resp. $(X,d)\in\mathscr{G}_\alpha(Y,d)$, then*

$$(X,\varrho_0^*)\in\Big[\bigcup_{\gamma<\alpha}\mathscr{G}_\gamma(Y,\varrho_0^*)\Big]_\delta,\quad \textit{resp. } (X,\varrho_0^*)\in\Big[\bigcup_{\gamma<\alpha}\mathscr{F}_\gamma(Y,\varrho_0^*)\Big]_\sigma;$$

(6.3) *if (X,d) is absolutely analytic, so is (X,ϱ_0^*).*

P r o o f. Let $\mathscr{B}=\bigcup_{n=1}^{\infty}\mathscr{B}_n$ serve as a base for (X,ϱ^*) such that each collection $\mathscr{B}_n$ is metrically discrete in (X,ϱ^*). In view of Lemma 3, for each $n\in\mathbb{N}$, there exist discrete in (X,d) collections $\mathscr{A}_{m,n}$ of sets of type F_σ in (X,d) such that each member of $\mathscr{B}_n$ is expressible as the union of some members of the collection $\bigcup_{m=1}^{\infty}\mathscr{A}_{m,n}$. According to Lemma 5, there exist compatible bicomplete quasi-metrics $\varrho_{m,n}$ on (Y,d) such that $\dim(Y,\varrho_{m,n}^*)=0$ and $\mathscr{A}_{m,n}\subseteq\mathscr{G}(X,\varrho_{m,n}^*)$ for

$m,n \in \mathbb{N}$. Moreover, it was proved in [11] that, if $(X,d) \in \mathscr{F}_{\alpha+1}(Y,d)$, resp. $(X,d) \in \mathscr{G}_{\alpha+1}(Y,d)$, then there exists a compatible quasi-metric ϱ_1 on (Y,d) such that $(X,\varrho_1^*) \in \mathscr{G}_\alpha(Y,\varrho_1^*)$, resp. $(X,\varrho_1^*) \in \mathscr{F}_\alpha(Y,\varrho_1^*)$, and $\dim(Y,\varrho_1^*) = 0$. By Theorem 2, the space (Y,ϱ_1^*) is completely metrizable. In the light of Theorem 2.1 of [14], we may demand that ϱ_1 be bicomplete. Let us arrange the quasi-metrics $\varrho_{m,n}$ into a sequence $\varrho_2,\ \varrho_3,\ \ldots$ Define $\varrho_0 = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}\min\{\varrho_n, 1\}$ if $(X,d) \in \mathscr{F}_{\alpha+1}(Y,d) \cup \mathscr{G}_{\alpha+1}(Y,d)$, and $\varrho_0 = \sum\limits_{n=2}^{\infty} \frac{1}{2^n}\min\{\varrho_n, 1\}$ otherwise. Without any difficulties one can check that the quasi-metric ϱ_0 has all the properties required in (6.1). The proof of (6.2) is similar. In order to get (6.3), it suffices to apply Proposition 1. □

Theorem 7. *An arbitrary compatible quasi-metric ϱ on a metrizable space X has the following properties:*

(7.1) *If $1 < \alpha < \omega_1$ (resp. $0 < \alpha < \omega_1$) and X is an absolute $\mathscr{F}_\alpha$-set (resp. $\mathscr{G}_\alpha$-set), then (X,ϱ^*) is an absolute $\mathscr{F}_\alpha$-set (resp. $\mathscr{G}_\alpha$-set).*

(7.2) *If X is absolutely analytic, so is (X,ϱ^*).*

Proof. It follows from Theorem 2 that if X is an absolute G_δ-set or, respectively, $G_{\delta\sigma}$-set, so is (X,ϱ^*).

Assume that $1 < \alpha$ (resp. $2 < \alpha$) and that X is an absolute $\mathscr{F}_\alpha$-set (resp. $\mathscr{G}_\alpha$-set). Let us regard X as a subspace of a complete metric space (Y,d). Take a compatible bicomplete quasi-metric ϱ_0 on (Y,d) which satifies all the properties described in Theorem 6. Put $\varrho_1 = \max\{\varrho^*, \varrho_0 \restriction_{X\times X}\}$. Since $\mathscr{G}(X,\varrho_0) \subseteq \mathscr{G}(X,\varrho^*) \subseteq \mathscr{G}(X,\varrho_0^*)$, the function ϱ_1 is a compatible quasi-metric on (X,ϱ^*) such that $\mathscr{G}(X,\varrho_1^*) = \mathscr{G}(X,\varrho_0^*)$. Without loss of generality, we may assume that $\varrho_1 \leqslant 1$. Denote by $\left(\tilde{X},\tilde{\varrho}^*\right)$ the metric completion of (X,ϱ^*). It was proved in [9] that there exist an $F_{\sigma\delta}$ subset A of $\left(\tilde{X},\tilde{\varrho}^*\right)$ and a compatible quasi-metric ϱ_2 on A, such that $X \subseteq A$ and $\varrho_1(x,y) = \varrho_2(x,y)$ for all $x,y \in X$. Then (X,ϱ_1^*) is a subspace of the metric space (A,ϱ_2^*). Since the metric ϱ_0^* is compatible on (X,ϱ_1^*), while $1 < \alpha$ (resp. $2 < \alpha$), to conclude that (X,ϱ^*) is an absolute $\mathscr{F}_\alpha$-set (resp. $\mathscr{G}_\alpha$-set), it suffices to observe that $\mathscr{G}(A,\varrho_2^*) \subseteq \mathscr{F}_1(A,\tilde{\varrho}^*)$. Using similar arguments as above, we can show (7.2). □

