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ON BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 70th birthday

Introduction

Consider the system of functional differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t) \tag{0.1}$$

with the boundary condition

$$l(x) = c_0, \tag{0.2}$$

where $p\colon C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $l\colon C(I;\mathbb{R}^n) \to \mathbb{R}^n$ are linear bounded operators, $q \in L(I;\mathbb{R}^n)$, $I = [a,b]$ and $c_0 \in \mathbb{R}^n$. The condition (0.2) includes, in particular, the initial condition

$$x(t_0) = c_0, \tag{0.3}$$

where $t_0 \in I$, and the periodic boundary condition

$$x(b) - x(a) = c_0. \tag{0.4}$$

By a solution of (0.1), (0.2) we understand an absolutely continuous vector function $x\colon I \to \mathbb{R}^n$ which satisfies (0.1) a. e. on I and verifies (0.2).

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The problems of existence of a solution of the differential system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P(t)x(\tau(t)) + q_0(t) \tag{0.5}$$

satisfying one of the conditions

$$x(t) = u(t) \quad \text{for } t \notin I, \quad l(x) = c_0, \tag{0.6}$$

$$x(t) = u(t) \quad \text{for } t \notin I, \quad x(t_0) = c_0, \tag{0.7}$$

$$x(t) = u(t) \quad \text{for } t \notin I, \quad x(b) - x(a) = c_0, \tag{0.8}$$

where $P \in L(I; \mathbb{R}^{n\times n})$, $q_0 \in L(I; \mathbb{R}^n)$, $\tau\colon I \to \mathbb{R}$ is a measurable function and $u\colon \mathbb{R} \to \mathbb{R}^n$ is a continuous and bounded vector function,[2] are reduced to the problems (0.1), (0.k) ($k = 2, 3, 4$). To see this, set

$$\tau_0(t) = \begin{cases} a & \text{for } \tau(t) < a, \\ \tau(t) & \text{for } a \leqslant \tau(t) \leqslant b, \\ b & \text{for } \tau(t) > b, \end{cases} \tag{0.9}$$

$$p(x)(t) = \chi_I(\tau(t))P(t)x(\tau_0(t)) \tag{0.10}$$

and

$$q(t) = (1 - \chi_I(\tau(t)))P(t)u(\tau(t)) + q_0(t), \tag{0.11}$$

where χ_I is the characteristic function of the interval I.

If $p(x)(t) \equiv P(t)x(t)$, the problem (0.1), (0.2) and analogous problems for systems of nonlinear ordinary differential equations have been studied in detail [2–5, 9, 11, 12, 17–19, 21–24]. Foundations of the theory of general boundary value problems for functional-differential equations, in particular of the problem (0.1), (0.2), were laid in monographs by Š. Schwabik, M. Tvrdý, O. Vejvoda [19] and N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina [1]. Similar problems are also considered in [2, 6, 7, 20].

In spite of a large number of papers devoted to problems of the type (0.1), (0.2) (see e.g. references in [1, 2, 19]), only a few efficient sufficient conditions for unique solvability of this problem are known at present. Here we try to fill this gap in a certain way.

[2] If $\tau(t) \in I$ for almost all $t \in I$, the condition $x(t) = u(t)$ for $t \notin I$ is to be dropped.

In Section 1 of the present paper we give necessary and sufficient conditions for unique solvability of the problem (0.1), (0.2) and prove also the J. Kurzweil – Z. Vorel [14, 15, 22] and Z. Opial [17] type theorems concerning correctness of this problem. These general results are applied in Section 2 to the problem (0.5), (0.6).

Throughout this paper the following notation is used:

$I = [a, b]$, $\mathbb{R} =]-\infty, \infty[$, $\mathbb{R}_+ = [0, \infty[$,

$\mathbb{R}^n$—the space of n dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|;$$

$\mathbb{R}^{n\times n}$—the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with elements $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^{n} |x_{ik}|;$$

$$\mathbb{R}_+^n = \{(x_i)_{i=1}^n \in \mathbb{R}^n \colon x_i \geqslant 0 \ (i = 1, \dots, n)\};$$
$$\mathbb{R}_+^{n\times n} = \{(x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n} \colon x_{ik} \geqslant 0 \ (i, k = 1, \dots, n)\};$$

if $x, y \in \mathbb{R}^n$ and $X, Y \in \mathbb{R}^{n\times n}$ then

$$x \leqslant y \Leftrightarrow y - x \in \mathbb{R}_+^n, \ X \leqslant Y \Leftrightarrow Y - X \in \mathbb{R}_+^{n\times n};$$

if $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ and $X = (x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$ then

$$|x| = (|x_i|)_{i=1}^n, \ |X| = (|x_{ik}|)_{i,k=1}^n;$$

$\det(X)$—the determinant of the matrix X;

X^{-1}—the inverse matrix to X;

$r(X)$—the spectral radius of the matrix X;

E—the unit matrix;

Θ—the zero matrix;

$C(I, \mathbb{R}^n)$—the space of continuous vector functions $x \colon I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\{\|x(t)\| \colon t \in I\};$$

if $x = (x_i)_{i=1}^n \in C(I, \mathbb{R}^n)$ then

$$|x|_C = (\|x_i\|_C)_{i=1}^n;$$

$L^\mu(I,\mathbb{R}^n)$, where $1 \leqslant \mu < +\infty$—the space of vector functions $x\colon I \to \mathbb{R}^n$ with elements integrable in the μ-th power with the norm

$$\|x\|_{L^\mu} = \left(\int_a^b \|x(t)\|^\mu \,\mathrm{d}t\right)^{1/\mu};$$

if $x = (x_i)_{i=1}^n \in L^\mu(I;\mathbb{R}^n)$ then

$$|x|_{L^\mu} = (\|x_i\|_{L^\mu})_{i=1}^n;$$

$L(I,\mathbb{R}^{n\times n})$—the space of integrable matrix functions $X\colon I \to \mathbb{R}^{n\times n}$;
if $X = (x_{ik})_{i,k=1}^n\colon I \to \mathbb{R}^{n\times n}$ then

$$\max_{t\in I}\{X(t)\} = \bigl(\max_{t\in I}\{x_{ik}(t)\}\bigr)_{i,k=1}^n,$$
$$\operatorname*{ess\,sup}_{t\in I}\{X(t)\} = \bigl(\operatorname*{ess\,sup}_{t\in I}\{x_{ik}(t)\}\bigr)_{i,k=1}^n.$$

If $Z \in C(I;\mathbb{R}^{n\times n})$ is a matrix function with columns $z_1,\dots,z_n$ and $g\colon C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ is a linear operator then $g(Z)$ stands for the matrix function with columns $g(z_1),\dots,g(z_n)$.

1. Problem (0.1), (0.2)

In this section, along with the problem (0.1), (0.2), we have to consider the corresponding homogeneous problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t), \tag{1.1}$$

$$l(x) = 0. \tag{1.2}$$

Throughout this section we will assume

(i) $p\colon C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ is a linear operator for which there is an integrable function $\eta\colon I \to \mathbb{R}$ such that

$$\|p(x)(t)\| \leqslant \eta(t)\|x\|_C \quad \text{for } t \in I,\ x \in C(I;\mathbb{R}^n);$$

(ii) $l\colon C(I;\mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator;

(iii) $q \in L(I;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$.

According to the terminology of the monograph by L.V. Kantorovič, B.Z. Vulich and A.G. Pinsker [10], p is an operator of the class H_b^0 and admits the representation by means of the Stieltjes integral (see [10], p. 317). In particular, Chapter V of the monograph [19] is dedicated to boundary value problems for the system (0.1) with an operator p from the class H_b^0.

1.1. Existence and uniqueness theorems. Let $\mathbf{B} = C(I;\mathbb{R}^n) \times \mathbb{R}^n$ be the Banach space containing elements $u = (x;c)$, where $x \in C(I;\mathbb{R}^n)$ and $c \in \mathbb{R}^n$, with the norm

$$\|u\|_{\mathbf{B}} = \|x\|_C + \|c\|.$$

For an arbitrary $u = (x;c) \in \mathbf{B}$ and an arbitrary but fixed point $t_0 \in I$ we set

$$f(u)(t) = \left(c + x(t_0) + \int_{t_0}^t p(x)(s)\,\mathrm{d}s,\ c - l(x)\right) \quad \text{for } t \in I, \tag{1.3}$$

$$h(t) = \left(\int_{t_0}^t q(s)\,\mathrm{d}s,\ c_0\right) \quad \text{for } t \in I.$$

The problem (0.1), (0.2) is equivalent to the following equation in $\mathbf{B}$:

$$u = f(u) + h, \tag{1.4}$$

since $u = (x,c)$ is a solution of (1.4) iff $c = 0$ and x is a solution of the problem (0.1), (0.2).

However, in view of (i)-(iii) and (1.3), $f\colon \mathbf{B} \to \mathbf{B}$ is a linear compact operator. Therefore by the Fredholm alternative for operator equations ([16], p. 275) it is necessary and sufficient for unique solvability of (1.4) that the operator equation

$$u = f(u) \tag{1.5}$$

have only the trivial solution. However, (1.5) is equivalent to the problem (1.1), (1.2). Thus the following theorem is valid.

