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A CHARACTERIZATION OF TRIBES WITH RESPECT TO THE ŁUKASIEWICZ t-NORM

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Abstract. We give a complete characterization of tribes with respect to the Łukasiewicz t-norm, i. e., of systems of fuzzy sets which are closed with respect to the complement of fuzzy sets and with respect to countably many applications of the Łukasiewicz t-norm. We also characterize all operations with respect to which all such tribes are closed. This generalizes the characterizations obtained so far for other fundamental t-norms, e. g., for the product t-norm.

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1. INTRODUCTION

The concept of T-tribes on a universe, i. e., a nonempty crisp set X, where T is a t-norm and the elements of the T-tribes are fuzzy subsets of X, was introduced in [1, 3] in order to have a proper generalization of the classical σ-algebras. The goal of [1] is to investigate measures on T-tribes, in particular, on tribes with respect to the Łukasiewicz t-norm T_{L}, and to use these results when studying cooperative games with fuzzy coalitions. On the other hand, Pykacz [8] suggested to use structures, especially tribes, based on T_{L} in order to find a description of quantum mechanical systems using fuzzy sets. Besides their axiomatic definition, no characterization of T_{L}-tribes has been given so far. This, and recent characterizations of tribes with respect to the Frank t-norms [4, 5, 6, 7] motivated us to find such a representation of T_{L}-tribes.

Let us first recall some basic notions and facts from [1, 9]. A *t-norm* (triangular norm) is a function $T\colon [0,1]^2 \to [0,1]$ which is commutative, associative, monotone

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in each component, and satisfies the boundary condition $T(1,x) = x$. Throughout this paper, T denotes a t-norm. In most of this paper, we will deal only with the Łukasiewicz t-norm $T_{\mathbf{L}}\colon (x,y) \mapsto \max(x+y-1,0)$ and with the minimum t-norm $T_{\mathbf{M}}\colon (x,y) \mapsto x \wedge y$.

For fuzzy subsets of X, say $f,g \in [0,1]^X$, we extend a given t-norm T pointwise, i. e.,

$$T(f,g)(x) = T(f(x),g(x)).$$

This operation may be viewed as an "intersection" of fuzzy sets. Defining the complement by $x \mapsto 1-x$, the corresponding *t-conorm* is defined using the de Morgan law (its pointwise extension serves as a "union" of fuzzy sets):

$$S(x,y) = 1 - T(1-x, 1-y).$$

Restricting them to crisp sets, i. e., to characteristic functions, we obtain the usual set-theoretical operations.

Since they are associative, the binary operations T, S can be naturally extended to functions of finitely or countably many variables, denoted $T_{m\in M}, S_{m\in M}$.

Definition 1.1. A *T-tribe* or *T-clan* on X is a collection $\mathscr{T} \subseteq [0,1]^X$ such that

1. $1 \in \mathscr{T}$,
2. if $f \in \mathscr{T}$ then $1-f \in \mathscr{T}$,
3. if $(f_m)_{m\in M} \subseteq \mathscr{T}$ then $T_{m\in M}\, f_m \in \mathscr{T}$ for countable or finite M, respectively.

We denote by $\mathbf{1}_B$ the characteristic function of a crisp set B (its domain, X or $[0,1]$, will be clear from the context). For each T-tribe $\mathscr{T}$ on X, we define

$$\mathscr{T}^{\vee} = \{Y \subseteq X \mid \mathbf{1}_Y \in \mathscr{T}\},$$

which is a σ-algebra on X, showing that T-tribes are proper generalizations of σ-algebras. If $Y \subseteq X$, the restriction

$$\mathscr{T}|Y = \{f|Y \mid f \in \mathscr{T}\}$$

is a T-tribe on Y.

Definition 1.2. A collection $\mathscr{T} \subseteq [0,1]^X$ is called a *generated tribe* if there exists a σ-algebra $\mathscr{B}$ of subsets of X such that

$$\mathscr{T} = \{f \in [0,1]^X \mid f \text{ is } \mathscr{B}\text{-measurable}\}.$$

In this case, we have $\mathscr{T}^{\vee} = \mathscr{B}$. A generated tribe is a T-tribe for any measurable t-norm T.

