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A NOTE ON DISTRIBUTIVE DOUBLE p-ALGEBRAS[1]

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In this paper we prove a congruence representation theorem for distributive double p-algebras, which is a natural analogous to the representation theorem given by Lakser [5] for p-algebras. This theorem induces a natural approach to the study of existence of solutions of systems of congruences. Also, we obtain a new characterization of the subdirectly irreducible distributive double p-algebras, which were characterized by Katriňák [4].

Preliminaries

For notation and basic facts on distributive p-algebras we refer the reader to [4]. Through the paper L will denote a distributive double p-algebra $\langle L, \vee, \wedge, ^*, ^+, 0, 1\rangle$, where * is the pseudocomplementation operation and $^+$ is the dual pseudocomplementation operation. As usual, $D(L)$ (resp. $\overline{D}(L)$) will denote the filter (ideal) of *dense* (*dual dense*) *elements* of L. $B(L)$ ($\overline{B}(L)$) will be the *skeleton* (*dual skeleton*) and $\dot{\vee}$ ($\dot{\wedge}$) will denote the join (meet) operation of $B(L)$ ($\overline{B}(L)$). The relation ϱ^L defined by $(x,y) \in \varrho^L$ iff $x^* = y^*$ and $x^+ = y^+$ is easily seen to be a congruence relation on L, the *determination congruence relation.* We use $G[x]$ to denote the sublattice $\{z \in L\colon\ (x,z) \in \varrho^L\}$. If $x \in L$, then we denote $d_x = x \vee x^*$ and $b_x = x \wedge x^+$. For any algebra A, $\mathrm{Con}(A)$ denotes the congruence lattice of A. Let $(\varrho^L] = \{\theta \in \mathrm{Con}(L)\colon\ \theta \subseteq \varrho^L\}$.

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By $\mathrm{Ct}(L)$ we denote the set of all 3-tuples $(\gamma,\delta,\sigma)\in\mathrm{Con}(B(L))\times\mathrm{Con}(\overline{B}(L))\times(\varrho^L]$ which satisfy

(T1) $(a,1)\in\gamma$, $d\in D(L)$ and $b\in\overline{D}(L)$ imply $((a\wedge b)\vee d, b\vee d)\in\sigma$,

(T2) $(a,0)\in\delta$, $d\in D(L)$ and $b\in\overline{D}(L)$ imply $((a\vee d)\wedge b, b\wedge d)\in\sigma$,

(T3) $(a,1)\in\gamma$ implies $(a^{++},1)\in\delta$,

(T4) $(a,0)\in\delta$ implies $(a^{**},0)\in\gamma$.

If $\theta\in\mathrm{Con}(L)$ then by θ_B, $\theta_{\overline{B}}$ we denote the restriction of θ to $B(L)$ and $\overline{B}(L)$, respectively.

Theorem 1. *Let L be a distributive double* p-*algebra. Then, the map*

$$\mathrm{Con}(L)\longrightarrow\mathrm{Con}(B(L))\times\mathrm{Con}(\overline{B}(L))\times(\varrho^L]$$
$$\theta\longrightarrow(\theta_B,\theta_{\overline{B}},\theta\wedge\varrho^L).$$

is a 1-1 homomorphism which maps $\mathrm{Con}(L)$ *onto* $\mathrm{Ct}(L)$*. If* $(\gamma,\delta,\sigma)\in\mathrm{Ct}(L)$ *then the corresponding congruence* $\theta\in\mathrm{Con}(L)$ *is determined by*

(I) $(x,y)\in\theta$ *iff* $(x^{**},y^{**})\in\gamma$, $(x^{++},y^{++})\in\delta$ *and*

$$(b_x\vee(d_x\wedge d_y), b_y\vee(d_x\wedge d_y))\in\sigma.$$

P r o o f. Since $x=x^{++}\vee(x^{**}\wedge(b_x\vee(d_x\wedge d_y)))$ and $y=y^{++}\vee(y^{**}\wedge(b_y\vee(d_x\wedge d_y)))$ for every $x,y\in L$, we have that for every $\theta\in\mathrm{Con}(L)$