Let us mention that it was proved in [11] that if ϱ is a compatible quasi-metric on a metric space (X,d) such that (X,ϱ^*) is an absolute $\mathscr{F}_\alpha$-set (resp. $\mathscr{G}_\alpha$-set), then (X,d) is an absolute $\mathscr{G}_{\alpha+1}$-set (resp. $\mathscr{F}_{\alpha+1}$-set).

The following lemma can be proved by transfinite induction:

Lemma 8. *Let $0 < \alpha < \omega_1$ and let $\mathscr{D}$ be a collection of closed subsets of a topological space X. If $\mathscr{T}$ is a topology on the set X such that $\mathscr{D} \subseteq \mathscr{T}$, then*

$$\mathscr{F}_\alpha(\mathscr{D}) = \begin{cases} \Big[\bigcup_{\gamma<\alpha} \mathscr{G}_\gamma(X,\mathscr{T})\Big]_\delta & \text{when } \alpha \text{ is even,} \\ \Big[\bigcup_{\gamma<\alpha} \mathscr{G}_\gamma(X,\mathscr{T})\Big]_\sigma & \text{when } \alpha \text{ is odd.} \end{cases}$$

and

$$\mathscr{G}_\alpha(\mathscr{D}_c) = \begin{cases} \Big[\bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(X,\mathscr{T})\Big]_\sigma & \text{when } \alpha \text{ is even,} \\ \Big[\bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(X,\mathscr{T})\Big]_\delta & \text{when } \alpha \text{ is odd.} \end{cases}$$

Lemma 9. *Let X be a topological space and let $\alpha < \omega_1$. Then, for every discrete collection $\mathscr{A} = \{A_s\colon s \in S\} \subseteq \mathscr{F}_\alpha(X)$, there exists a σ-discrete collection $\mathscr{D}$ of closed subsets of X such that $\mathscr{A} \subseteq \mathscr{F}_\alpha(\mathscr{D})$.*

P r o o f. Our lemma holds for $\alpha = 0$. Suppose that it holds for all $\gamma < \alpha$ where $\alpha \neq 0$. Assume first that α is even. For each $s \in S$, there exist sets $A_{n,s} \in \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(X)$ such that $A_s = \bigcap_{n=1}^{\infty} A_{n,s}$. The collections $\mathscr{B}_n = \{A_{n,s} \cap \mathrm{cl}_X A_s\colon s \in S\}$ are discrete in X. For $n \in \mathbb{N}$ and $\gamma < \alpha$, put $S_{n,\gamma} = \{s \in S\colon A_{n,s} \in \mathscr{F}_\gamma(X)\}$ and $\mathscr{C}_{n,\gamma} = \{A_{n,s} \cap \mathrm{cl}_X A_s\colon s \in S_{n,\gamma}\}$. Under the inductive assumption, for any $n \in \mathbb{N}$ and $\gamma < \alpha$, there exists a σ-discrete collection $\mathscr{D}_{n,\gamma}$ of closed subsets of X such that $\mathscr{C}_{n,\gamma} \subseteq \mathscr{F}_\gamma(\mathscr{D}_{n,\gamma})$. The collection $\mathscr{D} = \bigcup_{n=1}^{\infty} \bigcup_{\gamma<\alpha} \mathscr{D}_{n,\gamma}$ is σ-discrete and $\bigcup_{n=1}^{\infty} \mathscr{B}_n \subseteq \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(\mathscr{D})$. Since $A_s = \bigcap_{n=1}^{\infty} (A_{n,s} \cap \mathrm{cl}_X A_s)$, we have $\mathscr{A} \subseteq [\bigcup_{\gamma<\alpha} \mathscr{F}_\gamma(\mathscr{D})]_\delta = \mathscr{F}_\alpha(\mathscr{D})$.