Theorem 1.1[3]. *The problem* (0.1), (0.2) *is uniquely solvable if and only if the corresponding homogeneous problem* (1.1), (1.2) *has only the trivial solution.*

Let t_0 be an arbitrary but fixed point from I. We define the following sequences of operators $p^k\colon C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n)$ and matrices $\Lambda_k \in \mathbb{R}^{n\times n}$:

$$p^0(x)(t) = x(t),\ p^k(x)(t) = \int_{t_0}^t p(p^{k-1}(x))(s)\,\mathrm{d}s\ (k = 1,2,\ldots), \tag{1.6}$$

$$\Lambda_k = l(p^0(E) + p^1(E) + \ldots + p^{k-1}(E))\ (k = 1,2,\ldots). \tag{1.7}$$

[3] See [19], p. 179, V.3.5.

If the matrix Λ_k is non-singular for some k then we set

$$
\begin{aligned}
(1.8)\qquad & p^{k,0}(x)(t) = x(t),\ p^{k,m}(x)(t) = p^m(x)(t) \\
& -[p^0(E)(t) + \ldots + p^{m-1}(E)(t)]\Lambda_k^{-1} l(p^k(x)).
\end{aligned}
$$

Theorem 1.2. *Let there be positive integers k and m, a nonnegative integer m_0 and a matrix $A \in \mathbb{R}^{n\times n}$ such that*

$$
(1.9)\qquad r(A) < 1,
$$

where the matrix Λ_k is non-singular and the following inequality holds for an arbitrary solution x of (1.1), (1.2)*:*

$$
(1.10)\qquad |p^{k,m}(x)|_C \leqslant A|p^{k,m_0}(x)|_C.
$$

Then the problem (0.1), (0.2) *has a unique solution.*

Proof. According to Theorem 1.1 it is sufficient to show that the homogeneous problem (1.1.), (1.2) has only the trivial solution.

Let x be an arbitrary solution of (1.1), (1.2). It is clear that

$$
x(t) = c + p^1(x)(t),
$$

where $c = x(t_0)$. Consequently,

$$
\begin{aligned}
x(t) &= c + p^1(c + p^1(x))(t) \\
&= c + p^1(c)(t) + p^2(x)(t) \\
&= [p^0(E)(t) + p^1(E)(t)]c + p^2(x)(t).
\end{aligned}
$$

If we continue this process infinitely many times, then we obtain

$$
(1.11)\qquad x(t) = [p^0(E)(t) + \ldots + p^{i-1}(E)(t)]c + p^i(x)(t)
$$

for an arbitrary positive integer i.

(1.2), (1.7) and (1.11) yield

$$
0 = \Lambda_k c + l(p^k(x)).
$$

Since the matrix Λ_k is non-singular, we derive from the last equation

$$
c = -\Lambda_k^{-1} l(p^k(x)).
$$

Using this equality, we find from (1.8) and (1.11)

$$x(t) = p^{k,m_0}(x)(t), x(t) = p^{k,m}(x)(t)$$

and consequently

$$p^{k,m_0}(x)(t) = p^{k,m}(x)(t).$$

The last equality and (1.10) imply

$$|p^{k,m_0}(x)|_C \leqslant A|p^{k,m_0}(x)|_C$$

and

$$(E - A)|p^{k,m_0}(x)|_C \leqslant 0.$$

However, since A is nonnegative and the condition (1.9) holds, the inverse $(E-A)^{-1}$ to the matrix $E - A$ is nonnegative.

Multiplying both parts of the last inequality by $(E - A)^{-1}$, we obtain

$$|p^{k,m_0}(x)|_C \leqslant 0$$

and therefore

$$p^{k,m_0}(x)(t) \equiv 0.$$

Consequently, $x(t) \equiv 0$. □

If $l(x) = x(t_0)$ then by virtue of (1.6)–(1.8) we have for arbitrary positive integers k and m

$$\Lambda_k = E, \; l(p^k(x)) = 0, \; p^{k,m}(x)(t) = p^m(x)(t).$$

This is why Theorem 1.2 implies

Corollary 1.1. *Let there be a positive integer m, a nonnegative integer m_0 and a matrix $A \in \mathbb{R}^{n \times n}$ satisfying* (1.9) *such that the inequality*

$$|p^m(x)|_C \leqslant A|p^{m_0}(x)|_C \tag{1.12}$$

holds for an arbitrary solution x of the system (1.1) *with the initial condition $x(t_0) = 0$.*

Then the problem (0.1), (0.3) *has a unique solution.*

Let us give two examples showing that the condition (1.9) imposed on the matrix A in Corollary 1.1 is optimal and can not be replaced by the condition

$$r(A) \leqslant 1. \tag{1.13}$$

Let us consider differential systems

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(1) \tag{1.14}$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 2\int_0^1 x(s)\,\mathrm{d}s \tag{1.15}$$

on the interval $I = [0,1]$ with the initial condition

$$x(0) = 1. \tag{1.16}$$

Each solution of the system (1.14) as well as of (1.15) has the form

$$x(t) = ct,$$

where $c \in \mathbb{R}^n$ is an arbitrary constant vector. Consequently, the initial value problems (1.14), (1.16) and (1.15), (1.16) have no solution. On the other hand, we have

$$p^1(x)(t) = tx(1)$$

for the system (1.14) and

$$p^2(x)(t) = p^1(x)(t) = 2t\int_0^1 x(s)\,\mathrm{d}s$$

for the system (1.15). Consequently, the conditions (1.12) and (1.13) with $m = 1$, $m_0 = 0$ ($m = 2$, $m_0 = 1$) and $A = E$ are satisfied for the system (1.14) (for the system (1.15)).

Corollary 1.2. *Let there be nonnegative integers m and m_0 and a matrix $A \in \mathbb{R}_+^{n\times n}$ such that*

$$r(A) < \frac{\pi}{2(b-a)}, \tag{1.17}$$

and let

$$|p(p^m(x))|_{L^2} \leqslant A|p^{m_0}(x)|_{L^2} \tag{1.18}$$

be satisfied for an arbitrary solution x of the system (1.1) *with the initial condition $x(t_0) = 0$. Then the problem* (0.1), (0.3) *has a unique solution.*

Proof. We have to prove that the system (1.1) with the initial condition

$$x(t_0) = 0 \tag{1.19}$$

has the zero solution only.

Let x be an arbitrary solution of the problem (1.1), (1.19). Then according to (1.6),

$$x(t) = p^{m_0}(x)(t) = p^{m+1}(x)(t)$$

and

$$|p^{m_0}(x)|_{L^2} = |p^{m+1}(x)|_{L^2}. \tag{1.20}$$

However,

$$p^{m+1}(x)(t_0) = 0, \ \frac{\mathrm{d}}{\mathrm{d}t}[p^{m+1}(x)(t)] = p(p^m(x))(t).$$

Therefore in accordance with Wirtinger's inequality ([8], p. 409)

$$|p^{m+1}(x)|_{L^2} \leqslant \frac{2(b-a)}{\pi}|p(p^m(x))|_{L^2}.$$

This inequality together with (1.18) and (1.20) implies

$$|p^{m_0}(x)|_{L^2} \leqslant B|p^{m_0}(x)|_{L^2}$$

and

$$(E-B)|p^{m_0}(x)|_{L^2} \leqslant 0,$$

where

$$B = \frac{2(b-a)}{\pi}A.$$

However, by virtue of (1.17) we have $r(B) < 1$. Thus the previous inequality yields $|p^{m_0}(x)| \leqslant 0$. Consequently, $x(t) = p^{m_0}(x)(t) \equiv 0$. □

The condition (1.17) in Corollary 1.2 is also optimal. Indeed, let us consider the homogeneous problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x\left(\frac{\pi}{2} - t\right), \tag{1.21}$$
$$x(0) = 0$$

on the interval $I = [0, \frac{\pi}{2}]$. It has a nontrivial solution

$$x(t) = E \sin t,$$

in spite of the fact that the condition (1.18) with $m_0 = m = 0$ and $A = E$ is satisfied for the system (1.21), but the matrix A satisfies the equality

$$r(A) = \frac{\pi}{2(b-a)}$$

instead of (1.17).