We will make use of the following properties of the Łukasiewicz t-norm $T_{\mathbf{L}}$ and its corresponding t-conorm $S_{\mathbf{L}}\colon (x,y) \mapsto \min(x+y,1)$: If $\mathcal{T}$ is a $T_{\mathbf{L}}$-tribe, $f,g,h \in \mathcal{T}$ and $n \in \mathbb{N}$ such that $f+g \leqslant 1$, $f \leqslant h$ and $n \cdot f \leqslant 1$, then $f+g = S_{\mathbf{L}}(f,g) \in \mathcal{T}$, $h-f = T_{\mathbf{L}}(h, 1-f) \in \mathcal{T}$ and $n \cdot f = (S_{\mathbf{L}})^n f \in \mathcal{T}$. In this case we continue to use the algebraic notation $f+g$, $h-f$, $n \cdot f$.

The following results are taken from [1].

Theorem 1.3.

(i) *Every $T_{\mathbf{L}}$-tribe is a $T_{\mathbf{M}}$-tribe* ([1, Proposition 2.7]).

(ii) *All elements of a $T_{\mathbf{L}}$-tribe $\mathcal{T}$ are $\mathcal{T}^{\vee}$-measurable* ([3], [1, Proposition 3.2]).

(iii) *A $T_{\mathbf{L}}$-tribe on X is generated if and only if it contains all constant fuzzy subsets of X* ([1, Proposition 3.3]).

In view of (i), all $T_{\mathbf{L}}$-tribes are closed with respect to countable pointwise suprema $\bigvee$ and infima $\bigwedge$.

2. Preliminaries

In this section we introduce some notions—applicable in any T-tribe—that will be used in the sequel.

Definition 2.1. A subset G of a T-tribe $\mathcal{T}$ on X is called a *generating set* if $\mathcal{T}$ is the smallest T-tribe on X containing G. If G is a singleton, $G = \{g\}$, we say that g is a *generator* of $\mathcal{T}$.

Note that we do not use here the usual algebraic terminology "$\mathcal{T}$ is generated by G" in order to avoid confusion with the term "generated tribe" (Definition 1.2).

We define $T_{\mathbf{L}}$-*terms* as elements of the smallest set of n-ary terms in a $T_{\mathbf{L}}$-tribe ($n \in \mathbb{N}$ arbitrary) containing the projections and being closed with respect to the operations $T_{\mathbf{L}}$, $S_{\mathbf{L}}$ and complement. More exactly:

1. the projection $\mathrm{pr}_i\colon (x_1,\ldots,x_n) \mapsto x_i$ onto the i-th component is a $T_{\mathbf{L}}$-term for all $i \in \{1,\ldots,n\}$,
2. the complement of a $T_{\mathbf{L}}$-term is a $T_{\mathbf{L}}$-term,
3. the application of the t-norm $T_{\mathbf{L}}$ to a finite set of $T_{\mathbf{L}}$-terms gives a $T_{\mathbf{L}}$-term.

Since we may view the elements of $[0,1]$ also as constant functions on a singleton domain, we define a *T-tribe of constants* or *T-clan of constants* as a set $K \subseteq [0,1]$ such that

1. $1 \in K$,
2. if $r \in K$ then $1-r \in K$,
3. if $(r_m)_{m \in M} \subseteq K$ then $T_{m \in M}\, r_m \in K$ for countable or finite M, respectively.

In particular, the only $T_{\mathbf{L}}$-tribes of constants are

$$K_n = \left\{ \frac{i}{n} \mid i = 0, \dots, n \right\}$$

for $n \in \mathbb{N}$ and

$$K_\infty = [0,1].$$

Each $T_{\mathbf{L}}$-tribe of constants has a generator: For a given $n \in \mathbb{N}$ and each $i \in \{0, \dots, n\}$ with $\gcd(i,n) = 1$, i/n is a generator of K_n (gcd denotes the greatest common divisor). To see this, we apply the Euclidean division algorithm to find the greatest common divisor to i/n and 1, obtaining $1/n$. The algorithm gives us a $T_{\mathbf{L}}$-term E such that $E(i/n, 1) = 1/n$. All elements of K_n are obtained as integer multiples of $1/n$. Similarly, each irrational number in $[0,1]$ is a generator of K_∞.

