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*Czechoslovak Mathematical Journal*, Vol. 48 (1998), No. 2, 243–252

Persistent URL: http://dml.cz/dmlcz/127414

Czechoslovak Mathematical Journal, 48 (123) (1998), 243–252

# THE $\mathscr{A}r$-FREE PRODUCTS OF ARCHIMEDEAN $l$-GROUPS

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(Received July 10, 1995)

*Abstract.* The objective of this paper is to give two descriptions of the $\mathscr{A}r$-free products of archimedean $\ell$-groups and to establish some properties for the $\mathscr{A}r$-free products. Specifically, it is proved that $\mathscr{A}r$-free products satisfy the weak subalgebra property.

## 1. Introduction

We use the standard terminology and notation of [1, 3, 4]. All groups in this paper are abelian. The group operation of an $l$-group is written by additive notation. We use $\mathbb{N}$ and $\mathbb{Z}$ for the natural numbers and the integers, respectively. The symbol $\oplus$ refers to the group theoretic direct sum while $\boxplus$ denotes the cardinal sum of $l$-groups.

A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G \mid x \geqslant 0\}$ is the positive semigroup of $G$. A totally ordered group is called an 0-group. Let $G$ and $H$ be two po-groups. A map $\varphi$ from $G$ into $H$ is called a po-group homomorphism, if $\varphi$ is a group homomorphism and $x \geqslant y$ implies $\varphi(x) \geqslant \varphi(y)$ for any $x$, $y \in G$. A po-group homomorphism $\varphi$ is called a po-group isomorphism if $\varphi$ is an injection and $\varphi^{-1}$ is also a po-group homomorphism from $\varphi(G)$ to $G$.

Let $\mathscr{U}$ be a class of $l$-groups and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathscr{U}$. The $\mathscr{U}$-free product of $G_\lambda$ is an $l$-group $G \in \mathscr{U}$, denoted by ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$, together with a family of injective $l$-homomorphisms $\alpha_\lambda\colon G_\lambda \to G$ (call coprojections) such that

1. $\bigcup\limits_{\lambda\in\Lambda} \alpha_\lambda(G_\lambda)$ generates $G$ as an $l$-group;

2. if $H \in \mathscr{U}$ and $\{\beta_\lambda\colon G_\lambda \to H \mid \lambda \in \Lambda\}$ is a family of $l$-homomorphisms, then there exists a (necessarily) unique $l$-homomorphism $\gamma\colon G \to H$ satisfying $\beta_\lambda = \gamma\alpha_\lambda$ for all $\lambda \in \lambda$.

We often identify each free factor $G_\lambda$ with its image $\alpha_\lambda(G_\lambda)$ in ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ and thus view each $G_\lambda$ as an $l$-subgroup of ${}^{\mathscr{U}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$. By the Sikorski existence theorem [6],

$\mathscr{U}$-free products always exist if $\mathscr{U}$ is a class of $l$-groups closed under $l$-subgroups and direct products. Consequently, if $\mathscr{U}$ is a variety of $l$-groups, $\mathscr{U}$-free products always exist. Let $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{A}$ be the varieties of all $l$-groups, representable $l$-groups and abelian $l$-groups, respectively. In [10–13] Powell and Tsinakis have given several descriptions and some properties for free products in the varieties $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{A}$. Let $\mathscr{A}r$ be the class of all archimedean $l$-groups. Clearly, $\mathscr{A}r$ is closed under taking $l$-subgroups and direct products. Hence $\mathscr{A}r$-free products always exist. In this paper we will give two descriptions of the $\mathscr{A}r$-free products of archimedean $l$-groups and discuss some of their properties.

## 2. Descriptions for $\mathscr{A}r$-free products

First of all we consider $\mathscr{A}r$-free products of archimedean 0-groups (which, by Hölder's Theorem, are subgroups of the additive reals).

We recall some definitions. Let $\mathscr{U}$ be a class of $l$-groups and $[G, P]$ a po-group. The $\mathscr{U}$-free extension of $G$ is an $l$-group $\mathscr{F}_{\mathscr{U}}(G) \in \mathscr{U}$ for which there exists an injective po-group homomorphism $\alpha\colon G \to \mathscr{F}_{\mathscr{U}}(G)$ such that

1. $\alpha(G)$ generates $\mathscr{F}_{\mathscr{U}}(G)$ as an $l$-group;
2. if $H \in \mathscr{U}$ and $\beta\colon G \to H$ is a po-group homomorphism, then there exists an $l$-homomorphism $\gamma\colon \mathscr{F}_{\mathscr{U}}(G) \to H$ satisfying $\gamma\alpha = \beta$.