Corollary 1.3. *Let a matrix*

$$B_i = \sum_{j=1}^{i} \int_a^b p(p^j(E))(s)\,\mathrm{d}s \tag{1.22}$$

be non-singular for some nonnegative integer i and let there exist a matrix $B \in \mathbb{R}_+^{n\times n}$ such that the inequalities

$$\int_a^b |p(x)(t)|\,\mathrm{d}t \leqslant B|x|_C \tag{1.23}$$

hold for each solution x of the system (1.1) *with the condition $x(b) = x(a)$ and*

$$r(B + |B_i^{-1}|B^{i+2}) < 1. \tag{1.24}$$

Then the problem (0.1), (0.4) *has a unique solution.*

Proof. It is sufficient to prove that all assumptions of Theorem 1.2 are satisfied for $l(x) = x(b) - x(a)$, $k = i + 2$, $m = 1$ and $m_0 = 0$. Indeed, as a consequence of (1.22)–(1.24), (1.6)–(1.8) yield $B_i = \Lambda_k$,

$$|p^1(x)|_C \leqslant \int_a^b |p(x)(s)|\,\mathrm{d}s \leqslant B|x|_C,$$
$$|p^j(x)|_C \leqslant \int_a^b |p(p^{j-1}(x))(s)|\,\mathrm{d}s \leqslant B|p^{j-1}(x)|_C \leqslant B^j|x|_C \quad (j = 1, 2, \ldots),$$
$$|l(p^k(x))| = \left|\int_a^b p(p^{i+1}(x))(s)\,\mathrm{d}s\right|$$
$$\leqslant \int_a^b |p(p^{i+1}(x))(s)|\,\mathrm{d}s \leqslant B|p^{i+1}(x)|_C \leqslant B^{i+2}|x|_C$$

and

$$|p^{k,1}(x)|_C = |p^1(x) - B_i^{-1}l(p^k(x))|_C \leqslant A|x|_C,$$

where $A = B + |B_i^{-1}|B^{i+2}$ satisfies the inequality (1.9). $\square$

For arbitrary t_0 and $t \in I$ and $x \in C(I;\mathbb{R}^n)$, let us set

$$\alpha_*(t_0,t) = \min\{t_0,t\}; \alpha^*(t_0,t) = \max\{t_0,t\},$$
$$I_{t_0,t} = [\alpha_*(t_0,t), \alpha^*(t_0,t)]$$

and

$$\|x\|_{t_0,t} = \max\{\|x(s)\|\colon s \in I_{t_0,t}\}.$$

Definition 1.1[4]**.** An operator p is called *a Volterra operator with respect to* $\mathbf{t}_0 \in I$ if for arbitrary $t \in I$ and $x \in C(I;\mathbb{R}^n)$ satisfying the condition

$$x(s) = 0 \quad \text{for} \quad s \in I_{t_0,t},$$

we have

$$p(x)(s) = 0 \quad \text{for almost all} \quad s \in I_{t_0,t}.$$

Lemma 1.1. *If $p\colon C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ is a Volterra operator with respect to $t_0 \in I$ then the following inequalities hold for arbitrary $x \in C(I;\mathbb{R}^n)$:*

$$\|p(x)(t)\| \leqslant \eta(t)\|x\|_{t_0,t} \text{ for almost all } t \in I, \tag{1.25}$$

$$\|p^k(x)(t)\| \leqslant \frac{1}{k!}\left|\int_{t_0}^t \eta(s)\,\mathrm{d}s\right|^k \|x\|_{t_0,t} \text{ for } t \in I \ (k = 1,2,\ldots), \tag{1.26}$$

where η is the function appearing in the condition (i), *$p^k\colon C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n)$ $(k = 1,2,\ldots)$ are operators given by the equalities* (1.6).

Proof. For arbitrary $t \in I$ and $x \in C(I;\mathbb{R}^n)$ let us set

$$x_{t_0,t}(s) = \begin{cases} x(\alpha_*(t_0,t)) & \text{for } s < \alpha_*(t_0,t), \\ x(s) & \text{for } \alpha_*(t_0,t) \leqslant s \leqslant \alpha^*(t_0,t), \\ x(\alpha^*(t_0,t)) & \text{for } s > \alpha^*(t_0,t). \end{cases}$$

Then since p is a Volterra operator with respect to t_0, we have

$$p(x)(s) = (p(x_{t_0,t}))(s) \text{ for almost all } s \in I_{t_0,t}.$$

[4] See [1, 19, 23].

Thus using the condition (i) we find

$$\|p(x)(s)\| \leqslant \eta(s)\|x_{t_0,t}\|_C = \eta(s)\|x\|_{t_0,t} \text{ for almost all } s \in I_{t_0,t}.$$

However, this estimate implies the estimate (1.25) since $t \in I$ is arbitrary.

According to (1.6) and (1.25)

$$\begin{aligned}\|p^k(x)(t)\| &\leqslant \left|\int_{t_0}^t \|p(p^{k-1}(x))(s)\|\,\mathrm{d}s\right| \\ &\leqslant \left|\int_{t_0}^t \eta(s)\|p^{k-1}(x)\|_{t_0,s}\,\mathrm{d}s\right| \text{ for } t \in I \ (k = 1, 2, \ldots).\end{aligned}$$

Now, by induction, we get the estimates (1.26). □

Lemma 1.1 immediately implies

Lemma 1.2. *If p is a Volterra operator with respect to t_0 then the operator $E - p^1$ is invertible and*

$$(E - p^1)^{-1} = \sum_{k=0}^{+\infty} p^k, \tag{1.27}$$

where p^k $(k = 0, 1, \ldots)$ are operators defined by the equalities (1.6).

Theorem 1.2'. *If p is a Volterra operator with respect to t_0 then the problem (0.1), (0.2) has a unique solution if and only if there exist positive integers k, m and a matrix $A \in \mathbb{R}_+^{n\times n}$ such that Λ_k is non-singular, the equality (1.9) is satisfied and*

$$|p^{k,m}(x)|_C \leqslant A|x|_C \text{ for } x \in C(I; \mathbb{R}^n). \tag{1.28}$$

P r o o f. The sufficiency of the condition is implied by Theorem 1.2. Thus we need to prove the neccessity.

Assume that the problem (0.1), (0.2) has a unique solution, that is, the problem (1.1), (1.2) has the trivial solution only.

If x is an arbitrary solution of the system (1.1) then

$$x(t) = c + p^1(x)(t),$$

where

$$c = x(t_0).$$

Thus by virtue of the equality (1.27) we have

$$x(t) = X(t)c,$$

where

$$X(t) = \sum_{i=0}^{\infty} p^i(E)(t).$$

Therefore the system of algebraic equations

$$l(X)c = 0$$

has the trivial solution only, otherwise the problem (1.1), (1.2) would have a non-trivial solution. Hence

$$\det(l(X)) \neq 0. \tag{1.29}$$

Set

$$X_k(t) = \sum_{i=0}^{k-1} p^i(E)(t).$$

Then $\Lambda_k = l(X_k)$, $\lim\limits_{k\to+\infty} \|X - X_k\|_C = 0$. This and the continuity of l yields

$$\lim_{k\to+\infty} \Lambda_k = l(X). \tag{1.30}$$

(1.29) and (1.30) imply that there are a positive integer k_0 and a positive real number ϱ such that

$$\det(\Lambda_k) \neq 0, \ \|X_m\|_C |\!|\!| l |\!|\!| \|\Lambda_k^{-1}\| < \varrho \ (k = k_0, k_0+1, \dots; m = 1, 2, \dots), \tag{1.31}$$

where $|\!|\!| l |\!|\!|$ is a norm of the operator l. On the other hand, by Lemma 1.1 we have

$$\|p^k(x)\|_C \leqslant \frac{\varrho_0^k}{k!} \|x\|_C \ (k = 1, 2, \dots), \tag{1.32}$$

where

$$\varrho_0 = \int_a^b \eta(t)\,\mathrm{d}t.$$

Having in mind (1.31) and (1.32), we find from (1.8)

$$\|p^{k,m}(x)\|_C \leqslant \left(\frac{\varrho_0^m}{m!} + \varrho\frac{\varrho_0^k}{k!}\right) \|x\|_C \ (k = k_0, k_0+1, \dots; m = 1, 2, \dots). \tag{1.33}$$

Choose a positive integer $m_0 \geqslant k_0$ such that

$$\frac{\varrho_0^m}{m!} + \varrho\frac{\varrho_0^k}{k!} < \frac{1}{2n} \ (k = m_0, m_0+1, \dots; m = m_0, m_0+1, \dots).$$

Then for arbitrary fixed $k \geqslant m_0$ and $m \geqslant m_0$, (1.33) implies the inequality (1.28), where $A \in \mathbb{R}_+^{n\times n}$ is a matrix with the elements $\frac{1}{2n}$ and consequently, the inequality (1.9) is satisfied. $\square$

Theorem 1.3. *Let there exist a matrix function $P_0 \in L(I, \mathbb{R}^{n\times n})$ such that the differential system*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P_0(t)x(t) \tag{1.34}$$

with boundary conditions (1.2) *has the trivial solution only and that for an arbitrary solution x of the problem* (1.1), (1.2), *the following inequality holds:*

$$\int_a^b |G_0(t,s)[p(x)(s) + P_0(s)x(s)]| \, \mathrm{d}s \leqslant A|x|_C, \tag{1.35}$$

where G_0 is the Green matrix of the problem (1.34), (1.2) *and $A \in \mathbb{R}_+^{n\times n}$ is a matrix satisfying the condition* (1.9). *Then the problem* (0.1), (0.2) *has a unique solution.*

Proof. According to Theorem 1.1, we have to show that the problem (1.1), (1.2) has the trivial solution only provided the assumptions of Theorem 1.3 are satisfied.