Now, assume that $\alpha = \tau + m$ where τ is a limit ordinal and $m \in \mathbb{N}$ is odd. Then $A_s = \bigcup_{n=1}^{\infty} A_{n,s}$ where $A_{n,s} \in \mathscr{F}_{\tau+m-1}(X)$. Put $\mathscr{B}_n = \{A_{n,s}\colon s \in S\}$ for $n \in \mathbb{N}$. The collections $\mathscr{B}_n$ are discrete, so there exist σ-discrete collections $\mathscr{D}_n$ of closed subsets of X such that $\mathscr{B}_n \subseteq \mathscr{F}_{\tau+m-1}(\mathscr{D}_n)$. The collection $\mathscr{D} = \bigcup_{n=1}^{\infty} \mathscr{D}_n$ is σ-discrete and has the property that $\bigcup_{n=1}^{\infty} \mathscr{B}_n \subseteq \mathscr{F}_{\tau+m-1}(\mathscr{D})$. Then $\mathscr{A} \subseteq \mathscr{F}_{\tau+m}(\mathscr{D})$. $\square$

Lemma 10. *Let X be a collectionwise normal space and let $\alpha < \omega_1$. Then, for every discrete collection $\mathscr{A} = \{A_s\colon s \in S\} \subseteq \mathscr{G}_\alpha(X)$, there exists a σ-discrete collection $\mathscr{D}$ of open subsets of X such that $\mathscr{A} \subseteq \mathscr{G}_\alpha(\mathscr{D})$.*

P r o o f. Let us modify a little the proof of Lemma 9. Namely, since the space X is collectionwise normal, we can find a discrete collection $\{U_s\colon s \in S\}$ of open subsets of X such that $A_s \subseteq U_s$ for $s \in S$. Assume that $\alpha = \tau + m$ where τ is a limit ordinal and $m \in \mathbb{N}$ is odd. Then $A_s = \bigcap_{n=1}^{\infty} A_{n,s}$ for some $A_{n,s} \in \mathscr{G}_{\tau+m-1}(X)$. We consider discrete collections $\mathscr{B}_n = \{A_{n,s} \cap U_s\colon s \in S\} \subseteq \mathscr{G}_{\tau+m-1}(X)$ in order to find σ-discrete collections $\mathscr{D}_n$ of open subsets of X such that $\mathscr{B}_n \subseteq \mathscr{G}_{\tau+m-1}(\mathscr{D}_n)$. Then $\mathscr{A} \subseteq \mathscr{G}_{\tau+m}(\bigcup_{n=1}^{\infty} \mathscr{D}_n)$.

If α is a non-zero even ordinal, then $A_s = \bigcup_{n=1}^{\infty} A_{n,s}$ for some $A_{n,s} \in \bigcup_{\gamma<\alpha} \mathscr{G}_\gamma(X)$. Put $S_{n,\gamma} = \{s \in S\colon A_{n,s} \in \mathscr{G}_\gamma(X)\}$ and $\mathscr{C}_{n,\gamma} = \{A_{n,s}\colon s \in S_{n,\gamma}\}$ for $n \in \mathbb{N}$ and $\gamma < \alpha$. Making use of the inductive assumption, for any $n \in \mathbb{N}$ and $\gamma < \alpha$, we can find a σ-discrete collection $\mathscr{D}_{n,\gamma}$ of open subsets of X such that $\mathscr{C}_{n,\gamma} \subseteq \mathscr{G}_\gamma(\mathscr{D}_{n,\gamma})$. The proof will be completed if we put $\mathscr{D} = \bigcup_{n=1}^{\infty} \bigcup_{\gamma<\alpha} \mathscr{D}_{n,\gamma}$. □

Our next lemma is a consequence of Lemma 4 of [11].

Lemma 11. *For every σ-discrete collection $\mathscr{D}$ of closed subsets of a metrizable space X, there exists a compatible quasi-metric ϱ on X such that $\mathscr{D} \subseteq \mathscr{G}(X, \varrho^*)$.*

Lemma 12. *For every σ-discrete collection $\mathscr{D}$ of open subsets of a metrizable space X, there exists a compatible quasi-metric ϱ on X such that $\mathscr{D} \subseteq \mathscr{F}(X, \varrho^*)$.*

P r o o f. Let $\mathscr{D} = \bigcup_{n=1}^{\infty} \mathscr{D}_n \subseteq \mathscr{G}(X)$ where each collection $\mathscr{D}_n$ is discrete in X. Take any compatible metric d on X that is bounded by 1. For any $n \in \mathbb{N}$ and $D \in \mathscr{D}_n$, define

$$\varrho_{n,D}(x,y) = \begin{cases} d(x,y) + 1 & \text{if } y \notin D \text{ and } x \in D,\\ d(x,y) & \text{otherwise.} \end{cases}$$