The $\mathscr{U}$-free extension $\mathscr{F}_{\mathscr{U}}(G)$ of a po-group $[G, P]$ is called the $\mathscr{U}$-free $l$-group generated by $[G, P]$, denoted by $\mathscr{F}_{\mathscr{U}}\big([G, P]\big)$, if the mapping $\alpha$ in the above definition is a po-group isomorphism between $G$ and $\alpha(G)$. By Grätzer existence theorem on a free algebra generated by a partial algebra (Theorem 28.2 of [5]) we have.

**Lemma 2.1.** *There exists an $\mathscr{A}r$-free $l$-group $\mathscr{F}_{\mathscr{A}r}\big([G, P]\big)$ generated by a po-group $[G, P]$ if and only if $[G, P]$ is a po-group isomorphic to a po-subgroup of an archimedean $l$-group.*

Let $\{R_\lambda \mid \lambda \in \Lambda\}$ be a family of archimedean 0-groups. $H = \oplus_{\lambda\in\Lambda} R_\lambda$ is the abelian group free product of this family. Let $H^+$ be the set of all sums of conjugates in $H$ of $\bigcup\limits_{\lambda\in\Lambda} R_\lambda^+$. Then $[H, H^+] = \mathop{\boxplus}\limits_{\lambda\in\Lambda} R_\lambda$ and $\mathop{\boxplus}\limits_{\lambda\in\Lambda} R_\lambda \in \mathscr{A}r$. By Theorem 11.2.4 of [5] and the above Lemma 2.1 we see that

$$\text{(1)} \qquad {}^{\mathscr{A}r}\!\!\coprod_{\lambda\in\Lambda} R_\lambda \cong \mathscr{F}_{\mathscr{A}r}\big(\mathop{\boxplus}_{\lambda\in\Lambda} R_\lambda\big).$$

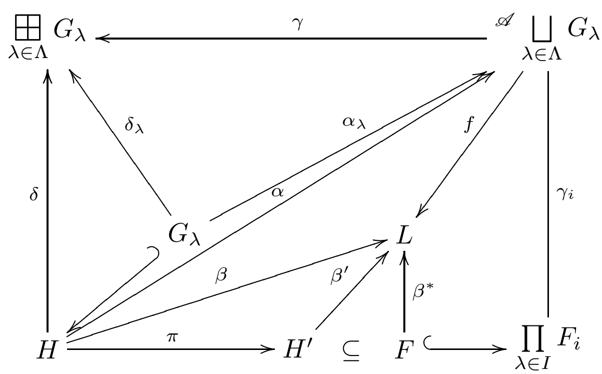
We now consider the description for $\mathscr{A}r$-free products of arbitrary archimedean $l$-groups. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of archimedean $l$-groups. Then the $\mathscr{A}$-free product $G = {}^{\mathscr{A}}\!\coprod\limits_{\lambda\in\Lambda} G_\lambda$ exists with the coprojections $\alpha_\lambda$, and we have several descriptions for $G$. Let $H = \oplus_{\lambda\in\Lambda} G_\lambda$ be the abelian group free product of $G_\lambda$. By the proof of Theorem 2.4 of [11] there exists a group isomorphism $\alpha\colon H \to \alpha(H) \subseteq {}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ such that the restriction of $\alpha$ onto each individual $G_\lambda$ is $\alpha_\lambda$. $G_\lambda(\lambda \in \Lambda)$ can be naturally embedded into the cardinal sum $\boxplus_{\lambda\in\Lambda} G_\lambda$ as $l$-groups with the embedding $\delta_\lambda\colon G_\lambda \to \boxplus_{\lambda\in\Lambda} G_\lambda$. Hence there exists a group homomorphism $\delta\colon H \to \boxplus_{\lambda\in\Lambda} G_\lambda$ which extends $\delta_\lambda(\lambda \in \Lambda)$, and there exists an $l$-homomorphism $\gamma\colon G \to \boxplus_{\lambda\in\Lambda} G_\lambda$ such that $\gamma\alpha_\lambda = \delta_\lambda(\lambda \in \Lambda)$. We declare two $\mathscr{A}r$-surjections $\beta_i\colon G \to F_i(i = 1, 2)$ to be equivalent if there exists an $l$-isomorphism $\gamma\colon F_1 \to F_2$ such that $\gamma\beta_1 = \beta_2$. Let