Let x be an arbitrary solution of (1.1), (1.2). Then since (1.34), (1.2) has a unique solution, we have

$$x(t) = \int_a^b G_0(t,s)[p(x)(s) - P_0(s)x(s)] \, \mathrm{d}s.$$

Thus by the inequality (1.35) we find

$$|x|_C \leqslant A|x|_C.$$

The last inequality implies $|x|_C = 0$ by (1.9). □

Corollary 1.4. *Let there exist a matrix function $P_0 \in L(I; \mathbb{R}^{n\times n})$ such that*

$$\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right) P_0(t) = P_0(t)\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right) \tag{1.36}$$

for almost all t and $s \in I$ and let the following inequality hold for any solution x of the system (1.1) *with the initial condition $x(t_0) = 0$:*

$$\left|\int_{t_0}^t \left|\exp\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right)[p(x)(s) - P_0(s)x(s)]\right| \mathrm{d}s\right| \leqslant A|x|_C \text{ for } t \in I,$$

where $A \in \mathbb{R}_+^{n\times n}$ is a matrix satisfying the condition (1.9). *Then the problem* (0.1), (0.3) *has a unique solution.*

It is sufficient to notice that by (1.36), the Cauchy matrix of the system (1.34) is of the form

$$C_0(t,s) = \exp\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right).$$

Corollary 1.5. *Let there exist a matrix function $P_0 \in L(I;\mathbb{R}^{n\times n})$ satisfying* (1.36) *and such that the matrix*

$$A_0 = E - \exp\left(\int_a^b P_0(s)\,\mathrm{d}s\right)$$

is non-singular and

$$\int_{t-b+a}^{t} \left| A_0^{-1} \exp\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right)[p(x)(s) - P_0(s)x(s)]\right| \mathrm{d}s \leqslant A|x|_C \text{ for } t \in I, \tag{1.37}$$

$$p(x)(t-b+a) \equiv p(x)(t), P_0(t-b+a) \equiv P_0(t), \tag{1.38}$$

where the matrix $A \in \mathbb{R}_+^{n\times n}$ satisfies the condition (1.9). *Then the problem* (0.1), (0.4) *has a unique solution.*

Proof. By (1.36) the non-singularity of the matrix A_0 is a necessary and sufficient condition for non-existence of a nontrivial solution of the problem (1.34), (0.4).

Let A_0 be a non-singular matrix and let G_0 be the Green matrix of the problem (1.34), (1.2), where $l(x) \equiv x(b) - x(a)$. Then by (1.36) we have for arbitrary $\tilde{q} \in L(I;\mathbb{R}^n)$

$$\int_a^b G_0(t,s)\tilde{q}(s)\,\mathrm{d}s = \int_{t-b+a}^{t} A_0^{-1}\exp\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right)\tilde{q}(s)\,\mathrm{d}s \text{ for } t \in I,$$

where $\tilde{q}(t-b+a) \equiv \tilde{q}(t)$. Therefore the inequality (1.37) implies the inequality (1.35). Consequently, all assumptions of Theorem 1.3 are satisfied. □

1.2. Correctness theorems for the problem (0.1), (0.2). Let k be an arbitrary positive integer and let us consider the perturbed problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p_k(x)(t) + q_k(t) \tag{1.39}$$

$$l_k(x) = c_{0k}, \tag{1.40}$$

together with the original problem (0.1), (0.2). Here

(i) $p_k\colon C(I;\mathbb{R}^n)\to L(I;\mathbb{R}^n)$ is a linear operator for which there is an integrable function $\eta_k\colon I\to\mathbb{R}_+$ such that

$$\|p_k(x)(t)\|\leqslant \eta_k(t)\|x\|_C \text{ for } t\in I,\ x\in C(I;\mathbb{R}^n),$$

(ii) $l_k\colon C(I,\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded operator,

(iii) $q_k\in L(I;\mathbb{R}^n)$, $c_{0k}\in\mathbb{R}^n$.

For an arbitrary bounded operator $g\colon C(I;\mathbb{R}^n)\to L(I;\mathbb{R}^n)$, we denote by $|\!|\!|g|\!|\!|$ its norm and by M_g the set of absolutely continuous vector functions $y\colon I\to\mathbb{R}^n$ that can be represented by

$$y(t)=z(a)+\int_a^t g(z)(s)\,\mathrm{d}s,$$

where $z\colon I\to\mathbb{R}^n$ is an arbitrary continuous vector function such that

$$\|z\|_C=1.$$

Theorem 1.4. *Let the problem* (0.1), (0.2) *have a unique solution* x,

$$\sup\left\{\left\|\int_a^t[p_k(y)(s)-p(y)(s)]\,\mathrm{d}s\right\|: t\in I, y\in M_{p_k}\right\}\to 0 \text{ for } k\to+\infty \tag{1.41}$$

and for an arbitrary absolutely continuous function $y\colon I\to\mathbb{R}^n$ *let*

$$\lim_{k\to+\infty}\left((1+|\!|\!|p_k|\!|\!|)\int_a^t[p_k(y)(s)-p(y)(s)]\,\mathrm{d}s\right)=0 \text{ uniformly on } I. \tag{1.42}$$

Suppose further that

$$\lim_{k\to+\infty}\left((1+|\!|\!|p_k|\!|\!|)\int_a^t[q_k(s)-q(s)]\,\mathrm{d}s\right)=0 \text{ uniformly on } I, \tag{1.43}$$

$$\lim_{k\to+\infty} l_k(y)=l(y) \text{ for } y\in C(I;\mathbb{R}^n) \tag{1.44}$$

and

$$\lim_{k\to+\infty} c_{0k}=c_0. \tag{1.45}$$

Then there is a positive integer k_0 *such that the problem* (1.39), (1.40) *has also a unique solution* x_k *for each* $k\geqslant k_0$ *and*

$$\lim_{k\to+\infty}\|x-x_k\|_C=0. \tag{1.46}$$

To prove Theorem 1.4. we will need the following lemma

Lemma 1.3. *Let the problem* (1.1), (1.2) *have the trivial solution only and let the sequences of operators p_k and l_k $(k = 1,2\dots)$ satisfy the conditions* (1.41) *and* (1.44), *respectively. Then there is a positive integer k_0 and a positive constant α such that an arbitrary absolutely continuous vector function $z\colon I \to \mathbb{R}^n$ can be estimated by*

$$\|z\|_C \leqslant \alpha\Delta_k(z) \quad (k = k_0, k_0+1,\dots), \tag{1.47}$$

where

$$\Delta_k(z) = \max_{t\in I}\left\{\|l_k(z)\| + (1 + \|\!|p_k|\!\|)\left\|\int_a^t [z'(s) - p_k(z)(s)]\,\mathrm{d}s\right\|\right\}. \tag{1.48}$$

Proof. Let us first mention that by the Banach-Steinhaus theorem ([16], p. 149), the condition (1.44) yields the boundedness of the sequence $|\!|\!|l_k|\!|\!|$ $(k = 1,2,\dots)$. Consequently, there is a positive constant β such that

$$\|l_k(y)\| \leqslant \beta\|y\|_C \text{ for } y \in C(I;\mathbb{R}^n) \ (k = 1,2,\dots). \tag{1.49}$$

Let us set

$$p^1(y)(t) = \int_a^t p(y)(s)\,\mathrm{d}s, \quad p_k^1(y)(t) = \int_a^t p_k(y)(s)\,\mathrm{d}s \ (k = 1,2,\dots).$$

Obviously, p^1 and $p_k^1\colon C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n)$ are linear bounded operators and

$$|\!|\!|p_k^1|\!|\!| \leqslant |\!|\!|p_k|\!|\!| \ (k = 1,2,\dots). \tag{1.50}$$

On the other hand, by (1.41),

$$\sup\{\|p_k^1(y) - p^1(y)\|_C\colon y \in M_{p_k}\} \to 0 \text{ for } k \to +\infty. \tag{1.51}$$

Let us suppose to the contrary that the lemma does not hold. Then there is an increasing sequence of positive integers $(k_m)_{m=1}^{+\infty}$ and a sequence of absolutely continuous vector functions $z_m : I \to \mathbb{R}^n$ $(m = 1,2,\dots)$ such that

$$\|z_m\|_C > m\Delta_{k_m}(z_m) \quad (m = 1,2,\dots). \tag{1.52}$$

Set

$$y_m(t) = \|z_m\|_C^{-1} z_m(t) \quad (m = 1,2,\dots),$$

$$v_m(t) = \int_a^t [y_m'(s) - p_{k_m}(y_m)(s)]\,\mathrm{d}s \quad (m = 1,2,\dots),$$

$$y_{0m}(t) = y_m(t) - v_m(t), \tag{1.53}$$

$$w_m(t) = p^1_{k_m}(y_{0m})(t) - p^1(y_{0m})(t) + p^1_{k_m}(v_m)(t).$$

Then

$$\|y_m\|_C = 1 \quad (m = 1, 2, \ldots), \tag{1.54}$$

$$y_{0m}(t) = y_m(a) + p^1_{k_m}(y_m)(t) \tag{1.55}$$

and

$$y_{0m}(t) = y_m(a) + p^1(y_{0m})(t) + w_m(t). \tag{1.56}$$

On the other hand, by virtue of (1.52) and (1.50) we have

$$\begin{aligned} \|v_m\|_C &\leqslant (1 + \|\!|p_{k_m}|\!\|)^{-1} \|z_m\|^{-1} \Delta_m(z_m) \\ &< \frac{1}{m}(1 + \|\!|p_{k_m}|\!\|)^{-1} \ (m = 1, 2, \ldots) \end{aligned} \tag{1.57}$$

and

$$\|p^1_{k_m}(v_m)\|_C \leqslant \|\!|p_{k_m}|\!\| \|v_m\|_C < \frac{1}{m} \ (m = 1, 2, \ldots). \tag{1.58}$$