(1) $(x,y)\in\theta$ iff $(x^{**},y^{**})\in\theta_B$, $(x^{++},y^{++})\in\theta_{\overline{B}}$ and

$$(b_x\vee(d_x\wedge d_y), b_y\vee(d_x\wedge d_y))\in\theta\wedge\varrho^L.$$

Let $(\gamma,\delta,\sigma)\in\mathrm{Ct}(L)$ and let θ_1 be the lattice congruence on L determined by $(x,y)\in\theta_1$ iff $(x^{**},y^{**})\in\gamma$, $(x^{++},y^{++})\in\delta$ and $((x\wedge b)\vee d,(y\wedge b)\vee d)\in\sigma$ for every $d\in D(L)$ and every $b\in\overline{D}(L)$. We claim that $\theta_1\in\mathrm{Con}(L)$. Let $(x,y)\in\theta_1$. Since $(x^*,y^*)\in\gamma$ and γ is a Boolean congruence, we have that $x^*\wedge a=y^*\wedge a$ for some $a\in B(L)$ such that $(a,1)\in\gamma$. Let $b\in\overline{D}(L)$ and $d\in D(L)$. By (T1) we have that $((a\wedge b)\vee d, b\vee d)\in\sigma$ and therefore it can be proved that

$$((x^*\wedge b)\vee d,(x^*\wedge a\wedge b)\vee d)\in\sigma.$$

In a similar manner we show that $((y^*\wedge b)\vee d$, $(y^*\wedge a\wedge b)\vee d)\in\sigma$ and therefore $((x^*\wedge b)\vee d,(y^*\wedge b)\vee d)\in\sigma$. Furthermore, since

$$(x^{*++}\wedge a^{++})^{++}=(y^{*++}\wedge a^{++})^{++}$$

and $(a^{++},1)\in\delta$, we have that $(x^{*++},y^{*++})\in\delta$ (use that δ is a Boolean congruence) and therefore $(x^*,y^*)\in\theta_1$. Note that it is readily proved that $\gamma\subseteq\theta_{1B}$. In a similar manner we show that θ_1 preserves $^+$ and that $\delta\subseteq\theta_{1\overline{B}}$. Thus the claim is established. Furthermore we have that

(2) $\theta_{1B}\subseteq\gamma$, $\theta_{1\overline{B}}\subseteq\delta$ and $\sigma=\theta_1\wedge\varrho^L$.

We will only prove that $\theta_1\wedge\varrho^L\subseteq\sigma$. The other inclusions are easy to check. Let $(x,y)\in\theta_1\wedge\varrho^L$ and let $d=(x\wedge y)\vee x^*$. Note that $(x,y)\in\theta_1\wedge\varrho^L$ implies $(x,x\wedge y)\in\sigma$. Therefore, $x^*=y^*$ and $x^+=y^+$, as well as

$$(r,s)=((x\wedge b_x)\vee d,(y\wedge b_x)\vee d)\in\sigma.$$

It follows that $(x^{**}\wedge r,x^{**}\wedge s)\in\sigma$ and consequently,

$$(x^{++}\vee(x^{**}\wedge r),x^{++}\vee(x^{**}\wedge s))=(x,x\wedge y)\in\sigma.$$

In a similar manner we show that $(x\wedge y,y)\in\sigma$ and therefore $(x,y)\in\sigma$. Thus we have proved (2) and the theorem follows from (1). □

Next, suppose that $D(L)$ has a least element z_0 and let $\mathrm{Ct}'(L)$ be the set of 3-tuples

$$(\gamma,\delta,\alpha)\in\mathrm{Con}(B(L))\times\mathrm{Con}(\overline{B}(L))\times\mathrm{Con}(G[z_0])$$

which satisfy (T3), (T4) and

(T1′) $(a,1)\in\gamma$ and $b\in\overline{D}(L)$ imply $((a\wedge b)\vee z_0,b\vee z_0)\in\alpha$,

(T2′) $(a,0)\in\delta$, and $b\in\overline{D}(L)$ imply $((a\vee z_0)\wedge b,b\wedge z_0)\in\alpha$.

Thus, we have the following result, which can be obtained from Theorem 1.

Corollary 2. *Suppose $D(L)$ has a least element z_0. The map*

$$\begin{aligned}\mathrm{Con}(L)&\longrightarrow\mathrm{Con}(B(L))\times\mathrm{Con}(\overline{B}(L))\times\mathrm{Con}(G[z_0])\\ \theta&\longrightarrow(\theta_B,\theta_{\overline{B}},\theta_{G[z_0]})\end{aligned}$$

is a 1-1 homomorphism which maps $\mathrm{Con}(L)$ onto $\mathrm{Ct}'(L)$. If $(\gamma,\delta,\alpha)\in\mathrm{Ct}'(L)$ then the corresponding congruence $\theta\in\mathrm{Con}(L)$ is determined by

(I) $(x,y)\in\theta$ *iff* $(x^{**},y^{**})\in\gamma$, $(x^{++},y^{++})\in\delta$ *and*

$$((x\wedge x^+)\vee z_0,(y\wedge y^+)\vee z_0)\in\alpha.$$

Note that the distributive double p-algebras having such least element z_0 form a variety (of type $(2,2,1,1,0,0,0)$) which contains the finite algebras.