$$D = \{\gamma_i\colon G \to F_i \mid i \in I\}$$

be the set of representatives of equivalence classes of $Ar$-surjections out of $G$. Thus, $\gamma \in D$ and $D$ is not empty. For each $\lambda \in \Lambda$ and each $i \in I$, $\gamma_i\alpha_\lambda$ is an $l$-homomorphism of $G_\lambda$ into $F_i$. The direct product $\prod_{i\in I} F_i$ is an archimedean $l$-group. For each $\lambda \in \Lambda$, let $\pi_\lambda$ be the natural $l$-homomorphism of $G_\lambda$ onto the $l$-subgroup $G'_\lambda$ of $\prod_{i\in I} F_i$. That is,

$$\pi_\lambda(g_\lambda) = \big(\dots, \gamma_i\alpha_\lambda(g_\lambda), \dots\big)$$

for $g_\lambda \in G_\lambda$. Let $H$ be the subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G'_\lambda$. Let $\pi$ be the group homomorphism of $H$ onto $H'$ which extends each $\pi_\lambda$ $(\lambda \in \Lambda)$.




That is,

$$\pi(h) = \big(\dots, \gamma_i\alpha(h), \dots\big)$$

for $h \in H$. Because $\gamma \in D$ and each $\delta_\lambda(\lambda \in \Lambda)$ is an $l$-isomorphism, $\pi$ is a group isomorphism of $H$ onto $H'$ and $\pi_\lambda$ is an $l$-isomorphism for $\lambda \in \Lambda$. Let $F$ be the sublattice of $\prod_{i\in I} F_i$ generated by $H'$. For each $h \in H$, let $h' = \pi(h)$. Since $\prod_{i\in I} F_i$ is a distributive lattice,

$$F\Big\{ \bigvee_{j\in J} \bigwedge_{k\in K} h'_{jk} \mid h_{jk} \in H,\ J \text{ and } K \text{ finite} \Big\}.$$

Thus we have the following result.

**Proposition 2.2.** *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of archimedean $l$-groups. Then the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is the sublattice $F$ of the direct product $\prod_{i\in I} F_i$ generated by the group isomorphic image $H'$ of the abelian group free product $H$ of $G_\lambda$, where $D = \{\gamma_i \colon G \to F_i \mid i \in I\}$ is the set of representatives of the equivalence classes of all $\mathscr{A}r$-surjections out of ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$.*

P r o o f. Suppose that $L \in \mathscr{A}r$ and that $\{\beta_\lambda \colon G_\lambda \to L \mid \lambda \in \Lambda\}$ is a family of $l$-homomorphisms. We shall show that there exists a unique $l$-homomorphism $\beta^* \colon F \to L$ such that $\beta^*\pi_\lambda = \beta_\lambda$ for each $\lambda \in \Lambda$. By the universal property of the group free product, there exists a group homomorphism $\beta \colon H \to L$ which extends each $\beta_\lambda(\lambda \in \Lambda)$. For any $h' = \pi(h) \in H'$, put

$$\beta'(h') = \beta(h).$$

Then $\beta'$ is a group homomorphism of $H$ into $L$. By the universal property of the $\mathscr{A}$-free product, there exists a unique $l$-homomorphism $f \colon G \to L$ such that $\beta_\lambda = f\alpha_\lambda$ for each $\lambda \in \Lambda$. Then $f\alpha = \beta'\pi = \beta$. By Lemma 11.3.1 of [4] we need only to show that for each finite subset $\{h_{jk} \mid j \in J,\ k \in K\} \subseteq H$, $\bigvee_{j\in J} \bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq 0$ implies $\bigvee_{j\in J} \bigwedge_{k\in K} \pi(h_{jk}) \neq 0$. In fact, $\bigvee_{j\in J} \bigwedge_{k\in K} f\alpha(h_{jk}) \neq 0$. Because $f \in D$, $\bigvee_{j\in J} \bigwedge_{k\in K} \gamma_i\alpha(h_{jk}) \neq 0$ for some $i \in I$. So

$$\bigvee_{j\in J} \bigwedge_{k\in K} \pi(h_{jk}) = \bigvee_{j\in J} \bigwedge_{k\in K} (\dots, \gamma_i\alpha(h_{jk}), \dots) = (\dots, \bigvee_{j\in J} \bigwedge_{k\in K} \gamma_i\alpha(H_{jk}), \dots) \neq 0.$$