It follows from (1.54) and (1.55) that

$$y_{0m} \in M_{k_m} \quad (m = 1, 2, \ldots).$$

This is why (1.51) implies

$$\lim_{m \to +\infty} \|p^1_{k_m}(y_{0m}) - p^1(y_{0m})\|_C = 0. \tag{1.59}$$

By (1.58) and (1.59) we obtain

$$\lim_{m \to +\infty} \|w_m\|_C = 0. \tag{1.60}$$

On the other hand, according to (1.53), (1.54) and (1.57) we have

$$\|y_{0m}\|_C \leqslant \|y_m\|_C + \|v_m\|_C \leqslant 2 \quad (m = 1, 2, \ldots).$$

Therefore there exists an integrable function $\gamma \colon I \to \mathbb{R}_+$ such that the following inequalities hold almost everywhere on I:

$$\|p(y_{0m})(t)\| \leqslant \gamma(t) \quad (m = 1, 2, \ldots).$$

Consequently,

$$\|p^1(y_{0m})(t) - p^1(y_{0m})(s)\| \leqslant \int_s^t \gamma(\xi)\,\mathrm{d}\xi \text{ for } a \leqslant s \leqslant t \leqslant b \ (m = 1, 2, \ldots). \tag{1.61}$$

By (1.60) and (1.61), the representation (1.56) obviously implies the equicontinuity of the sequence $(y_{0m})_{m=1}^{\infty}$. Hence by the Arcelà-Ascoli lemma we can assume without loss of generality that the sequence $(y_{0m})_{m=1}^{\infty}$ uniformly converges. Let us set

$$\lim_{m\to+\infty} y_{0m}(t) = y_0(t).$$

Then by (1.53), (1.54), (1.56) and (1.60), we obtain

$$\lim_{m\to+\infty} \|y_m - y_0\|_C = 0, \tag{1.62}$$
$$\|y_0\| = 1, y_0(t) = y_0(a) + p^1(y_0)(t).$$

Consequently, y_0 is a nontrivial solution of the system (1.1).

By (1.49) and (1.52)

$$\begin{aligned}\|l_{k_m}(y_0)\| &\leqslant l_{k_m}(y_m - y_0)\| + \|l_{k_m}(y_m)\| \\ &\leqslant \beta\|y_0 - y_m\| + \|z_m\|_C^{-1}\|l_{k_m}(z_m)\| \\ &\leqslant \beta\|y_0 - y_m\|_C + \frac{1}{m} \quad (m = 1, 2, \ldots).\end{aligned}$$

Using (1.44) and (1.62), we find

$$l(y_0) = 0,$$

that is, y_0 is a solution of the problem (1.1), (1.2). But this is not possible since the problem (1.1), (1.2) has no nontrivial solution. This contradiction completes the proof of the lemma. □

Proof of Theorem 1.4. Let k_0 be the positive integer appearing in Lemma 1.3. By this lemma, the homogeneous problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p_k(x)(t),$$
$$l_k(x) = 0$$

possesses the trivial solution only for each $k \geqslant k_0$. By Theorem 1.1 this fact guarantees the existence of a unique solution of the problem (1.39), (1.40).

To complete the proof, it remains to show that the equality (1.46) holds, where x and x_k are solutions of the problems (0.1), (0.2) and (1.39), (1.40), respectively.

Define

$$z_k(t) = x_k(t) - x(t).$$

Then for each $k \geqslant k_0$, we have

$$\frac{dz_k(t)}{dt} = p_k(z_k)(t) + \tilde{q}_k(t),$$
$$l_k(z_k) = \zeta_k,$$

where

$$\tilde{q}_k(t) = p_k(x)(t) - p(x)(t) + q_k(t) - q(t),$$
$$\zeta_k = c_{0k} - c_0 + l(x) - l_k(x).$$

By virtue of the conditions (1.42)–(1.45)

$$\delta_k = (1 + \|\!|p_k|\!\|) \max \left\{ \left\| \int_a^t \tilde{q}_k(s)\,\mathrm{d}s \right\| : t \in I \right\} \to 0 \text{ for } k \to +\infty,$$
$$\lim_{k\to+\infty} \zeta_k = 0.$$

On the other hand, by Lemma 1.3, there is a positive constant α such that

$$\|y_k\|_C \leqslant \alpha(\|\zeta_k\| + \delta_k) \ (k = k_0, k_0 + 1, \ldots).$$

Thus

$$\lim_{k\to+\infty} \|y_k\|_C = 0,$$

and therefore the equality (1.46) holds. □

Corollary 1.6. *Let the problem* (0.1), (0.2) *have a unique solution and let*

$$\lim_{k\to+\infty} \int_a^t [p_k(y)(s) - p(y)(s)]\,\mathrm{d}s = 0 \textit{ uniformly on } I \tag{1.63}$$

for any absolutely continuous $y\colon I \to \mathbb{R}^n$. *Assume further that*

$$\lim_{k\to+\infty} \int_a^t [q_k(s) - q(s)]\,\mathrm{d}s = 0 \textit{ uniformly on } I, \tag{1.64}$$

$$\lim_{k\to+\infty} l_k(y) = l(y) \textit{ for } y \in C(I;\mathbb{R}^n), \lim_{k\to+\infty} c_{0k} = c_0$$

and that there is an integrable function $\eta\colon I \to \mathbb{R}_+$ *such that the following inequalities hold almost everywhere on* I *for arbitrary* $y \in C(I;\mathbb{R}^n)$:

$$\|p_k(y)(t)\| \leqslant \eta(t)\|y\|_C \quad (k = 1, 2, \ldots). \tag{1.65}$$

Then the conclusion of Theorem 1.4 holds.

P r o o f. Without loss of generality, we can suppose

$$\|p(y)(t)\| \leqslant \eta(t)\|y\|_C. \tag{1.66}$$

From (1.65) we get

$$\|p_k\| \leqslant \int_a^b \eta(t)\,\mathrm{d}t \quad (k = 1, 2, \ldots).$$

This is why the conditions (1.42) and (1.43) follow from (1.63) and (1.64).

By Theorem 1.4, it is now sufficient to verify the condition (1.41) to complete the proof of the corollary. Let us suppose to the contrary that the condition (1.41) does not hold. Then there are $\varepsilon_0 > 0$, a sequence of positive integers $(k_m)_{m=1}^{+\infty}$ and a sequence of vector functions

$$y_m \in M_{p_{k_m}} \quad (m = 1, 2, \ldots) \tag{1.67}$$

such that

$$\max_{t \in I} \left\{ \left\| \int_a^t [p_{k_m}(y_m)(s) - p(y_m)(s)\,\mathrm{d}s \right\| \right\} > \varepsilon_0. \tag{1.68}$$

From (1.67) and (1.65) we get

$$y_m(t) = z_m(a) + \int_a^t p_{k_m}(z_m)(s)\,\mathrm{d}s \quad (m = 1, 2, \ldots),$$

where $z_m \in C(I; \mathbb{R}^n)$, $\|z_m\|_C = 1$,

$$\|y_m\|_C \leqslant 1 + \int_a^b \eta(s)\,\mathrm{d}s$$

and

$$\|y_m(t) - y_m(s)\| \leqslant \int_s^t \eta(\xi)\,\mathrm{d}\xi \quad \text{for } a \leqslant s \leqslant t \leqslant b. \tag{1.69}$$

Consequently, the sequence $(y_m)_{m=1}^{+\infty}$ is uniformly bounded and equicontinuous. Thus without loss of generality, we can consider it to be uniformly convergent. Set

$$\lim_{m \to +\infty} y_m(t) = y(t).$$

Then by (1.69) we have

$$\|y(t) - y(s)\| \leqslant \int_s^t \eta(\xi)\,\mathrm{d}\xi \quad \text{for } a \leqslant s \leqslant t \leqslant b.$$

Consequently, the function $y\colon I \to \mathbb{R}^n$ is absolutely continuous.

According to (1.63), (1.65) and (1.66) we have

$$\max\left\{\left\|\int_a^t [p_{k_m}(y_m)(s) - p(y_m)(s)]\,\mathrm{d}s\right\| : t \in I\right\} \leqslant \left(2\int_a^b \eta(s)\,\mathrm{d}s\right)\|y_m - y\|_C$$
$$+\max\left\{\left\|\int_a^t [p_{k_m}(y)(s) - p(y)(s)]\,\mathrm{d}s\right\| : t \in I\right\} \to 0 \text{ for } t \to +\infty,$$

and this contradicts (1.68). The proof of the corollary is now complete. □

A result similar to Corollary 1.6 is given in the paper by R. Tsitskishvili [20].