Since all numbers in this paper will be integers or elements of $[0,1]$, we denote by $\mathbb{Q}$ and $\mathbb{I}$ the set of all rational and irrational numbers in $[0,1]$, respectively. The symbol id denotes the identity function on $[0,1]$.

The following definition generalizes a notion introduced in [7, 4]. Its simplification has been suggested by Mesiar (private communication).

Definition 2.2. A function $a \colon [0,1] \to [0,1]$ is *T-admissible* if it belongs to the T-tribe on $[0,1]$ with the generator id.

The following is a full characterization of T-admissible functions:

Proposition 2.3. (composition principle, cf. [7]) *A function $a \colon [0,1] \to [0,1]$ is T-admissible if and only if for each T-tribe $\mathscr{T}$ and each $f \in \mathscr{T}$ the composition $a \circ f$, defined by $(a \circ f)(x) = a(f(x))$, is an element of $\mathscr{T}$.*

Proof. Suppose that a is T-admissible. Then there is a sequence of T-tribe operations which, applied to id, gives a. The same sequence of operations, applied to $f \in \mathscr{T}$, yields $a \circ f$.

Since we may take the T-tribe of T-admissible functions for $\mathscr{T}$ and id for f, sufficiency is obvious. □

Therefore T-admissible functions are exactly the functions (unary operations) such that each T-tribe is closed with respect to them. They may be considered "all possible" fuzzy generalizations of the unary Boolean operations. Note that we have $\{a(0), a(1)\} \subseteq \{0,1\}$ for each T-admissible function a.

Corollary 2.4. *The T-tribe $\mathscr{A}$ of T-admissible functions is closed with respect to composition, i. e., for all $a, b \in \mathscr{A}$ we have $a \circ b \in \mathscr{A}$.*

The aim of this section is to describe the tribe of T_{L}-admissible functions. We denote this tribe by $\mathscr{A}_{\mathrm{L}}$. Obviously, all T_{L}-admissible functions are Borel measurable. However, there are Borel measurable functions which are not T_{L}-admissible. For instance, if $a \in \mathscr{A}_{\mathrm{L}}$ and $n \in \mathbb{N}$, then $a(1/n) \in K_n$. Thus the only constant functions in $\mathscr{A}_{\mathrm{L}}$ are 0 and 1.

Lemma 3.1. *Let $B \subseteq [0,1]$ be a Borel set. Then $\mathbf{1}_B$ is T_{L}-admissible.*

P r o o f. According to the composition principle (Proposition 2.3), $\mathbf{1}_B$ is T_{L}-admissible if and only if for every T_{L}-tribe $\mathscr{T}$ and for each $f \in \mathscr{T}$ we have

$$\mathbf{1}_B \circ f = \mathbf{1}_{f^{-1}(B)} \in \mathscr{T}.$$

The latter relation is equivalent to $f^{-1}(B) \in \mathscr{T}^{\vee}$. Thus the statement of the lemma is equivalent to the claim that all elements of a T_{L}-tribe $\mathscr{T}$ are $\mathscr{T}^{\vee}$-measurable, which is true because of Theorem 1.3. □

The following lemma determines the possible values of T_{L}-admissible functions at rational points.

Lemma 3.2. *Let $n \in \mathbb{N}$, $i \in \{0,\dots,n\}$ such that* $\gcd(i,n) = 1$. *Then for each $j \in \{0,\dots,n\}$ the function $(j/n)\cdot \mathbf{1}_{\{i/n\}}$ is T_{L}-admissible.*

P r o o f. The value j/n belongs to K_n. Since i/n is a generator of K_n, there is a T_{L}-term E such that $E(i/n) = j/n$. The T_{L}-admissible function $f = E(\mathrm{id})$ satisfies $f(i/n) = j/n$. According to Lemma 3.1, $f \wedge \mathbf{1}_{\{i/n\}} = (j/n)\cdot \mathbf{1}_{\{i/n\}}$ is also T_{L}-admissible. □

Although $\mathscr{A}_{\mathrm{L}}$ does not contain nontrivial constants, its restriction $\mathscr{A}_{\mathrm{L}}|\mathbb{I}$ contains all constants, as a consequence of the following lemma.