Therefore $\beta'$ can be uniquely extended to an $l$-homomorphism $\beta^* \colon F \to L$. □

Below we will give another description for $\mathscr{A}r$-free products. Given $G \in \mathscr{A}r$, an $l$-ideal $K$ of $G$ will be called an archimedean kernel if $G/K \in \mathscr{A}r$. Let $AK(G)$ be the set of all archimedean kernels of $G$. For any $0 \neq g \in G$, there exists an archimedean kernel $K_g$ of $G$ such that $g \overline{\in} K_g$. $K_g$ is called an $AK$ excluding $g$. For example, 0 is always an $AK$ excluding $g \neq 0$, because $G \in \mathscr{A}r$.

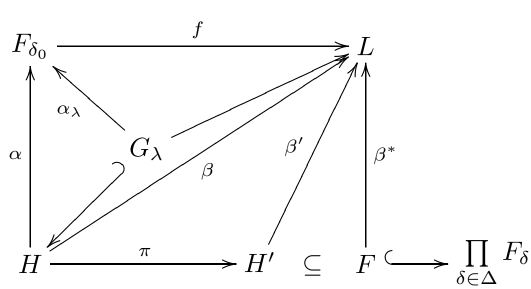
Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be family of $l$-groups in $\mathscr{A}r$. Let

$$\Gamma = \bigcup_{\lambda\in\Lambda} AK(G_\lambda)$$

and consider the set $\Delta$ of all choice functions $\delta\colon\ \Lambda \to \Gamma$. For each $\delta \in \Delta$ and each $\lambda \in \mathscr{L}$, let $K_{\delta(\lambda)} \in AK(G_\lambda)$. Then $\underset{\lambda\in\Lambda}{\boxplus}\big(G_\lambda/K_{\delta(\lambda)}\big) \in \mathscr{A}r$. $\underset{\lambda\in\Lambda}{\boxplus}\big(G_\lambda/K_{\delta(\lambda)}\big)$ can be also naturally viewed as a po-group. By Lemma 2.1 there exists an $\mathscr{A}r$-free $l$-group $F_\delta = \mathscr{F}_{\mathscr{A}r}\big(\underset{\lambda\in\Lambda}{\boxplus}(G_\lambda/K_{\delta(\lambda)})\big)$ generated by the po-group $\underset{\lambda\in\Lambda}{\boxplus}\big(G_\lambda/K_{\delta(\lambda)}\big)$ for each $\delta \in \Delta$. Then $\prod\limits_{i\in I} F_\delta$ is an archimedean $l$-group. We denote by $\varrho_\delta$ the projection of $\prod\limits_{i\in I} F_\delta$ onto $F_\delta$ for each $\delta \in \Delta$. For each $\lambda \in \Lambda$, let $\pi_\lambda$ be the $l$-homomorphism of $G_\lambda$ onto the $l$-subgroup $G'_\lambda$ of $\prod\limits_{i\in I} F_\delta$ satisfying $\varrho_\delta\pi_\lambda(g_\lambda) = g_\lambda + K_{\delta(\lambda)}$ for $g_\lambda \in G_\lambda$. $\pi_\lambda$ is an $l$-isomorphism for each $\lambda \in \Lambda$. In fact, for $0 \neq g_\lambda$ we take $K_{\delta(\lambda)} = K_{g_\lambda}$, an $AK$ excluding $g_\lambda$. Then $g_\lambda + K_{g_\lambda} \neq K_{g_\lambda}$, and so $\varrho_\delta\pi_\lambda(g_\lambda) \neq 0$. Let $H'$ be the subgroup of $\prod\limits_{i\in I} F_\delta$ generated by $\bigcup\limits_{\lambda\in\Lambda} G_\lambda$ and let $\pi$ be the group homomorphism of $H = \oplus_{\lambda\in\Lambda} G_\lambda$ onto $H'$ which extends each $\pi_\lambda(\lambda \in \Lambda)$. It is easy to see that $\pi$ is a group isomorphism. Then we have the following description of $\mathscr{A}r$-free products.