Put $\varrho_n = \sup\{\varrho_{n,D}\colon D \in \mathscr{D}_n\}$ and $\varrho = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \varrho_n$. It is obvious that ϱ_n are quasi-metrics on X such that $\mathscr{G}(X,d) \subseteq \mathscr{G}(X,\varrho_n)$. To show that $\mathscr{G}(X,\varrho_n) \subseteq \mathscr{G}(X,d)$, for a given $x \in X$, choose a neighbourhood U of x in (X,d) such that U meets at most one member of $\mathscr{D}_n$. Let $D_0 \in \mathscr{D}_n$ be such that $U \cap D = \emptyset$ for all $D \in \mathscr{D}_n \setminus \{D_0\}$. If $x \in D_0$, we can find $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq U \cap D_0$. If $x \notin D_0$, we choose $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq U$. In both cases, we have $\varrho_{n,D}(x,y) = d(x,y)$ for any $y \in B_d(x,\varepsilon)$ and $D \in \mathscr{D}_n$; hence $B_d(x,\varepsilon) \subseteq B_{\varrho_n}(x,\varepsilon)$ and, in consequence, $\mathscr{G}(X,\varrho_n) \subseteq \mathscr{G}(X,d)$. Now, using similar arguments as in the proofs of Lemmas 2 and 4 given in [11], we can show that ϱ is a compatible with d quasi-metric on X such that all members of $\mathscr{D}$ are clopen in (X,ϱ^*). □

Remark. In connection with Theorem 4, let us note that if $\mathscr{D}$ is a σ-discrete base for a metric space (X,d), then the construction of a compatible quasi-metric ϱ on (X,d) such that $\mathscr{D} \subseteq \mathscr{F}(X,\varrho^*)$ described in Lemma 12 need not lead to a quasi-metric ϱ such that $\mathscr{D}$ is a subbase for (X,ϱ^*) (cf. [11, construction of Lemma 4]).

Since σ-discretely decomposable families are of fundamental importance in descriptive set theory, it is worthwhile to investigate deeper their behaviour with respect to compatible quasi-metrics. Although not all of the properties of σ-discretely decomposable families described below are applied in this paper, we include them because they seem to be of independent interest.

The following theorem is a generalization of Proposition 9 of [11].

Theorem 13. *For every ordinal $\alpha < \omega_1$ and every σ-discrete collection $\mathscr{A}$ of subsets of a metric space (X,d), the following conditions are equivalent:*

(13.1) $\mathscr{A} \subseteq \mathscr{F}_{\alpha+1}(X,d)$, *resp.* $\mathscr{A} \subseteq \mathscr{G}_{\alpha+1}(X,d)$.

(13.2) *There exists a compatible quasi-metric ϱ on (X,d) such that $\mathscr{A} \subseteq \mathscr{G}_\alpha(X,\varrho^*)$, resp. $\mathscr{A} \subseteq \mathscr{F}_\alpha(X,\varrho^*)$.*

(13.3) *There exists a compatible quasi-metric ϱ on (X,d) such that $\mathscr{A} \subseteq \mathscr{G}_\alpha(X,\varrho^*)$, resp. $\mathscr{A} \subseteq \mathscr{F}_\alpha(X,\varrho^*)$, $\dim(X,\varrho^*) = 0$ and, moreover, if (X,d) is an absolute $F_{\sigma\delta}$-set, we may demand that ϱ be bicomplete.*

Furthermore, if $\mathscr{A} \subseteq \mathscr{F}_{\alpha+1}(X,d) \cap \mathscr{G}_{\alpha+1}(X,d)$, the quasi-metric ϱ appearing in (13.3) can be constructed in such a way that $\mathscr{A} \subseteq \mathscr{F}_\alpha(X,\varrho^*) \cap \mathscr{G}_\alpha(X,\varrho^*)$.

P r o o f. Assume (13.1). It follows from Lemmas 9 and 10 that there exists a σ-discrete collection $\mathscr{D}$ of closed (resp. open) subsets of (X,d) such that $\mathscr{A} \subseteq \mathscr{F}_{\alpha+1}(\mathscr{D})$, resp. $\mathscr{A} \subseteq \mathscr{G}_{\alpha+1}(\mathscr{D})$. According to Lemmas 11 and 12, there exists a compatible quasi-metric ϱ on (X,d) such that $\mathscr{D} \subseteq \mathscr{G}(X,\varrho^*)$, resp. $\mathscr{D} \subseteq \mathscr{F}(X,\varrho^*)$. Now, making use of Lemma 8, we obtain that (13.1) implies (13.2).

The implication (13.2) $\Rightarrow$ (13.3) is a consequence of Theorem 6 and [14, Thm. 2.1]. That (13.3) implies (13.1) follows from Proposition 1.