Lemma 3.3. For each *$r \in [0,1]$ there is a T_{L}-admissible function a such that $a(z) = r$ for all $z \in \mathbb{I}$.*

P r o o f. The case $r = 0$ is trivial; suppose that $r > 0$. It is sufficient to find, for each $\varepsilon > 0$, a T_{L}-admissible function b such that $\mathrm{Range}(b|\mathbb{I}) \subseteq (r-\varepsilon, r)$. Moreover, we will construct b such that $\mathrm{Range}(b|C) \subseteq (r-\varepsilon, r)$, where

$$C = [0,1] \setminus \bigcup_{n \leqslant 1/\varepsilon} K_n \supseteq \mathbb{I}.$$

Claim 1. For each $c \in C$, there is an open neighborhood N_c of c and a $T_{\mathbf{L}}$-admissible function f_c such that $\mathrm{Range}(f_c) \subseteq [0, r)$ and $\mathrm{Range}(f_c|N_c) \subseteq (r - \varepsilon, r)$.

The Euclidean division algorithm applied to c and 1 results, after finitely many steps, in a positive number less than ε. As a suitable integer multiple, we obtain an element of $(r - \varepsilon, r)$. Thus there is a $T_{\mathbf{L}}$-term E_c such that $E_c(c) \in (r - \varepsilon, r)$. Using the $T_{\mathbf{L}}$-admissible function $E_c(\mathrm{id})$, we define the open set $N_c = (E_c(\mathrm{id}))^{-1}((r - \varepsilon, r))$ and $f_c = E_c(\mathrm{id}) \wedge \mathbf{1}_{N_c}$.

Claim 2. There is a countable set $D \subset C$ such that $C \subseteq \bigcup_{c \in D} N_c$.

The collection $(N_c)_{c \in C}$ is an open covering of C and C is a finite union of open intervals. An open covering of an open interval (u, v) contains a countable subcovering. Indeed, for each $n \in \mathbb{N}$ the compact set $[u + 1/n, v - 1/n]$ has a finite subcovering and the union of these subcoverings is a countable subcovering of (u, v).

Applying these two claims and putting $b = \bigvee_{c \in D} f_c$, we complete the proof of the lemma. □

Theorem 3.4. *A function $a\colon [0, 1] \to [0, 1]$ is $T_{\mathbf{L}}$-admissible if and only if it is Borel measurable and*

$$\text{(K)} \quad a(i/n) \in K_n \text{ for all } n \in \mathbb{N} \text{ and } i \in \{0, \ldots, n\} \text{ with } \gcd(i, n) = 1.$$

Proof. According to Theorem 1.3, all $T_{\mathbf{L}}$-admissible functions are Borel measurable. The values at i/n belong necessarily to the $T_{\mathbf{L}}$-tribe of constants with the generator i/n, which is K_n.

To show sufficiency, let a be a Borel measurable function satisfying (K). We have to prove that it is admissible. According to Lemmas 3.1 and 3.2, the function $a \wedge \mathbf{1_Q}$ is $T_{\mathbf{L}}$-admissible. The function $a \wedge \mathbf{1_I}$ is a monotone limit of Borel measurable step functions which are $T_{\mathbf{L}}$-admissible due to Lemmas 3.1 and 3.3. Consequently, $a = (a \wedge \mathbf{1_Q}) \vee (a \wedge \mathbf{1_I})$ is $T_{\mathbf{L}}$-admissible. □

The following $T_{\mathbf{L}}$-admissible functions will play an important role in the sequel. They are "as close to constants as possible". We will write $\mathbb{N}_\infty$ for $\mathbb{N} \cup \{\infty\}$.

Corollary 3.5. *For each $z, r \in [0, 1]$, put $n_z = \min\{n \in \mathbb{N}_\infty \mid z \in K_n\}$ (i. e., z is a generator of K_{n_z}), and define the function $d_r\colon [0, 1] \to [0, 1]$ by*

$$d_r(z) = \min([r, 1] \cap K_{n_z}).$$

Then we have

(i) d_r *is the smallest* $T_{\mathbf{L}}$*-admissible function such that Range*$(d_r) \subseteq [r, 1]$,
(ii) *if* $r \in (0, 1)$, *then* r *is the only cluster point of Range*(d_r),
(iii) *if* $r \in \{0, 1\}$, *then Range*(d_r) *has no cluster point.*

4. Characterization of T_L-tribes

The characterization of $T_{\mathbf{L}}$-admissible functions gives us a tool for the characterization of $T_{\mathbf{L}}$-tribes. Before treating this problem in its full generality, we introduce the class of semigenerated $T_{\mathbf{L}}$-tribes. This class is smaller, but easier to work with, and sufficiently general for many applications.