**Theorem 2.3.** *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of archimedean $l$-groups. Then the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\coprod\limits_{\lambda\in\Lambda} G_\lambda$ is the sublattice $F$ of the direct product $\prod\limits_{\delta\in\Delta} F_\delta$ generated by the group isomorphic image $H'$ of the group free product $H$ of $G_\lambda$.*

P r o o f. We show the universal property. Suppose that $L \in \mathscr{A}r$ and that $\{\beta_\lambda\colon\ G_\lambda \to L \mid \lambda \in \Lambda\}$ is a family of $l$-homomorphisms. We shall show that there exists a unique $l$-homomorphism $\beta^*\colon\ F \to L$ such that $\beta^*\pi_\lambda = \beta_\lambda$. Clearly, there exists a group homomorphism $\beta\colon\ H \to L$ which extends each



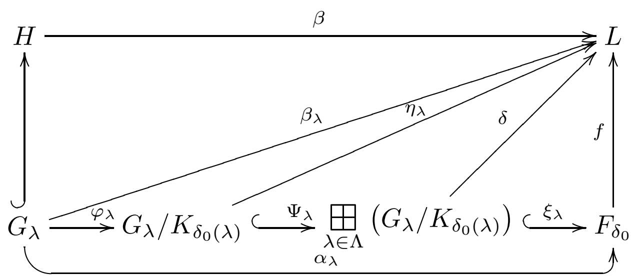

$\beta_\lambda(\lambda \subset \Lambda)$. For any $h' = \pi(h) \in H'(h \in H)$, put

$$\beta'(h') = \beta(h).$$

By Lemma 11.3.1 of [4] we need only to show that for each finite subset $\{h_{jk} \mid j \in J, k \in K\} \subseteq H$, $\bigvee_{j\in J}\bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq 0$ implies $\bigvee_{j\in J}\bigwedge_{k\in K} \pi(h_{jk}) \neq 0$. For each $\lambda \in \Lambda$, put $K_{\delta_0(\lambda)} = \beta_\lambda^{-1}(0)$ and $F_{\delta_0} = \mathscr{F}_{\mathscr{A}r}\big(\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})\big)$. Let $\varphi_\lambda$ be the natural $l$-homomorphism of $G_\lambda$ onto $G_\lambda/K_{\delta_0(\lambda)}$, let $\eta_\lambda$ be the $l$-isomorphism of $G_\lambda/K_{\delta_0(\lambda)}$ into $L$ such that $\eta_\lambda\varphi_\lambda = \beta_\lambda$, let $\Psi_\lambda$ be the embedding of $G_\lambda/K_{\delta_0(\lambda)}$ into $\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})$, let $\gamma$ be the group homomorphism of $\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})$ into $L$ such that $\gamma\Psi_\lambda = \eta_\lambda$ ($\gamma$ is also a po-group homomorphism), and let $\xi_\lambda$ be the po-group isomorphism of $\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})$ into $F_{\delta_0}$. Then there exists an $l$-homomorphism $f$ of $F_{\delta_0}$ into $L$ such that $f\xi_\lambda = \gamma$. Let $\alpha_\lambda = \xi_\lambda\Psi_\lambda\varphi_\lambda$. Then

$$\beta_\lambda = \eta_\lambda\varphi_\lambda = \gamma\Psi_\lambda\varphi_\lambda = f\xi_\lambda\Psi_\lambda\varphi_\lambda = f\alpha_\lambda$$

and $\varrho_{\delta_0}\pi_\lambda = \alpha_\lambda$ for each $\lambda \in \Lambda$. Let $\alpha$ be the unique group homomorphism




of $H$ into $F_{\delta_0}$ which extends each $\alpha_\lambda (\lambda \in \Lambda)$. It follows that

$$\beta'\pi = \beta = f\alpha \text{ and } \varrho_{\delta_0}\pi = \alpha.$$

Thus, $\bigvee_{j\in J}\bigwedge_{k\in K} f\alpha(h_{jk}) \neq 0$. That is, $f\big(\bigvee_{j\in J}\bigwedge_{k\in K} \alpha(h_{jk})\big) \neq 0$. Hence $\bigvee_{j\in J}\bigwedge_{k\in K} \alpha(h_{jk}) \neq 0$. So