2. Problem (0.5), (0.6)

2.1. Existence and uniqueness theorems. According to the note done in the introduction, the problem (0.5), (0.6) can be rewritten to the form (0.1), (0.2), where the operator p and the vector function q are given by the equalities (0.10) and (0.11) and the function τ_0 is given by the equality (0.9). That is why Theorem 1.1 for the problem (0.5), (0.6) assumes the following form.

Theorem 2.1. *The problem* (0.5), (0.6) *has a unique solution if and only if the corresponding homogeneous problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \chi_I(\tau(t))P(t)x(\tau_0(t)) \tag{2.1}$$

$$l(x) = 0 \tag{2.2}$$

has the trivial solution only.

According to the Riesz theorem (see [16], p. 184) there exists a unique matrix function $\Lambda\colon I \to \mathbb{R}^{n\times n}$ such that the elements of Λ have bounded variation on I,

$$\Lambda(b) = \Theta \tag{2.3}$$

and for arbitrary $x \in C(I;\mathbb{R}^n)$, the following representation is true:

$$l(x) = \int_a^b d\Lambda(t)x(t). \tag{2.4}$$

Obviously, (2.3) and (2.4) yield that if $x\colon I \to \mathbb{R}^n$ is absolutely continuous then

$$l(x) = -\Lambda(a)x(a) - \int_a^b \Lambda(t)x'(t)\,\mathrm{d}t. \tag{2.5}$$

Let t_0 be an arbitrary but fixed point from the interval I. For an arbitrary matrix function $V \in L(I;\mathbb{R}^{n\times n})$ set

$$[V(t)]_{\tau,0} = \Theta, [V(t)]_{\tau,1} = \chi_I(\tau(t))V(t),$$

$$[V(t)]_{\tau,i+1} = [V(t)]_{\tau,1} \int_{t_0}^{\tau_0(t)} [V(s)]_{\tau,i}\,\mathrm{d}s \quad (i = 1, 2, \ldots),$$

where τ_0 is the function given by (0.9). Then by (0.10), (1.6)–(1.8), and (2.5) we have

$$\Lambda_k = -\Lambda(a) - \sum_{i=0}^{k-1} \left[\int_a^b \Lambda(s)[P(s)]_{\tau,i}\,\mathrm{d}s - \Lambda(a) \int_a^{t_0} [P(s)]_{\tau,i}\,\mathrm{d}s \right] \tag{2.6}$$

and

$$|p^{k,m}(x)|_C \leqslant A_{k,m}|x|_C \text{ for } x \in C(I;\mathbb{R}^n),$$

where

$$A_{k,m} = A_m + \left(E + \sum_{i=0}^{m-1} A_i\right)|\Lambda_k^{-1}| \times \left(\int_a^{t_0} [|P(s)|]_{\tau,k}\,\mathrm{d}s + \int_a^b |\Lambda(s)|[|P(s)|]_{\tau,k}\,\mathrm{d}s \right) \tag{2.7}$$

and

$$A_i = \max\left\{ \left| \int_{t_0}^t [|P(s)|]_{\tau,i}\,\mathrm{d}s \right| : t \in I \right\} \quad (i = 0, \ldots, m). \tag{2.8}$$

Therefore the following statements follow from Theorems 1.2, 1.2' and Corollary 1.2.

Theorem 2.2.[5] *Assume that there exist positive integers k and m such that*

$$\det(\Lambda_k) \neq 0 \tag{2.9}$$

and

$$r(A_{k,m}) < 1, \tag{2.10}$$

[5] For $\tau(t) \equiv t$ an analogous result was obtained by T. Kiguradze ([13], Lemma 2.7).

where Λ_k and $A_{k,m}$ are matrices given by the equalities (2.6)–(2.8). Then the problem (0.5), (0.6) has a unique solution.

Theorem 2.2'. *If the inequality*

$$(\tau(t)-t)(t-t_0)\leqslant 0$$

holds almost everywhere on I, then the problem (0.5), (0.6) has a unique solution if and only if there are positive integers k and m such that the inequalities (2.9) and (2.10) are satisfied, where Λ_k and $A_{k,m}$ are matrices defined by the equalities (2.6)–(2.8).

Corollary 2.1. *If the inequality*

$$r(A_m)<1, \tag{2.11}$$

is satisfied with

$$A_m=\max\left\{\left|\int_{t_0}^{t}[|P(s)|]_{\tau,m}\,\mathrm{d}s\right| : t\in I\right\}$$

for some positive integer m, then the problem (0.5), (0.7) has a unique solution.

Corollary 2.2. *Let τ be absolutely continuous and monotone and suppose that there exists a matrix $A\in\mathbb{R}^{n\times n}$ such that*

$$r(A)<\frac{\pi}{2(b-a)} \tag{2.12}$$

and the inequality

$$\chi_I(\tau(t))|P(t)|\leqslant A|\tau'(t)|^{1/2}$$

holds almost everywhere on I. Then the problem (0.5), (0.7) has a unique solution.

The examples given in Section 1 show that the strict inequalities in Corollaries 2.1 and 2.2 cannot be replaced by nonstrict ones.

Corollary 2.3. *Assume that*

$$\det(B_0)\neq 0$$

and

$$r(B+|B_0^{-1}|B^2)<1,$$

where

$$B_0 = \int_a^b \chi_I(\tau(s))P(s)\,\mathrm{d}s, \; B = \int_a^b \chi_I(\tau(s))|P(s)|\,\mathrm{d}s.$$

Then the problem (0.5), (0.8) *has a unique solution.*

Theorem 2.3. *Let there exist a matrix function $P_0 \in L(I; \mathbb{R}^{n\times n})$ such that the differential system*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P_0(t)x(t) \tag{2.13}$$

with the boundary conditions (2.2) *has the trivial solution only and*

$$\int_a^b |G_0(t,s)|Q(s)\,\mathrm{d}s \leqslant A \text{ for } t \in I, \tag{2.14}$$

where

$$Q(t) = |\chi_I(\tau(t))P(t) - P_0(t)| + |P_0(t)| \left| \int_{t_0}^{\tau_0(t)} |P(s)|\,\mathrm{d}s \right|, \tag{2.15}$$

G_0 is the Green matrix of the problem (2.13), (2.2) *and $A \in \mathbb{R}_+^{n\times n}$ is a matrix satisfying the inequality*

$$r(A) < 1. \tag{2.16}$$

Then the problem (0.5), (0.6) *has a unique solution.*

Proof. Let x be a solution of the problem (1.1), (1.2), where p is the operator defined by the equality (0.10). Then using (2.15), we have

$$\begin{aligned} |p(x)(s) - P_0(s)x(s)| &= |[\chi_I(\tau(t)P(t) - P_0(t)]x(\tau_0(t)) \\ &\quad + P_0(t)\int_t^{\tau_0(t)} x'(s)\,\mathrm{d}s| \\ &= |[\chi_I(\tau(t))P(t) - P_0(t)]x(\tau_0(t)) \\ &\quad + P_0(t)\int_t^{\tau_0(t)} \chi_I(\tau(s))P(s)x(\tau_0(s))\,\mathrm{d}s| \leqslant Q(t)|x|_C. \end{aligned}$$

According to this estimate, (2.14) implies the inequality (1.35). Therefore, all assumptions of Theorem 1.3 are satisfied. □

Corollary 2.4. *Assume that there exists a matrix function* $P_0 \in L(I; \mathbb{R}^{n\times n})$ *such that the equality*

$$\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right) P_0(t) = P_0(t) \left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right) \tag{2.17}$$

is satisfied for almost all s *and* $t \in I$ *and that*

$$\left|\int_{t_0}^t \left|\exp\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right)\right| Q(s)\,\mathrm{d}s\right| \leqslant A \text{ for } t \in I, \tag{2.18}$$

where Q *is the matrix function defined by* (2.15) *and* $A \in \mathbb{R}_+^{n\times}$ *is a matrix satisfying the condition* (2.16). *Then the problem* (0.5), (0.7) *has a unique solution.*

Corollary 2.5. *Assume that there is a matrix function* $P_0 \in L(I; \mathbb{R}^{n\times n})$ *such that the equality* (2.17) *is satisfied for almost all* s *and* $t \in I$. *Let the matrix*

$$A_0 = E - \exp\left(\int_a^b P_0(s)\,\mathrm{d}s\right) \tag{2.19}$$

be non-singular and

$$\int_{t-b+a}^t \left|A_0^{-1} \exp\left(\int_s^t P_0(\xi)\,\mathrm{d}\xi\right)\right| Q(s)\,\mathrm{d}s \leqslant A \text{ for } t \in I, \tag{2.20}$$

where Q *is the matrix defined by* (2.15). *Suppose further that*

$$P_0(t-b+a) \equiv P_0(t), \;\; Q(t-b+a) \equiv Q(t) \tag{2.21}$$

and that $A \in \mathbb{R}_+^{n\times n}$ *is a matrix satisfying the condition* (2.16). *Then the problem* (0.5), (0.8) *has a unique solution.*

2.2. Correctness theorems. Let k be an arbitrary positive integer and, together with the problem (0.5), (0.6), let us consider the perturbed problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P_k(t)x(\tau_k(t)) + q_{0k}(t), \tag{2.22}$$

$$l_k(x) = c_{0k}, \;\; x(t) = u_k(t), \;\; t \notin I, \tag{2.23}$$

where $P_k \in L(I; \mathbb{R}^{n\times n})$, $q_{0k} \in L(I; \mathbb{R}^n)$, $c_{0k} \in \mathbb{R}^n$, $\tau_k \colon I \to \mathbb{R}$ is measurable, $u_k \colon \mathbb{R} \to \mathbb{R}^n$ is continuous and bounded, and $l_k \colon C(I; \mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator.