Definition 4.1. A collection $\mathscr{T} \subseteq [0,1]^X$ is called a *semigenerated* T*-tribe* if there exist a σ-algebra $\mathscr{B}$ of subsets of X, a sequence $(C_n)_{n\in\mathbf{N}}$ of T-tribes of constants and a countable $\mathscr{B}$-partition $(X_n)_{n\in\mathbf{N}}$ of X such that

$$\mathscr{T} = \left\{ f \in \prod_{n\in\mathbf{N}} C_n^{X_n} \mid f \text{ is } \mathscr{B}\text{-measurable} \right\}.$$

Obviously, a semigenerated T-tribe is a T-tribe. Moreover, if T_1 and T_2 are t-norms and if a T_1-tribe is a semigenerated T_2-tribe, then it is also a semigenerated T_1-tribe.

The following theorem asserts that any countable subset of a $T_{\mathbf{L}}$-tribe is contained in a semigenerated sub-$T_{\mathbf{L}}$-tribe.

Theorem 4.2. *Every* $T_{\mathbf{L}}$*-tribe on* X *with a countable generating set is semigenerated.*

Proof. Let $\{g_i \mid i \in \mathbb{N}\}$ be a generating set of a $T_{\mathbf{L}}$-tribe $\mathscr{T} \subseteq [0,1]^X$. We denote by $\mathscr{C}$ the $T_{\mathbf{L}}$-clan with the generating set $\{g_i \mid i \in \mathbb{N}\}$. Obviously we have $\mathscr{C} \subseteq \mathscr{T}$. Since $\mathscr{C}$ contains only $T_{\mathbf{L}}$-polynomials in $g_i, i \in \mathbb{N}$, it is a countable set.

We take $\mathscr{B} = \mathscr{T}^{\vee}$. All g_i, $i \in \mathbb{N}$, are $\mathscr{B}$-measurable (Theorem 1.3). The $\mathscr{B}$-partition $(X_n)_{n\in\mathbf{N}_\infty}$ of X is defined by

$$X_n = \bigcap_{i\in\mathbf{N}} g_i^{-1}(K_n) \setminus \bigcup_{k<n} X_k.$$

It is easy to see that, for each $n \in \mathbb{N}$, the following conditions are equivalent:
(E1) $x \in X_n$,
(E2) the $T_{\mathbf{L}}$-tribe of constants with the generating set $\{g_i(x) \mid i \in \mathbb{N}\}$ is K_n,
(E3) $\{c(x) \mid c \in \mathscr{C}\} = K_n$.

Thus $\mathscr{T} \subseteq \prod_{n \in \mathbf{N}_\infty} K_n^{X_n}$.

It remains to prove that each $\mathscr{B}$-measurable function $f \in \prod_{n \in \mathbf{N}_\infty} K_n^{X_n}$ belongs to $\mathscr{T}$.

Claim 1. The tribe $\mathscr{T}$ contains a function which coincides with f on $\bigcup_{n \in \mathbf{N}} X_n$.

We define a $T_{\mathbf{L}}$-admissible function d_{0+} by

$$d_{0+}(z) = \frac{1}{\min\{n \in \mathbb{N}_\infty \mid z \in K_n\}},$$

which is the smallest $T_{\mathbf{L}}$-admissible function attaining positive values on $\mathbb{Q}$. The function

$$g = \bigwedge_{c \in \mathscr{C}} (d_{0+} \circ c)$$

belongs to $\mathscr{T}$. For each $n \in \mathbb{N}$, each of (E1), (E2), (E3) is equivalent to

(E4) $g(x) = 1/n$,

and hence f coincides on $\bigcup_{n \in \mathbf{N}} X_n$ with

$$\bigvee_{n \in \mathbf{N}} \left(\mathbf{1}_{X_n} \wedge \bigvee_{k \in K_n} (\mathbf{1}_{f^{-1}(\{k\})} \wedge k \cdot n \cdot g) \right),$$

which belongs to $\mathscr{T}$.