$$\begin{aligned}\bigvee_{j\in J}\bigwedge_{k\in K} \pi(h_{jk}) &= \bigvee_{j\in J}\bigwedge_{k\in K} \big(\dots, \varrho_{\delta_0}\pi(h_{jk}), \dots\big) = \Big(\dots, \bigvee_{j\in J}\bigwedge_{k\in K} \varrho_{\delta_0}\pi(h_{jk}), \dots\Big)\\ &= \Big(\dots, \bigvee_{j\in J}\bigwedge_{k\in K} \alpha(h_{jk}), \dots\Big) \neq 0.\end{aligned}$$

Therefore $\beta'$ can be uniquely extended to an $l$-homomorphism $\beta^*\colon F \to L$. □

## 3. The relation between $\mathscr{A}$-free products and $\mathscr{A}r$-free products

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of archimedean $l$-groups. By universal properties there exists an $l$-homomorphism $\varphi$ of ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ onto ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$. If ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is archimedean, then ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda \cong {}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$. Now we consider the $\mathscr{A}r$-free product of two archimedean 0-groups $R_1$ and $R_2$. By Corollary 1.9.1 [7] and the above formula (1) we have

$$R_1{}^{\mathscr{A}} \sqcup R_2 \cong \mathscr{F}_{\mathscr{A}}(R_1 \boxplus R_2),$$
$$R_1{}^{\mathscr{A}r} \sqcup R_2 \cong \mathscr{F}_{\mathscr{A}r}(R_1 \boxplus R_2),$$

where $\mathscr{F}_{\mathscr{A}}(R_1 \boxplus R_2)$ and $\mathscr{F}_{\mathscr{A}r}(R_1 \boxplus R_2)$ are respectively the $\mathscr{A}$-free $l$-group and the $\mathscr{A}r$-free $l$-group generated by $R_1 \boxplus R_2$. So the problem is reduced to the following under what condition the $\mathscr{A}$-free $l$-group $\mathscr{F}_{\mathscr{A}}\big([G,P]\big)$ generated by a po-group $[G,P]$ is archimedean. In [2] S.J. Bernau established a necessary and sufficient condition under which the $\mathscr{A}$-free $l$-group generated by a po-group is archimedean. However, his proof contains an error. Namely, $[G,P]$ is a po-group and need not be a partially ordered vector space (see [14] for derails). The correct result is given in the following theorem. First we introduce some concepts.

Let $[G,P]$ be a po-group and $S$ a nonempty subset of $G$. $S$ is said to be positively independent if for any finite subset $\{x_1,\dots,x_k\}$ of $S$ and non-negative integers $\{\lambda_1,\dots,\lambda_k\}$, $\sum\limits_{i=1}^{k} \lambda_i x_i \in -P$ only if $\lambda_i = 0$ $(i=1,\dots,k)$. A po-group $[G,P]$ is said to be strongly uniformly archimedean if, given $u \in G$ and a positively independent subset $\{v_1,\dots,v_k\}$ of $G$, there exists $n \in \mathbb{N}$ such that if $\lambda_1,\dots,\lambda_k$ are non-negative integers and $\sum\limits_{i=1}^{k} \lambda_i \geqslant mn$ with $m \in \mathbb{N}$, then $\sum\limits_{i=1}^{k} \lambda_i v_i \nleqslant mu$. It is well known that if a po-group $[G,P]$ is semi-closed, then the $\mathscr{A}$-free $l$-group $\mathscr{F}_{\mathscr{A}}\big([G,P]\big)$ generated by $[G,P]$ exists (cf. [16]).

**Theorem 3.1.** *The $\mathscr{A}$-free $l$-group $\mathscr{F}_{\mathscr{A}}\big([G,P]\big)$ generated by a semi-closed po-group $[G,P]$ is archimedean if and only if $[G,P]$ is strongly uniformly archimedean.*

The proof of this theorem is similar to that of Theorem 4.3 of [2].

Now let $R_1$ and $R_2$ be two archimedean 0-groups. We call two nonzero elements $(a,b)$ and $(c,d)$ in $R_1 \times R_2$ separated if $(a,b) + \nu(c,d) = 0$ for a positive real number $\nu$. It is clear that $R_1 \boxplus R_2$ is semi-closed. So Theorem 2.6 of [8] and Theorem 3.1 yield.