Suppose now that $\mathscr{A} \subseteq \mathscr{F}_{\alpha+1}(X,d) \cap \mathscr{G}_{\alpha+1}(X,d)$. We can find compatible quasi-metrics ϱ_1, ϱ_2 on X such that $\mathscr{A} \subseteq \mathscr{G}_\alpha(X,\varrho_1^*) \cap \mathscr{F}_\alpha(X,\varrho_2^*)$. If we put $\varrho = \varrho_1 + \varrho_2$, then $\mathscr{A} \subseteq \mathscr{F}_\alpha(X,\varrho^*) \cap \mathscr{G}_\alpha(X,\varrho^*)$. □

Theorem 14. *For every ordinal $\alpha < \omega_1$ and every collection $\mathscr{A}$ of subsets of a metric space (X,d), the following conditions are equivalent:*

(14.1) *$\mathscr{A}$ is σ-discretely decomposable in (X,d) into members of $\mathscr{F}_{\alpha+1}(X,d)$, resp. $\mathscr{G}_{\alpha+1}(X,d)$.*

(14.2) *There exists a compatible quasi-metric ϱ on (X,d) such that $\mathscr{A}$ is σ-discretely decomposable in (X,ϱ^*) into members of $\mathscr{G}_\alpha(X,\varrho^*)$, resp. $\mathscr{F}_\alpha(X,\varrho^*)$.*

(14.3) *There exists a compatible quasi-metric ϱ on (X,d) such that $\mathscr{A}$ is σ-discretely decomposable in (X,ϱ^*) into members of $\mathscr{G}_\alpha(X,\varrho^*)$, resp. $\mathscr{F}_\alpha(X,\varrho^*)$,* $\dim(X,\varrho^*)=0$ *and, moreover, if (X,d) is an absolute $F_{\sigma\delta}$-set, we may demand that ϱ be bicomplete.*

Furthermore, if $\mathscr{A}$ is σ-discretely decomposable in (X,d) into Borel sets of ambiguous class $\alpha+1$ in (X,d), we can construct a compatible quasi-metric ϱ on (X,d) such that $\mathscr{A}$ is σ-discretely decomposable in (X,ϱ^*) into Borel sets of ambiguous class α in (X,ϱ^*) and ϱ has all other properties described in (14.3).

Proof. The implications (14.1) $\Longrightarrow$ (14.2) $\Longrightarrow$ (14.3) follow from Theorem 13. Assume (14.3). Using the same technique as in [6, proof of Lemma 2], we can find a sequence of collections $\mathscr{D}_n \subseteq \mathscr{G}_\alpha(X,\varrho^*)$, resp. $\mathscr{D}_n \subseteq \mathscr{F}_\alpha(X,\varrho^*)$ that are metrically discrete in (X,ϱ^*), such that each member of $\mathscr{A}$ is the union of some members of $\bigcup_{n=1}^{\infty} \mathscr{D}_n$. According to Lemma 3, each collection $\mathscr{D}_n$ is σ-discretely decomposable in (X,d) into members of $\mathscr{F}_{\alpha+1}(X,d)$, resp. $\mathscr{G}_{\alpha+1}(X,d)$. This shows that (14.3) implies (14.1).

3. Making a mapping continuous

Let us pass to a characterization of those mappings f from a metrizable space X to a topological space Y for which there exists a compatible quasi-metric ϱ on X such that $f\colon (X,\varrho^*)\to Y$ is continuous. □

Definitions. Let X, Y be topological spaces and let $f\colon X\to Y$. A collection $\mathscr{B}$ of subsets of X is called a *base for the mapping* f if, for every open set $V\subseteq Y$, the inverse image $f^{-1}(V)$ is the union of some sets from $\mathscr{B}$. If, in addition, the collection $\mathscr{B}$ is σ-discrete, we say that $\mathscr{B}$ is a *σ-discrete (in X) base for f*.

The mapping f is called *σ-discrete* if f has a σ-discrete base (cf. e.g. [6] and [7]).

Theorem 15. *For a mapping f from a metric space (X,d) to a topological space Y, the following conditions are equivalent:*

(15.1) *The mapping f has a σ-discrete in (X,d) base consisting of F_σ-sets of (X,d).*

(15.2) *There exists a compatible quasi-metric ϱ on (X,d) such that $f\colon (X,\varrho^*)\to Y$ is continuous.*

(15.3) *There exists a compatible quasi-metric ϱ on (X,d) such that* $\dim(X,\varrho^*)=0$ *and $f\colon (X,\varrho^*)\to Y$ is continuous; furthermore, if (X,d) is an absolute $F_{\sigma\delta}$-set, we may demand that ϱ be bicomplete.*

Proof. To show that (15.2) is a consequence of (15.1), it suffices to apply Theorem 13. That (15.2) implies (15.3) follows from Theorem 6 and [14, Thm. 2.1]. If ϱ

is a compatible quasi-metric on (X,d) such that $f\colon (X,\varrho^*) \to Y$ is continuous, then a σ-metrically discrete base $\mathscr{B}$ for the metric space (X,ϱ^*) is a base for f. It follows from Lemma 3 that $\mathscr{B}$ is σ-discretely decomposable in (X,d) into F_σ-sets of (X,d); hence (15.3) implies (15.1), which completes the proof. $\square$

Let us recall that a mapping f from a topological space X to a topological space Y is of *Borel class* α, $\alpha < \omega_1$, if the inverse image $f^{-1}(V)$ of every open set $V \subseteq Y$ is of additive class α in X.