Claim 2. For each $r \in [0,1]$, $\mathscr{T}|X_\infty$ contains the constant function with value r.

We define the $T_{\mathbf{L}}$-admissible function d_r as in Corollary 3.5 and put

$$e_r = \bigwedge_{c \in \mathscr{C}} (d_r \circ c).$$

Since $\mathscr{C}$ is countable, we have $e_r \in \mathscr{T}$. We will prove that e_r attains the constant value r on X_∞. Evidently, Range$(e_r|X_\infty) \subseteq [r,1]$. Suppose that $e_r(x) = q > r$ for some $x \in X_\infty$. The set $D =$ Range$(d_r) \cap [q,1]$ is finite and

$$\{(d_r \circ c)(x) \mid c \in \mathscr{C}\} \subseteq D.$$

Taking the preimages under d_r, we obtain

$$\{c(x) \mid c \in \mathscr{C}\} \subseteq d_r^{-1}(D).$$

On the right-hand side we have a finite set of rational numbers. So also the left-hand side is a generating set of a finite $T_{\mathbf{L}}$-tribe of constants, say K_n with $n \in \mathbb{N}$. We obtain $x \in X_n$, contradicting the assumption $x \in X_\infty$.

Claim 3. There is an $h \in \mathscr{T}$ which coincides with f on X_∞.

Since $\mathscr{T}|X_\infty$ contains all constant functions, it is a generated tribe (Theorem 1.3), i. e., it contains all $\mathscr{B}$-measurable elements of $[0,1]^{X_\infty}$. □

Now we are able to give a full characterization of $T_{\mathbf{L}}$-tribes, using the $T_{\mathbf{L}}$-tribes of constants K_n, $n \in \mathbb{N}$.

Theorem 4.3. *A collection $\mathscr{T} \subseteq [0,1]^X$ is a $T_{\mathbf{L}}$-tribe if and only if there exist a σ-algebra $\mathscr{B}$ of subsets of X and a sequence $(\nabla_n)_{n\in\mathbf{N}}$ of σ-filters in $\mathscr{B}$ with $\nabla_m \subseteq \nabla_n$ whenever n is a divisor of m, such that*

$$\mathscr{T} = \{f \in [0,1]^X \mid f \text{ is } \mathscr{B}\text{-measurable},\ f^{-1}(K_n) \in \nabla_n \text{ for all } n \in \mathbb{N}\}.$$

P r o o f . For sufficiency, the only nontrivial point is to show that $\mathscr{T}$ is closed with respect to $T_{\mathbf{L}}$. Let $(g_i)_{i\in\mathbf{N}} \subseteq \mathscr{T}$ and put

$$g = \underset{i\in\mathbf{N}}{T_{\mathbf{L}}}\, g_i.$$

The measurability of g is evident. For all $n \in \mathbb{N}$,

$$g^{-1}(K_n) \supseteq \bigcap_{i\in\mathbf{N}} g_i^{-1}(K_n) \in \nabla_n,$$

hence $g^{-1}(K_n) \in \nabla_n$.

In order to show necessity, let $\mathscr{T}$ be a $T_{\mathbf{L}}$-tribe. We define $\mathscr{B} = \mathscr{T}^\vee$ and, for $n \in \mathbb{N}$,

$$\nabla_n = \{g^{-1}(K_n) \mid g \in \mathscr{T}\}.$$

Claim 1. For each $n \in \mathbb{N}$, ∇_n is a σ-filter in $\mathscr{B}$.

Since all elements of $\mathscr{T}$ are $\mathscr{B}$-measurable (Theorem 1.3), we have $\nabla_n \subseteq \mathscr{B}$ and $X = \mathbf{1}_X^{-1}(K_1) \in \nabla_n$. Suppose that $A, B \in \mathscr{B}$, $A \subseteq B$ and $A \in \nabla_n$. Then $A = g^{-1}(K_n)$ for some $g \in \mathscr{T}$. Taking $h = g \vee \mathbf{1}_B$, we obtain $B = h^{-1}(K_n) \in \nabla_n$. Suppose finally that $\{A_i \mid i \in \mathbb{N}\} \subseteq \nabla_n$. For each $i \in \mathbb{N}$, there is a $g_i \in \mathscr{T}$ satisfying $g_i^{-1}(K_n) = A_i$. We fix an r, $0 < r < 1/(2n)$, and define $a = \mathbf{1}_{K_n} \vee (1 - d_r)$, where d_r is defined as in Corollary 3.5. The function a is $T_{\mathbf{L}}$-admissible, its range has no cluster point in K_n and $a^{-1}(K_n) = K_n = a^{-1}(\{1\})$. These properties imply