**Theorem 3.2.** *The following are equivalent:*

1. ${R_1}^{\mathscr{A}} \sqcup R_2$ *is archimedean,*
2. $R_1 \boxplus R_2$ *is strongly uniformly archimedean,*
3. $R_1 \boxplus R_2$ *has no separated, positively independent pairs,*
4. ${R_1}^{\mathscr{A}} \sqcup R_2 \cong {R_1}^{\mathscr{A}r} \sqcup R_2$.

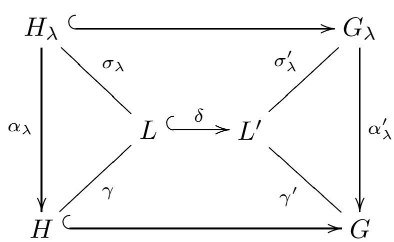
## 4. The weak subalgebra property

Let $\mathscr{U}$ be a class of $l$-groups closed under $l$-subgroups and direct products. $\mathscr{U}$-free products are said to have the subalgebra property if for any family $\{G_\lambda \mid \lambda \in \Lambda\}$ in $\mathscr{U}$ with $l$-subgroups $H_\lambda \subseteq G_\lambda$, ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} H_\lambda$ is simply the $l$-subgroup of ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup\limits_{\lambda\in\Lambda} H_\lambda$. It is well known that $\mathscr{A}$-free products satisfy the subalgebra property [11]. $\mathscr{U}$-free products are said to have the weak subalgebra property if, whenever $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of $l$-groups in $\mathscr{U}$ with $l$-subgroups $H_\lambda \subseteq G_\lambda$ and any family of $l$-homomorphisms $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{U}$ can be extended to a family of $l$-homomorphisms $\sigma'_\lambda\colon G_\lambda \to L' \in \mathscr{U}$ and there exists a $\mathscr{U}$-injection $\delta\colon L \to L'$ such that $\sigma'_\lambda\big|_{H_\lambda} = \delta\sigma_\lambda$, then ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} H_\lambda$ is the $l$-subgroup of ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup\limits_{\lambda\in\Lambda} H_\lambda$.

**Theorem 4.1.** *$\mathscr{A}r$-free products satisfy the weak subalgebra property.*

Proof. Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of $l$-groups in $\mathscr{A}r$ with $l$-subgroups $H_\lambda \subseteq G_\lambda$, any family of $l$-homomorphisms $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{A}r$ can be extended to a family of $l$-homomorphisms $\sigma'_\lambda\colon G_\lambda \to L' \in \mathscr{A}r$ and there exists an $\mathscr{A}r$-injection $\delta\colon L \to L'$ such that $\sigma'_\lambda\big|_{H_\lambda} = \delta\sigma_\lambda$. We see that $H = {}^{\mathscr{A}}\bigsqcup\limits_{\lambda\in\Lambda} H_\lambda$ is the $l$-subgroup of $G = {}^{\mathscr{A}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup\limits_{\lambda\in\Lambda} H_\lambda$.

(1) First we show that any $l$-homomorphism $\gamma\colon H \to L \in \mathscr{A}r$ can be extended to an $l$-homomorphism $\gamma'\colon G \to L' \in \mathscr{A}r$ and there exists an $\mathscr{A}r$-injection $\delta\colon L \to L'$ such that $\gamma'\big|_H = \delta\gamma$. In fact, any $l$-homomorphism $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{A}r$ induces a family of $l$-homomorphisms $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{A}r$ such that $\gamma\alpha_\lambda = \sigma_\lambda$ for each $\lambda \in \Lambda$ where $\alpha_\lambda$ is the inclusion map. Then $\sigma_\lambda$ can