Theorem 2.4. *Let the problem* (0.5), (0.6) *have a unique solution* x,

$$\lim_{k\to+\infty}\int_a^t [i+(-1)^i\chi_I(\tau_k(s))]P_k(s)\,\mathrm{d}s = \int_a^t [i+(-1)^i\chi_I(\tau(s))]P(s)\,\mathrm{d}s \text{ uniformly on } I \ (i=0,1), \tag{2.24}$$

$$\lim_{k\to+\infty}\int_a^t q_{0k}(s)\,\mathrm{d}s = \int_a^t q(s)\,\mathrm{d}s \text{ uniformly on } I, \tag{2.25}$$

$$\operatorname{ess\,sup}\{|\tau_k(t)-\tau(t)|\colon t\in I\}\to 0 \text{ for } k\to+\infty, \tag{2.26}$$

$$\lim_{k\to+\infty} u_k(t) = u(t) \text{ uniformly on } \mathbb{R}, \tag{2.27}$$

$$\lim_{k\to+\infty} l_k(y) = l(y) \text{ for } y\in C(I;\mathbb{R}^n),\ \lim_{k\to+\infty} c_{0k} = c_0. \tag{2.28}$$

Let further $\eta\colon I\to\mathbb{R}_+$ *be an integrable function such that*

$$\|P_k(t)\| \leqslant \eta(t) \ (k=1,2,\ldots) \tag{2.29}$$

is satisfied almost everywhere on I. *Then there is a positive integer* k_0 *such that for each* $k\geqslant k_0$, *the problem* (2.22), (2.23) *has also a unique solution* x_k *and*

$$\lim_{k\to+\infty}\|x-x_k\|_C = 0. \tag{2.30}$$

The following lemma will be useful for proving Theorem 2.4.

Lemma 2.1. *Assume that* H *and* $H_k\in L(I;\mathbb{R}^{n\times n})$, V *and* $V_k\in L^{+\infty}(I;\mathbb{R}^n)$ $(k=1,2,\ldots)$,

$$\lim_{k\to+\infty}\int_a^t H_k(s)\,\mathrm{d}s = \int_a^t H(s)\,\mathrm{d}s \ \text{ uniformly on } I, \tag{2.31}$$

$$\operatorname{ess\,sup}\{\|V_k(t)-V(t)\|\colon t\in I\}\to 0 \text{ for } k\to+\infty \tag{2.32}$$

and that there exists an integrable function $\eta\colon I\to\mathbb{R}^n$ *such that*

$$\|H_k(t)\| \leqslant \eta(t) \ (k=1,2,\ldots) \tag{2.33}$$

holds almost everywhere on I. *Then*

$$\lim_{k\to+\infty}\int_a^t H_k(s)V_k(s)\,\mathrm{d}s = \int_a^t H(s)V(s)\,\mathrm{d}s \text{ uniformly on } I. \tag{2.34}$$

P r o o f. Let us first note that (2.31) and (2.33) yield

$$\|H(t)\| \leqslant \eta(t) \tag{2.35}$$

almost everywhere on I.

Let $\varepsilon > 0$ be an arbitrary number. Since V is essentially bounded, there is a continuously differentiable vector function $V_0 \colon I \to \mathbb{R}^n$ such that

$$\int_a^b \eta(t)\|V(s) - V_0(s)\| \,\mathrm{d}s < \frac{\varepsilon}{4}. \tag{2.36}$$

Let us set

$$G_k(t) = \int_a^t [H_k(s) - H(s)] \,\mathrm{d}s.$$

Then by (2.31)

$$\lim_{k\to+\infty} \|G_k\|_C = 0.$$

Hence

$$\int_a^t [H_k(s) - H(s)]V_0(s) \,\mathrm{d}s$$
$$= G_k(t)V_0(t) - \int_a^t G_k(s)V_0'(s) \,\mathrm{d}s \to 0 \text{ for } k \to +\infty \quad \text{uniformly on } I.$$

Consequently, there is a positive integer k_0 such that

$$\left\| \int_a^t [H_k(s) - H(s)]V_0(s) \,\mathrm{d}s \right\| < \frac{\varepsilon}{4} \quad \text{for } t \in I,\ k \geqslant k_0. \tag{2.37}$$

On the other hand, by virtue of (2.32) we can assume without loss of generality, that

$$\int_a^b \eta(s)\|V_k(s) - V(s)\| \,\mathrm{d}s < \frac{\varepsilon}{4} \quad \text{for } k \geqslant k_0. \tag{2.38}$$

By virtue of the conditions (2.33) and (2.35)–(2.38), the equality

$$\int_a^t [H_k(s)V_k(s) - H(s)V(s)] \,\mathrm{d}s = \int_a^t H_k(s)(V_k(s) - V(s)) \,\mathrm{d}s$$
$$+ \int_a^t [H_k(s) - H(s)]V_0(s) \,\mathrm{d}s + \int_a^t [H_k(s) - H(s)](V(s) - V_0(s)) \,\mathrm{d}s$$

implies

$$\left\| \int_a^t [H_k(s)V_k(s) - H(s)V(s)] \,\mathrm{d}s \right\| \leqslant \int_a^b \eta(s)\|V_k(s) - V(s)\| \,\mathrm{d}s$$
$$+ \left\| \int_a^t [H_k(s) - H(s)]V_0(s) \,\mathrm{d}s \right\| + 2\int_a^b \eta(s)\|V(s) - V_0(s)\| \,\mathrm{d}s < \varepsilon \text{ for } k \geqslant k_0, t \in I.$$

Consequently, the relation (2.34) is verified. □

Proof of Theorem 2.4. By remarks given in the introduction, the problems (0.5), (0.6) and (2.28), (2.29) can be rewritten to the form (0.1), (0.2) and (1.39), (1.40), respectively, where

$$\tau_{0k}(t) = \begin{cases} \tau_k(t) & \text{for } \tau_k(t) \in [a,b], \\ a & \text{for } \tau_k(t) < a, \\ b & \text{for } \tau_k(t) > b, \end{cases} \tag{2.39}$$

$$p_k(x)(t) = P_k(t)\chi_I(\tau_k(t))x(\tau_{0k}(t)), \tag{2.40}$$

$$q_k(t) = (1 - \chi_I(\tau_k(t))P_k(t)u_k(\tau_k(t)) + q_{0k}(t) \tag{2.41}$$

and τ_0, p and q are defined by equalities (0.9)–(0.11).

In virtue of Lemma 2.1 and the conditions (2.24)–(2.27), (2.29), the equalities (0.11) and (2.41) yield the condition (1.64). On the other hand, the inequalities (1.65) follow from (2.29) and (2.40). Therefore, for completing the proof of Theorem 2.4, it remains to show by Corollary 1.6 that the condition (1.63) is satisfied for an arbitrary absolutely continuous vector function $y\colon I \to \mathbb{R}^n$.

In view of (0.9), (2.24), (2.26), (2.29) and (2.39), the matrix and vector functions

$$H_k(t) = \chi_I(\tau_k(t))P_k(t), \quad H(t) = \chi_I(\tau(t))P(t),$$
$$V_k(t) = Y(\tau_{0k}(t)), \quad V(t) = Y(\tau_0(t))$$

satisfy the conditions (2.31)–(2.33). Now according to Lemma 2.1, the condition (2.34) holds. The condition (2.34) is equivalent to the condition (1.63) by (0.10) and (2.40). □

Theorem 2.5. *Assume that the problem* (0.5), (0.6) *has a unique solution,*

$$\lim_{k\to+\infty} \left(\varrho_k \int_a^t [P_k(s) - P(s)]\,\mathrm{d}s \right) = 0 \ \textit{uniformly on } I, \tag{2.42}$$

$$\lim_{k\to+\infty} \left(\varrho_k \int_a^t [q_{0k}(s) - q_0(s)]\,\mathrm{d}s \right) = 0 \ \textit{uniformly on } I, \tag{2.43}$$

$$\lim_{k\to+\infty} (\varrho_k^2[u_k(t) - u(t)]) = 0 \ \textit{uniformly on } \mathbb{R}, \tag{2.44}$$

where

$$\varrho_k = 1 + \int_a^b \|P_k(t)\|\,\mathrm{d}t,$$

and suppose that the conditions (2.28) are fulfilled. Let further the function $\tau\colon I \to \mathbb{R}$ be continuous and monotone, let the components of the vector function u have bounded variations and

$$\tau_k(t) \equiv \tau(t) \ (k = 1, 2, \ldots). \tag{2.45}$$

Then the conclusion of Theorem 2.4 holds.