$$(a \circ g_i)^{-1}(\{1\}) = (a \circ g_i)^{-1}(K_n) = A_i$$

and, because of $\bigwedge_{i\in\mathbf{N}} (a \circ g_i) \in \mathscr{T}$,

$$\bigcap_{i\in\mathbf{N}} A_i = \bigcap_{i\in\mathbf{N}} (a \circ g_i)^{-1}(\{1\}) = \left(\bigwedge_{i\in\mathbf{N}} (a \circ g_i)\right)^{-1}(\{1\}) \in \nabla_n.$$

Claim 2. Each $f \in \mathscr{T}$ is $\mathscr{B}$-measurable and $f^{-1}(K_n) \in \nabla_n$ for all $n \in \mathbb{N}$.

This follows easily from Theorem 1.3 and the definition of ∇_n.

Claim 3. Let $f \in [0,1]^X$ be a $\mathscr{B}$-measurable function and assume $f^{-1}(K_n) \in \nabla_n$ for all $n \in \mathbb{N}$. Then $f \in \mathscr{T}$.

For each $n \in \mathbb{N}$, there exists a $g_n \in \mathscr{T}$ satisfying $g^{-1}(K_n) = f^{-1}(K_n)$. We will prove that f belongs to the $T_{\mathbf{L}}$-tribe $\mathscr{T}_g \subseteq \mathscr{T}$ with the generating set $\{g_n \mid n \in \mathbb{N}\} \cup \{\mathbf{1}_{f^{-1}([q,1])} \mid q \in \mathbb{Q}\}$. According to Theorem 4.2, $\mathscr{T}_g$ is a semigenerated tribe. There is a σ-algebra $\mathscr{B}_g \subseteq \mathscr{B}$ and a $\mathscr{B}_g$-partition $(X_n)_{n\in\mathbf{N}_\infty}$ of X such that $\mathscr{T}_g$ consists of all $\mathscr{B}_g$-measurable elements of $\prod_{n\in\mathbf{N}_\infty} K_n^{X_n}$. Since $\mathscr{B}_g$ contains all $f^{-1}([q,1])$, $q \in \mathbb{Q}$, the function f is $\mathscr{B}_g$-measurable. It remains to prove that $f \in \prod_{n\in\mathbf{N}_\infty} K_n^{X_n}$, i. e., $f(x) \in K_n$ for each $n \in \mathbb{N}$ (the relation $f(x) \in K_\infty$ is always satisfied). We have defined X_n such that $\{g_i(x) \mid i \in \mathbb{N}\}$ is a generating set of K_n. Thus $g_n(x) \in K_n$ and $x \in g_n^{-1}(K_n) = f^{-1}(K_n)$, implying $f(x) \in K_n$. □

5. Consequences and extensions

Here we collect some implications of the characterization of $T_{\mathbf{L}}$-tribes which might be of independent interest. In particular, we clarify the possibility of approximation by functions with finite or countable range.

Corollary 5.1. *Every element of a $T_{\mathbf{L}}$-tribe $\mathscr{T}$ is a uniform limit of a monotone sequence of elements of $\mathscr{T}$ with countable range.*

However, functions with finite range only are not sufficient in Corollary 5.1:

Example 5.2. Let $\mathscr{T}$ be the $T_{\mathbf{L}}$-tribe of all $T_{\mathbf{L}}$-admissible functions and let P be an infinite set of odd primes. According to Theorem 3.4, there is a $T_{\mathbf{L}}$-admissible function f such that $f(1/p) = (p-1)/(2p)$ for all $p \in P$. We have $\mathrm{Range}(f|P) \subseteq [1/3, 1/2]$. If $g \in \mathscr{T}$ such that $\mathrm{Range}(g|P) \subseteq (0,1)$, then $\mathrm{Range}(g|P)$ is infinite, so f is not a uniform limit of functions with finite range.