In the light of Theorem 15, if for a mapping f from a metric space (X,d) to a topological space Y there exists a compatible quasi-metric ϱ on (X,d) such that $f\colon (X,\varrho^*) \to Y$ is continuous, then $f\colon (X,d) \to Y$ is a σ-discrete mapping of Borel class 1. We do not know if the σ-discreteness of a mapping f of Borel class 1 from a metric space (X,d) to a topological space Y is a sufficient condition in order to find a compatible quasi-metric ϱ on (X,d) such that $f\colon (X,\varrho^*) \to Y$ is continuous; however, if the space Y is also metrizable, we can apply Hansell's result of [6; 3.4, Lemma 10] to deduce the following corollary to Theorem 15:

Corollary 16. *For a mapping f from a metric space (X,d) to a metrizable space Y, conditions* (15.2) *and* (15.3) *are equivalent to the following:*

(16.1) $f\colon (X,d) \to Y$ *is a σ-discrete mapping of Borel class* 1.

Theorem 3 of [6] asserts that every Borel mapping from an absolutely analytic metrizable space X to a metrizable space Y is σ-discrete; accordingly, we can state the following:

Corollary 17. *For a mapping f from an absolutely analytic metric space (X,d) to a metrizable space Y, conditions* (15.2) *and* (15.3) *are equivalent to the following:*

(17.1) $f\colon (X,d) \to Y$ *is a mapping of Borel class* 1.

Let us say that a collection $\mathscr{A}$ of subsets of a metrizable space X is *analytic-additive* if, for each subcollection $\mathscr{A}_0$ of $\mathscr{A}$, the union $A_0 = \bigcup\{A\colon A \in \mathscr{A}_0\}$ can be obtained by applying the Suslin operation to closed subsets of X, i. e. A_0 can be represented in the form

$$A_0 = \bigcup_{\tau\in\mathbb{N}^\omega} \bigcap_{n=1}^{\infty} F(\tau \mid n)$$

where $F(\tau \mid n)$ are closed subsets of X for $\tau \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$.

Fleissner's Proposition P asserts that every point-finite analytic-additive collection in a metrizable space is σ-discretely decomposable (cf. [2]). It is not clear whether Proposition P is consistent with ZFC; however, it was proved in [3] that the Product

Measure Extension Axiom (PMEA) implies Proposition P. On the other hand, if, for instance, a Q-space exists (i. e. a metrizable space X which cannot be represented as the countable union of closed discrete subspaces but each subset of which is of type F_σ in X), then there exists a non-σ-discrete mapping between metrizable spaces which is of Borel class 1 (cf. [11]); therefore the existence of a Q-space implies that Proposition P is false. It seems unknown whether the existence of a non-σ-discrete mapping between metrizable spaces which is of Borel class 1 is equivalent to the existence of a Q-space. Some other results related to the above-mentioned set-theoretic problems can be found, e. g. in [2], [3], [5] and [11].

It is evident that if $\mathscr{B}$ is a σ-discrete base for a metrizable space Y and if $f\colon X \to Y$, then the collection $\{f^{-1}(B)\colon B \in \mathscr{B}\}$ forms a base for f which is the countable union of point-finite subcollections; therefore, with Proposition P in hand, we can establish the following:

Corollary 18. *If we assume Proposition P, then, for every mapping f from a metric space (X, d) to a metrizable space Y, conditions* (15.2), (15.3) *and* (17.1) *are all equivalent.*

Our investigations lead us in a natural way to the problem of how and when it is possible to lower the Borel class of a σ-discrete Borel map f if one replaces the original topology on the domain X of f by the topology determined by the metric ϱ^* where ϱ is a compatible quasi-metric on X. Let us devote our next section to this question.

4. Lowering the Borel classes of σ-discrete Borel maps

Let $\alpha < \omega_1$. For topological spaces X and Y, denote by $\mathscr{B}_\alpha(X, Y)$ the collection of all mappings $f\colon X \to Y$ of Borel class α. Let the symbol $\mathscr{B}_\alpha^{\sigma-d}(X, Y)$ stand for the collection of all those mappings $f\colon X \to Y$ which have σ-discrete bases in X consisting of sets of additive class α.

It follows from [6; 3.4, Lemma 10] that if X, Y are metrizable spaces, then the class $\mathscr{B}_\alpha^{\sigma-d}(X, Y)$ coincides with the class of all σ-discrete mappings which are in $\mathscr{B}_\alpha(X, Y)$.