Proof. By (2.45), the problems (0.5), (0.6) and (2.22), (2.23) are equivalent to the problems (0.1), (0.2) and (1.39), (1.40), respectively, where τ_0, p and q are defined by equations (0.9)–(0.11),

$$p_k(x)(t) = P_k(t)\chi_I(\tau(t))x(\tau_0(t)), \tag{2.46}$$

$$q_k(t) = (1 - \chi_I(\tau(t))P_k(t)u_k(\tau(t)) + q_{0k}(t). \tag{2.47}$$

By Theorem 1.4, to complete the proof it suffices to verify that the sequences p_k and q_k $(k = 1, 2, \ldots)$ satisfy the conditions (1.41)–(1.43).

Let us first note that

$$1 + |\!|\!| p_k |\!|\!| \leqslant \varrho_k \quad (k = 1, 2, \ldots). \tag{2.48}$$

On the other hand, the monotonicity and continuity of τ and the condition (2.42) imply

$$\lim_{k \to +\infty} (\varrho_k \|Q_{ik}\|_C) = 0 \quad (i = 0, 1), \tag{2.49}$$

where

$$Q_{ik}(t) = \int_a^t [i + (-1)^i \chi_I(\tau(s))[P_k(s) - P(s)]\,\mathrm{d}s \quad (i = 0, 1).$$

In view of (0.10) and (2.46), for an arbitrary absolutely continuous $y\colon I \to \mathbb{R}^n$ we have

$$\begin{aligned}\int_a^t [p_k(y)(s) - p(y)(s)]\,\mathrm{d}s &= \int_a^t Q'_{0k}(s)y(\tau_0(s))\,\mathrm{d}s \\ &= Q_{0k}(t)y(\tau_0(t)) + \int_a^t Q_{0k}(s)\,\mathrm{d}y(\tau_0(s)).\end{aligned} \tag{2.50}$$

If $y \in M_{p_k}$ then

$$y(t) = z(a) + \int_a^t \chi_I(\tau(s))P_k(s)z(\tau_0(s))\,\mathrm{d}s,$$

where

$$z \in C(I; \mathbb{R}^n), \|z\|_C \leqslant 1.$$

Therefore

$$\|y\|_C \leqslant 1 + \int_a^b \|P_k(s)\| \, \mathrm{d}s = \varrho_k \tag{2.51}$$

and

$$\int_a^b \| \, \mathrm{d}y(s)\| \leqslant \int_a^b \|P_k(s)\| \, \mathrm{d}s < \varrho_k.$$

The last inequality together with the monotonicity of τ_0 yields

$$\left| \int_a^b \mathrm{d}y(\tau_0(s)) \, \mathrm{d}s \right| \leqslant \int_a^b \| \, \mathrm{d}y(s)\| < \varrho_k. \tag{2.52}$$

By (2.51) and (2.52), (2.50) implies

$$\left\| \int_a^t [p_k(y)(s) - p(y)(s)] \, \mathrm{d}s \right\| \leqslant 2\varrho_k \|Q_{0k}\|_C \text{ for } t \in I, y \in M_{p_k}.$$

Therefore, according to (2.49), the condition (1.41) holds.

Let $y\colon I \to \mathbb{R}^n$ be an arbitrary absolutely continuous function. Set

$$\varrho_0 = \|y\|_C + \int_a^b \| \, \mathrm{d}y(\tau_0(s))\|.$$

Then by (2.50), we obtain

$$\left\| \int_a^t [p_k(y)(s) - p(y)(s)] \, \mathrm{d}s \right\| \leqslant \varrho_0 \|Q_{0k}\|_C \text{ for } t \in I.$$

According to (2.48) and (2.49) this estimate implies the condition (1.42).

According to (0.11) and (2.47),

$$\int_a^t [q_k(s) - q(s)] \, \mathrm{d}s = \int_a^t (1 - \chi_I(\tau(s)) P_k(s)[u_k(\tau(s)) - u(\tau(s))] \, \mathrm{d}s + \int_a^t Q'_{1k}(s) u(\tau(s)) \, \mathrm{d}s.$$

However,

$$\int_a^t Q'_{1k}(s) u(\tau(s)) \, \mathrm{d}s = Q_{1k}(t) u(\tau(t)) - \int_a^t Q_{1k}(s) du(\tau(s)).$$

Therefore

$$\left\| \int_a^t [q_k(s) - q(s)]\, ds \right\| \leqslant \varrho_k \|u_k - u\|_C + \varrho \|Q_{1k}\|_C,$$

where

$$\varrho = \|u\|_C + \int_a^b \|du(\tau(s))\|.$$

In view of (2.44), (2.48) and (2.49), we conclude that the condition (1.43) is fulfilled. □

The assertion of Theorem 2.5 implies the theorems of Z. Opial given in [17] for the case $\tau(t) \equiv t$.

References

[1] *N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina*: Introduction to the Theory of Functional Differential Equations. Nauka, Moscow, 1991. (In Russian.)

[2] *S.R. Bernfeld and V. Lakshmikantham*: An Introduction to Nonlinear Boundary Value Problems. Academic Press, Inc., New York and London, 1974.

[3] *E.A. Coddington and N. Levinson*: Theory of Ordinary Differential Equations. McGraw-Hill Book Company, Inc., New York Toronto London, 1955.

[4] *R. Conti*: Recent trends in the theory of boundary value problems for ordinary differential equations. Boll. Unione mat. ital. *22* (1967), 135–178.

[5] *R. Conti*: Problémes lineaires pour les equations differentielles ordinaires. Math. Nachr. *23* (1961), 161–178.

[6] *S.M. Gelashvili*: On a boundary value problem for systems of functional differential equations. Arch. Math. (Brno) *20* (1964), 157–168. (In Russian.)

[7] *J. Hale*: Theory of Functional Differential Equations. Springer-Verlag, New York Heidelberg Berlin, 1977.

[8] *G.H. Hardy, J.E. Littlewood and G. Pólya*: Inequalities. Cambridge Univ. Press, Cambridge, 1934.

[9] *P. Hartman*: Ordinary Differential Equations. John Wiley, New York, 1964.

[10] *L.V. Kantorovič, B.Z. Vulich and A.G. Pinsker*: Functional Analysis in Semiordered Spaces. GITTL, Leningrad, 1950. (In Russian.)

[11] *I. Kiguradze*: Some Singular Boundary Value Problems for Ordinary Differential Equations. Tbilisi Univ. Press, Tbilisi, 1975. (In Russian.)

[12] *I. Kiguradze*: Boundary value problems for systems of ordinary differential equations. Current problems in mathematics. Newest results, Vol. 30, 3–103, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vses. Inst. Nauchn. i Tekh. Inform., Moscow (1978). (In Russian.)

[13] *T. Kiguradze*: Some boundary value problems for systems of linear differential equations of hyperbolic type. Memoirs on Diff. Eq. and Math. Physics *1* (1994), 1–144.

[14] *J. Kurzweil*: Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. *7* (1957), 418–449.

[15] *J. Kurzweil and Z. Vorel*: Continuous dependence of solutions of differential equations on a parameter. Czechoslovak Math. J. *7* (1957), 568–583. (In Russian.)

[16] *L.A. Ljusternik and V.I. Sobolev*: Elements of Functional Analysis. Nauka, Moscow, 1965.

[17] *Z. Opial*: Continuous parameter dependence in linear systems of differential equations. J. Diff. Eqs. *3* (1967), 571–579.

[18] *Z. Opial*: Linear problems for systems of nonlinear differential equations. J. Diff. Eqs. *3* (1967), 580–594.
[19] *Š. Schwabik, M. Tvrdý and O. Vejvoda*: Differential and Integral Equations: Boundary Value Problems and Adjoints. Academia, Praha, 1979.
[20] *R.A. Tsitskishvili*: Unique solvability and correctness of a linear boundary value problem for functional-differential equations. Rep. Enlarged Sessions Sem. I.N. Vekua Inst. Appl. Math. *5* (1990), no. 3, 195–198. (In Russian.)
[21] *N.I. Vasiljev and J.A. Klokov*: Foundations of the Theory of Boundary Value Problems for Ordinary Differential Equations. Zinatne, Riga, 1978. (In Russian.)
[22] *Z. Vorel*: Continuous dependence of parameters. Nonlinear. Anal. Theory Meth. and Appl. *5* (1981), no. 4, 373–380.
[23] *W. Walter*: Differential and Integral Inequalities. Springer-Verlag, Berlin Heidelberg New York, 1970.
[24] *W.M. Whyburn*: Differential equations with general boundary conditions. Bull. Am. Math. Soc. *48* (1942), 691–704.

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