By a *system on L* we understand a $2n$-tuple $(\theta_1, \dots, \theta_n, x_1, \dots, x_n)$, where $\theta_1, \dots, \theta_n \in \mathrm{Con}(L)$, $x_1, \dots, x_n \in L$ and $(x_i, x_j) \in \theta_i \vee \theta_j$ for every $1 \leqslant i, j \leqslant n$. A *solution* of a system $(\theta_1, \dots, \theta_n, x_1, \dots, x_n)$ is an element $x \in L$ such that $(x, x_i) \in \theta_i$ for every $1 \leqslant i \leqslant n$. We remember that an algebra is arithmetical (i.e. congruence permutable and congruence distributive) iff every system has a solution. (See [3].) In particular we have that every system on a Boolean algebra has a solution.

For $1 \leqslant i \leqslant n$ we define the terms t_i^n as follows:

$$t_i^n = b_{y_i} \vee \bigwedge\nolimits_{j=1}^{n} d_{y_j}.$$

It is easy to check that $t_i^n(\vec{x}) \in G[z]$ for every $\vec{x} \in L^n$, where $z = \bigwedge\limits_{j=1}^{n} d_{x_j}$.

Lemma 3. *If $\vec{x} \in L^n$, $d \in D(L)$ and $d \leqslant \bigwedge\limits_{j=1}^{n} d_{x_j}$ then*

$$x_i = \big((d \vee d_{x_i}) \wedge x_i^{**}\big) \vee x_i^{++},$$

for $i = 1, \dots, n$.

Consequently, for $1 \leqslant i \leqslant n$, $x_i = \big(t_i^n(\vec{x}) \wedge x_i^{**}\big) \vee x_i^{++}$.

Theorem 4. *Let $S = (\theta_1, \dots, \theta_n, x_1, \dots, x_n)$ be a system on L. Consider the following systems associated with S:*

$$S_B = (\theta_{1B}, \dots, \theta_{nB}, x_1^{**}, \dots, x_n^{**}) \text{ on } B(L),$$
$$S_{\overline{B}} = (\theta_{1\overline{B}}, \dots, \theta_{n\overline{B}}, x_1^{++}, \dots, x_n^{++}) \text{ on } \overline{B}(L)$$

and

$$S_{G[z]} = (\theta_{1G[z]}, \dots, \theta_{nG[z]}, t_1^n(\vec{x}), \dots, t_n^n(\vec{x}))$$

on $G[z]$, where $z = \bigwedge\limits_{j=1}^{n} d_{x_j}$. If $a \in B(L)$, $b \in \overline{B}(L)$ and $t \in G[z]$ are solutions of S_B, $S_{\overline{B}}$ and $S_{G[z]}$, respectively, then $(t \wedge a) \vee b$ is a solution of S. Reciprocally, if x is a solution of S, then $x^{**} \in B(L)$, $x^{++} \in \overline{B}(L)$ and $b_x \vee z \in G[z]$ are solutions of S_B, $S_{\overline{B}}$ and $S_{G[z]}$, respectively. Consequently, S has a solution in L if and only if $S_{G[z]}$ has a solution in $G[z]$.

P r o o f. For the if part, note that, by the above lemma, $((t \wedge a) \vee b, x_i) = ((t \wedge a) \vee b, (t_i^n(\vec{x}) \wedge x_i^{**}) \vee x_i^{++}) \in \theta_i$.

To prove the only if part, note that $b \vee z \in G[z]$ for every $b \in \overline{D}(L)$. Furthermore, $(b_x \vee z, t_i^n(\vec{x})) = (b_x \vee z, b_{x_i} \vee z) \in \theta_{iG[z]}$. □

Corollary 5. *(Adams and Beazer* [1]*) A distributive double p-algebra L is congruence permutable if and only if $G[x]$ is relatively complemented for every $x \in L$.*

P r o o f. It is well known that a lattice is congruence permutable (i.e. every system (θ, δ, x, y) has a solution) iff it is relatively complemented. □

Subdirectly irreducibles

In [4], Katriňák characterizes the subdirectly irreducible distributive double p-algebras. Now, using Theorem 1, we will obtain a new characterization of the non regular subdirectly irreducible distributive double p-algebras.