Theorem 19. *For every ordinal $\alpha < \omega_1$ and every mapping f from a metric space (X, d) to a topological space Y, the following conditions are equivalent:*

(19.1) $f \in \mathscr{B}_{\alpha+1}^{\sigma-d}\big((X, d), Y\big)$.

(19.2) *There exists a compatible quasi-metric ϱ on (X, d) such that*

$$f \in \mathscr{B}_\alpha^{\sigma-d}\big((X, \varrho^*), Y\big).$$

(19.3) *There exists a compatible quasi-metric* ϱ *on* (X,d) *such that*

$$f \in \mathscr{B}_\alpha^{\sigma-d}\big((X,\varrho^*),Y\big),$$

$\dim(X,\varrho^*) = 0$ *and, in addition, if* (X,d) *is an absolute* $F_{\sigma\delta}$*-set, we may demand that* ϱ *be bicomplete.*

P r o o f. It suffices to apply Theorem 14. □

Remark. Let X and Y be metrizable spaces and let $\alpha < \omega_1$. Theorem 15 describes a relatively simple natural necessary and sufficient condition for a mapping $f \in \mathscr{B}_1(X,Y)$ to find a compatible quasi-metric ϱ on X such that $f \in \mathscr{B}_0\big((X,\varrho^*),Y\big)$, while Theorem 19 gives us only a sufficient condition for $f \in \mathscr{B}_{\alpha+1}(X,Y)$ in order to exist a compatible quasi-metric ϱ on X such that $f \in \mathscr{B}_\alpha\big((X,\varrho^*),Y\big)$. The following problem seems interesting:

Problem. Let $0 < \alpha < \omega_1$. If, for a mapping $f \in \mathscr{B}_{\alpha+1}\big((X,d),Y\big) \setminus \mathscr{B}_\alpha\big((X,d),Y\big)$, there exists a compatible quasi-metric ϱ on the metric space (X,d) such that $f \in \mathscr{B}_\alpha\big((X,\varrho^*),Y\big)$, must f be necessarily σ-discrete?

Let us turn our attention to those Borel mappings between metrizable spaces which can be represented analytically. Namely, for metrizable spaces X and Y, put $\mathscr{B}_1^*(X,Y) = \mathscr{B}_1^{\sigma-d}(X,Y)$ and suppose that, for a countable ordinal $\alpha > 1$, we have already defined the classes $\mathscr{B}_\gamma^*(X,Y)$ where $1 \leqslant \gamma < \alpha$. Then $\mathscr{B}_\alpha^*(X,Y)$ denotes the collection of all those mappings $f\colon X \to Y$ which are the pointwise limits of convergent sequences of mappings belonging to $\bigcup_{\gamma<\alpha} \mathscr{B}_\gamma^*(X,Y)$. For convenience, denote by $\mathscr{B}_0^*(X,Y)$ the collection of all continuous mappings from X to Y. Obviously, in general, not all mappings from $\mathscr{B}_1^*(X,Y)$ are the pointwise limits of convergent sequences of continuous functions; however, for every ordinal $\alpha < \omega_1$, we have $\mathscr{B}_\alpha^*(X,Y) \subseteq \mathscr{B}_\alpha(X,Y)$ if α is finite, while $\mathscr{B}_\alpha^*(X,Y) \subseteq \mathscr{B}_{\alpha+1}(X,Y)$ if α is infinite. Hansell proved in [7] that, for $\alpha < \omega_1$, the following inclusions hold: $\mathscr{B}_\alpha^{\sigma-d}(X,Y) \subseteq \mathscr{B}_\alpha^*(X,Y)$ if α is finite, while $\mathscr{B}_{\alpha+1}^{\sigma-d}(X,Y) \subseteq \mathscr{B}_\alpha^*(X,Y)$ if α is infinite. Since it seems still unknown whether the pointwise limit of a convergent sequence of σ-discrete mappings is necessarily σ-discrete (cf. [7, p. 210] and [15, pp. 476–477]) let us state a lemma which will allow us to replace the last two inclusions by equalities. This lemma might be hidden somewhere in the literature, but we are unable to locate it, so we shall include its proof for completeness.