The notion of T-admissibility can be naturally generalized to functions of more than one variable. The composition principle remains valid and characterizations similar to Theorem 3.4 can be derived. We demonstrate this generalization for binary operations.

Definition 5.3. A (binary) *T-admissible operation* is a function $\square : [0,1]^2 \to [0,1]$ which belongs to the T-tribe $\mathscr{T}$ on $[0,1]^2$ with the generating set $\{\mathrm{pr}_1, \mathrm{pr}_2\}$, where $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections onto the first and second component, respectively.

In complete analogy to Theorem 3.4 we get

Theorem 5.4. *An operation $\square : [0,1]^2 \to [0,1]$ is $T_{\mathbf{L}}$-admissible if and only if it is Borel measurable and for each $p, q \in \mathbb{Q}$ the value $p \square q$ belongs to the $T_{\mathbf{L}}$-tribe of constants with the generating set $\{p, q\}$.*

6. Special case—F_s-tribes

The family of Frank t-norms F_s is defined [2], for $s \in (0, \infty) \setminus \{1\}$, by the formula

$$F_s(x, y) = \log_s \left(1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}\right).$$

The limit cases are: $F_0 = T_{\mathbf{M}}$, $F_\infty = T_{\mathbf{L}}$, and $F_1 = T_{\mathbf{P}}$, where $T_{\mathbf{P}}$ is the product t-norm $(x, y) \mapsto x \cdot y$. It is known that, for $s \in (0, \infty)$, each F_s-tribe is a $T_{\mathbf{L}}$-tribe [1, Proposition 2.7]. So our results may be applied to these F_s-tribes as a special case. The historical development was reverse—the study of characterizations of F_s-tribes (denoted as T_s-tribes) preceded and inspired this paper (cf. [4, 5, 7, 6]). Nonetheless, some results of this paper (e. g., Theorem 4.2) are not only true for $T_{\mathbf{L}}$-tribes, but surprisingly have additional consequences for F_s-tribes.

Theorem 6.1. *For all $s \in (0, \infty)$ we have:*

(i) *There are only two F_s-tribes of constants, $K_1 = \{0, 1\}$ and $K_\infty = [0, 1]$.*

(ii) *A collection $\mathscr{T} \subseteq [0,1]^X$ is a semigenerated F_s-tribe if and only if there is an $X_1 \in \mathscr{T}^\vee$ such that $\mathscr{T}$ is the set of all $\mathscr{T}^\vee$-measurable elements of $\{0,1\}^{X_1} \times [0,1]^{X \setminus X_1}$.*

(iii) *An F_s-tribe with a countable generating set is semigenerated.*

(iv) *A collection $\mathscr{T} \subseteq [0,1]^X$ is an F_s-tribe if and only if there is a σ-filter ∇_1 in $\mathscr{T}^\vee$ such that*

$$\mathscr{T} = \{f \in [0,1]^X \mid f \text{ is } \mathscr{T}^\vee\text{-measurable}, \ f^{-1}(\{0,1\}) \in \nabla_1\}.$$

(v) *A function $a\colon [0,1] \to [0,1]$ is F_s-admissible if and only if it is Borel measurable and $\{a(0), a(1)\} \subseteq \{0,1\}$.*

(vi) *Every element of an F_s-tribe $\mathscr{T}$ is a uniform limit of a monotone sequence of elements of $\mathscr{T}$ with finite range.*

(vii) *An operation $\Box : [0,1]^2 \to [0,1]$ is F_s-admissible if and only if it is Borel measurable and $x \Box y \in \{0,1\}$ for all $x, y \in \{0,1\}$. In particular, all measurable t-norms are F_s-admissible.*

(viii) *An F_s-tribe is a T-tribe for any measurable t-norm T.*

Semigenerated tribes were first introduced in [4] in the special form of (ii); (iii) is a corollary of Theorem 4.2 and a generalization of [4]; (iv) is a corollary of Theorem 4.3 which appears in [5, 7]. In the form of (v), F_s-admissible functions were first introduced in [5, 7]; (vi) is a strengthening of Corollary 5.1 (it is a consequence of the characterization of F_s-admis-sible functions). Finally, (viii) is a consequence of (vii); it is mentioned in [5].

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