Remember that L is said to be *regular* if ϱ^L is the trivial congruence. Katriňák [4] calls L *nearly regular* if $|G[x]| \leqslant 2$ for every $x \in L$. By $M(L)$ (resp. $m(L)$) we denote the set of *maximal* (*minimal*) prime filters of L. It is well known that a prime filter p is maximal (minimal) if and only if $D(L) \subseteq p$ ($\overline{D}(L) \subseteq L - p$). (See [2].)

By $\theta_{lat}(x, y)$ we denote the principal lattice congruence on L generated by (x, y).

Lemma 6 (Katriňák [4]). *The following are equivalent:*

(1) *L is nearly regular,*

(2) $(\varrho^L] = \{\Delta^L, \varrho^L\}$, *where* $\Delta^L = \{(x, x)\colon\ x \in L\}$.

P r o o f. (2)⇒(1). If $x < y \leqslant z$ and $x, y \in G[z]$ then $\theta_{lat}(x, y)$, $\theta_{lat}(y, z) \in \mathrm{Con}(L)$ and $\theta_{lat}(x, y) \neq \theta_{lat}(y, z)$. Thus $y = z$.

(1)⇒(2). Suppose that L is proper nearly regular. Let p_i be prime filters such that $p_i \notin M(L) \cup m(L)$ for $i =$ 1,2. It can be checked that there exist $z \in D(L)$ and $w \in \overline{D}(L)$ such that $z \notin p_i$ and $w \in p_i$, for every $1 \leqslant i \leqslant 2$. Thus, $x \in p_i$ iff $(x \vee z) \wedge w \in p_i \cap G[w]$ for every $x \in L$ and $1 \leqslant i \leqslant 2$. Since $|G[w]| \leqslant 2$, we have that $p_1 = p_2$. Thus, we have proved that there exists at most one prime filter p such that $p \notin M(L) \cup m(L)$. Let $(z, w), (x, y) \in \varrho^L$ be such that $x < y$ and $z < w$. Since $p_1 = \{t \in L\colon\ (t \vee z) \wedge w = w\}$ and $p_2 = \{t \in L\colon\ (t \vee x) \wedge y = y\}$ are prime filters, we have that $p_1 = p_2$ and hence $(w \vee x) \wedge y = y$ and $(z \vee x) \wedge y \neq y$. Since L is nearly regular we have that $(z \vee x) \wedge y = x$ and hence $(x, y) \in \theta_{lat}(z, w)$. Thus (2) follows. The case L of regular is trivial. □

Given any $a \in L$, $a^{n(+*)}$ is defined inductively as follows:

$$a^{0(+*)} = a,\ a^{(n+1)+*} = a^{n(+*)+*},\ \text{for every } n \geqslant 0.$$

The elements $a^{n(*+)}$ are defined in a similar fashion.

Let $x \in L$. We denote $F_x = \{a \in B(L)\colon\ a \geqslant x^{n(+*)} \text{ for some } n \geqslant 1\}$ and $I_x = \{b \in \overline{B}(L)\colon\ b \leqslant x^{+n(*+)} \text{ for some } n \geqslant 1\}$. It is easy to check that F_x is a filter of $B(L)$ and I_x is an ideal of $\overline{B}(L)$. Let Θ_x (resp. Γ_x) be the congruence on $B(L)$ ($\overline{B}(L)$) associated with the filter F_x (ideal I_x).

We will say that $x \in L$ is *transversal* if for every $n \geqslant 1$, $d \in D(L)$ and $b \in \overline{D}(L)$ we have that

$$(x^{n(+*)} \wedge b) \vee d = b \vee d,$$
$$(x^{+n(*+)} \vee d) \wedge b = b \wedge d.$$

It is easy to check that

(I) x is transversal iff $(\Theta_x, \Gamma_x, \Delta^L) \in \mathrm{Ct}(L)$.