Lemma 20. *If* X, Y *are metrizable spaces and a mapping* $f\colon X \to Y$ *is the pointwise limit of a sequence* $\langle f_n\rangle_{n\in\mathbb{N}}$ *of* σ*-discrete Borel mappings* $f_n\colon X \to Y$, *then* f *is* σ*-discrete.*

P r o o f. Let d_X and d_Y be compatible metrics on the spaces X and Y, respectively. Denote by $(\tilde{X}, \tilde{d_X})$ and $(\tilde{Y}, \tilde{d_Y})$ the completions of (X, d_X) and (Y, d_Y), respectively. Choose $\alpha_n < \omega_1$ such that $f_n \in \mathscr{B}_{\alpha_n+1}(X, Y)$ for $n \in \mathbb{N}$. By Theorem 9 of [7], for each $n \in \mathbb{N}$, there exist a set A_n of multiplicative class $\alpha_n + 2$ in $(\tilde{X}, \tilde{d_X})$ and a mapping $\tilde{f_n} \in \mathscr{B}_{\alpha_n+1}\big(A_n, (\tilde{Y}, \tilde{d_Y})\big)$, such that $X \subseteq A_n$ and $\tilde{f_n}(x) = f_n(x)$ for $x \in X$. Put $A = \bigcap\limits_{n=1}^{\infty} A_n$ and, for $n, m, k \in \mathbb{N}$, define

$$C_{n,m,k} = \Big\{x \in A\colon\ \tilde{d_Y}\big(\tilde{f}_n(x), \tilde{f}_{n+m}(x)\big) < \frac{1}{k}\Big\}.$$

The set A being absolutely Borel, it follows from Theorem 3 of [6] that all the mappings $g_n = \tilde{f_n} \restriction_A$ are σ-discrete in A. Therefore, according to Theorem 4 of [6], the diagonal mappings $g_n \bigtriangleup g_{n+m}$ are Borel in A and, in consequence, the sets $C_{n,m,k}$ are Borel in A. This implies that the set

$$B = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} C_{n,m,k}$$

is absolutely Borel. Clearly, $X \subseteq B$. If $x \in B$, then $\big\langle g_n(x) \big\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\tilde{Y}, \tilde{d_Y})$, so that it converges in $(\tilde{Y}, \tilde{d_Y})$ to some point $g(x) \in \tilde{Y}$. In this way, we obtain a Borel mapping $g\colon (B, \tilde{d_X}) \to (\tilde{Y}, \tilde{d_Y})$ such that $g(x) = f(x)$ for $x \in X$ (cf. [7, proof of Theorem 9]). In view of Theorem 3 of [6], the mapping g is σ-discrete, which yields that f is σ-discrete. $\square$

Corollary 21. *Let X and Y be metrizable spaces. For every ordinal $\alpha < \omega_1$, we have: $\mathscr{B}^*_\alpha(X, Y) = \mathscr{B}^{\sigma-d}_\alpha(X, Y)$ if α is finite, and $\mathscr{B}^*_\alpha(X, Y) = \mathscr{B}^{\sigma-d}_{\alpha+1}(X, Y)$ if α is infinite.*

As an immediate consequence of Theorem 19 and Corollary 21, we can state the following:

Theorem 22. *For every ordinal $\alpha < \omega_1$ and every mapping f from a metric space (X, d) to a metrizable space Y, the following conditions are equivalent:*

(22.1) $f \in \mathscr{B}^*_{\alpha+1}\big((X, d), Y\big)$.

(22.2) *There exists a compatible quasi-metric ϱ on (X, d) such that*

$$f \in \mathscr{B}^*_\alpha\big((X, \varrho^*), Y\big).$$

(22.3) *There exists a compatible quasi-metric ϱ on (X, d) such that*

$$f \in \mathscr{B}^*_\alpha\big((X, \varrho^*), Y\big),$$

$\dim(X, \varrho^*) = 0$ *and, in addition, if* (X, d) *is an absolute* $F_{\sigma\delta}$*-set, we may demand that* ϱ *be bicomplete.*

Corollary 23. *For every ordinal* $\alpha < \omega_1$ *and every mapping* f *from an absolutely analytic metric space* (X, d) *to a metrizable space* Y*, the following conditions are equivalent:*

(23.1) $f \in \mathscr{B}_{\alpha+1}\big((X, d), Y\big)$.

(23.2) *There exists a compatible quasi-metric* ϱ *on* (X, d) *such that*

$$f \in \mathscr{B}_\alpha\big((X, \varrho^*), Y\big).$$

(23.3) *There exists a compatible quasi-metric* ϱ *on* (X, d) *such that*

$$f \in \mathscr{B}_\alpha\big((X, \varrho^*), Y\big),$$

$\dim(X, \varrho^*) = 0$ *and, in addition, if* (X, d) *is an absolute* $F_{\sigma\delta}$*-set, we may demand that* ϱ *be bicomplete.*

Corollary 24. *If we assume Proposition P, then, for every ordinal* $\alpha < \omega_1$ *and every mapping* f *from a metric space* (X, d) *to a metrizable space* Y*, conditions* (23.1)–(23.3) *are equivalent.*

Finally, let us note that, in conditions (14.3), (15.3), (19.3), (22.3) and (23.3) we may always require of ϱ to share all the properties of ϱ_0 described in Theorem 6.

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