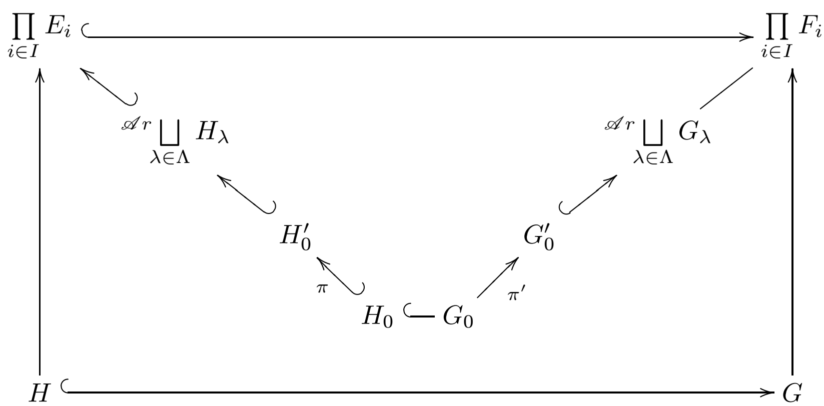


be extended to a family of $l$-homomorphisms $\sigma'_\lambda\colon G_\lambda \to L' \in \mathscr{A}r$ and there exists an $\mathscr{A}r$-injection $\delta\colon L \to L'$ such that $\sigma'_\lambda\big|_{H_\lambda} = \delta\sigma_\lambda$. By the universal property there exists an $l$-homomorphism $\gamma'\colon G \to L'$ such that $\gamma'\alpha'_\lambda = \sigma'_\lambda$ for each $\lambda \in \Lambda$ where $\alpha'_\lambda$ is the inclusion map. Hence

$$\delta\sigma_\lambda = \sigma'_\lambda\big|_{H_\lambda} = (\gamma'\alpha'_\lambda)\big|_{H_\lambda} = \gamma'\big|_{H_\lambda}$$

for each $\lambda \in \Lambda$. By virtue of the uniqueness, $\gamma'\big|_{H_\lambda} = \delta\gamma$.

(2) Now we show that ${}^{\mathscr{A}r}\coprod_{\lambda\in\Lambda} H_\lambda$ is the $l$-subgroup of ${}^{\mathscr{A}r}\coprod_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda$. Let $G_0 = \oplus_{\lambda\in\Lambda} G_\lambda$, $H_0 = \oplus_{\lambda\in\Lambda} H_\lambda$. Then $G_0$ and $H_0$ are subgroups of $G$ and $H$, respectively, and $H_0$ is a subgroup of $G_0$, $H$ is an $l$-subgroup




of $G$. Let

$$D = \left\{\gamma'_i\colon G \to F_i \mid i \to I\right\}$$

be the set of representatives of equivalence classes of $\mathscr{A}r$-surjections out of $G$. For each $i \in I$, $\gamma'_i\big|_H$ is an $\mathscr{A}r$-surjection out of $H$. Conversely, for an arbitrary $\mathscr{A}r$-surjection $\gamma\colon H \to E$ there exists by paragraph (1) an $i \in I$ and an $\mathscr{A}r$-injection $\delta\colon E \to F_i$ such that $\delta\gamma = \gamma'_i\big|_H$. Hence the set

$$C = \left\{\gamma'_i\big|_H\colon H \to E_i \leqslant F_i \mid i \in I\right\}$$

contains at least one element of each equivalence class of $\mathscr{A}r$-surjections out of $H$. But many different $\gamma'_s$ may give rise to the same $\gamma$. If $C$ contains more than one representative of some of the classes then the result of the construction is still the $\mathscr{A}r$-coproduct. So redundancy in $C$ does not harm the result. By Proposition 2.2 we see that the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\coprod_{\lambda\in\Lambda} G_\lambda$ is the sublattice of the direct product $\prod_{i\in I} F_i$

generated by the group isomorphic image $G_0'$ of $G_0$ with the group isomorphism $\pi'$, and the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the sublattice of the direct product $\prod_{i\in I} E_i$ generated by the group isomorphic image $H_0'$ of $H_0$ with the group isomorphism $\pi$. $\pi'\big|_{G_\lambda}$ and $\pi\big|_{H_\lambda}$ are all $l$-isomorphisms for each $\lambda \in \Lambda$. Hence ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is the $l$-subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G_\lambda$ where $G_\lambda' = \pi'(G_\lambda) \cong G_\lambda$ and ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the $l$-subgroup of $\prod_{i\in I} E_i$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$ where $H_\lambda' = \pi(H_\lambda) \cong H_\lambda$. From the above we see that $\prod_{i\in I} E_i$ is an $l$-subgroup of $\prod_{i\in I} F_i$ and $\pi'\big|_{H_0} = \pi$. Therefore ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the $l$-subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$, and so ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is also the $l$-subgroup of ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$. $\square$

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