Theorem 7. *Suppose that L is not regular. Then L is (finitely) subdirectly irreducible if and only if L is nearly regular and 1 is the only transversal element.*

Proof. Suppose that L is finitely subdirectly irreducible. We claim that 1 is the only transversal element. Suppose that $(\Theta_x, \Gamma_x,\ \Delta^L) \in \mathrm{Ct}(L)$. Let θ be the congruence associated with the triple $(\Theta_x, \Gamma_x,\ \Delta^L)$. Since, by Theorem 1, $\theta \wedge \varrho^L = \Delta^L$, we have that $\theta = \Delta^L$ and hence $x = 1$. The claim follows from (I). To prove that L is nearly regular, note that if $x < y < z$ and $y, z \in G[x]$ then $\theta_{lat}(x, y)$, $\theta_{lat}(y, z) \in \mathrm{Con}(L)$ and $\theta_{lat}(x, y) \cap \theta_{lat}(y, z) = \Delta^L$.

Suppose now that L is nearly regular and 1 is the only transversal element. We will prove that ϱ^L is a monolite in $\mathrm{Con}(L)$. Let $\Delta^L \neq \theta \in \mathrm{Con}(L)$. Note that, for every $x \in [1]\theta$, $(\Theta_x, \Gamma_x, \theta(x, 1) \wedge \varrho^L) \in \mathrm{Ct}(L)$. Thus, by (I), $\theta(x, 1) \wedge \varrho^L \neq \Delta^L$ and hence, by Lemma 4, $\varrho^L \subseteq \theta(x, 1) \subseteq \theta$. □

We conclude the paper by giving a lemma from which the characterization given by Katriňák in [4] can be obtained.

Lemma 8. *If L is proper nearly regular and $x \in L$ then the following are equivalent:*

i) *x is transversal,*

ii) *$|G[d]| = 1$ for every $d \in [F_{d_x}) \cap D(L)$.*

P r o o f. i)⇒ii). Let $d \in [F_{d_x}) \cap D(L)$ and suppose that $d_1 \in G[d]$, $d \leqslant d_1$. Since x is transversal, we have that $d = (x^{n(+*)} \wedge b_{d_1}) \vee d = b_{d_1} \vee d = d_1$, where $n \geqslant 1$ is such that $x^{n(+*)} \leqslant d$. Suppose now that $d_1 \leqslant d$. We will prove that $d_1 \in [F_{d_x}) \cap D(L)$. Let z be such that $z \in F_{d_x}$ and $z \leqslant d$. Since $z \wedge d_1 \in G[z \wedge d] = G[z]$, we have that $z^{n(+*)} = (z \wedge d_1)^{n(+*)}$ for every $n \geqslant 1$. Thus, $z \wedge d_1 \in F_{d_x}$ and hence, $d_1 \in [F_{d_x}) \cap D(L)$.

ii)⇒i). Suppose that x is not transversal. We will prove that ii) is not true. We consider two cases:

CASE $(x^{n(+*)} \wedge b) \vee d \neq b \vee d$ for some $n \geqslant 1$, $d \in D(L)$ and $b \in \overline{D}(L)$. Let $z = (x^{n(+*)} \wedge b) \vee d$ and $w = b \vee d$. Since $x^{n(+*)} \wedge z = x^{n(+*)} \wedge w$, we have that $x^{n(+*)} \vee z \neq x^{n(+*)} \vee w$. Now the case follows from the observation that $(x^{n(+*)} \vee z, x^{n(+*)} \vee w) \in \varrho^L$.

CASE $(x^{+n(*+)} \vee d) \wedge b = b \wedge d$ for some $n \geqslant 1$, $d \in D(L)$ and $b \in \overline{D}(L)$. Using similar arguments as above we can show that there exists $b \in \overline{D}(L)$ satisfying $|G[b]| \neq 1$ such that $x^{+n(*+)} \geqslant b$ for some $n \geqslant 0$. Thus $x^{n+1(+*)} \leqslant b^* \leqslant b \vee b^* \in D(L)$. Since, for every $b_1 \in G[b]$, $b_1 \wedge b^* = 0$, we have that $\big|G[b \vee b^*]\big| \neq 1$ and therefore we have completed the last possible case. □

By $C(L)$ we denote the set $\{x \in L\colon\ x^* \vee x = 1 \text{ and } x^* \wedge x = 0\}$.

Corollary 9 (Katriňák [4]). *Let L be non regular. L is (finitely) subdirectly irreducible if and only if L is nearly regular, $C(L) = \{0, 1\}$ and for every $1 \neq x \in D(L)$ with $|G[x]| = 1$ there exists $d \in D(L)$ satisfying $|G[d]| \neq 1$ such that $x^{n(+*)} \leqslant d$ for some $n \geqslant 0$.*

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