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THE ISÉKI-TYPE CHARACTERIZATION OF CERTAIN REGULAR ORDERED SEMIGROUPS

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Dedicated to Professor Josef Novák on the occasion of his 90th birthday

1. INTRODUCTION

Using Iséki characterization of regular semigroups by their one-sided ideals ([3], p. 34, exercise 11), Bedřich Pondělíček introduced and investigated in [19] a certain binary relation on the system of all closure operations on the carrier set of a semigroup and then characterized among others regular and one-sided regular semigroups. In the paper [20] the same author obtained analogous results for topological semigroups with continuous closed translations which possess namely compact topological semigroups.

The present contribution aims at characterizing regular ordered semigroups (similarly as in the paper [20]) under the assumption that left and right translations are not only isotone (which is the usual compatibility condition of orderings with binary operations in ordered semigroups) but even strongly isotone mappings. Such mappings were introduced by L. L. Esakia—[4], [5] and also used by him in connection with investigations of non-classical logics and generalizations of the Stone duality. It is to be noted that closure operations which are tools of characterizations of regularity of ordered semigroups following ideas of [19] and [20] have their origin as object of special investigations in the well known Čech's topological seminar, one of whose excellent active members was Professor Josef Novák.

As a basic fact let us recall the Iséki characterization theorem ([10], [3], p. 34, [19], p. 220):

1.1. *A semigroup $(S, \cdot)$ is regular if and only if $A \cap B = A \cdot B$ for every right ideal A and every left ideal B of $(S, \cdot)$.*

A strongly isotone mapping of a (quasi) ordered set into another one ([4], [5]) is a special case of a strong homomorphism of relational systems in the sense of papers [15]–[18] modified for ordered sets. Recall that a mapping f of an ordered set $(X, \leqq_X)$ into an ordered set $(Y, \leqq_Y)$ is said to be strongly isotone if $f(x) \leqq_Y y$ holds for $(x, y) \in X \times Y$ if and only if there exists $x' \in X$ such that $x \leqq_X x'$ and $f(x') = y$.

If $[M)_{\leqq}$ means the end of $(X, \leqq)$ (or the upper set or the dual ideal) generated by M (in particular, $[x)_{\leqq}$ is the principal end—or the principal dual ideal—or the principal upper cone generated by $x \in X$) we can easily obtain the following characterization ([5], I., §4, Proposition 4.14):

1.2. *Let $(X_i, \leqq_i)$, $i = 1, 2$ be (quasi) ordered sets, let $f\colon X_1 \to X_2$ be a mapping. Then the following conditions are equivalent:*

1° *The mapping $f\colon (X_1, \leqq_1) \to (X_2, \leqq_2)$ is strongly isotone.*

2° *For any end A of $(X_1, \leqq_1)$ the image $f(A)$ is an end of $(X_2, \leqq_2)$ and for any end B of $(X_2, \leqq_2)$ the preimage $f^{-1}(B)$ is an end of $(X_1, \leqq_1)$.*

3° *For any element $x \in X_2$ we have $(f^{-1}(x)]_{\leqq_1} = f^{-1}((x]_{\leqq_2})$.*

4° *For any element $x \in X_1$ we have $f([x)_{\leqq_1}) = [f(x))_{\leqq_2}$.*

Terminology concerning semigroups in general is taken over from the basic monograph [3] or [9]; the terms of the ordered semigroups theory can be found in [6].

2. SC-ordered semigroups

Definition. An ordered semigroup $(S, \cdot, \leqq)$ is said to satisfy the condition of a left (right) strong compatibility of the ordering with the binary operation—briefly the LSC (RSC)-condition, or $(S, \cdot, \leqq)$ is called an LSC-ordered (RSC-ordered) semigroup if for any triad of elements $a, b, c \in S$ such that $a \cdot b \leqq c$ there exists an element $c' \in S$, $c' \geqslant b$ ($c' \geqslant a$) with the property $c = a \cdot c'$ ($c = c' \cdot b$). If an ordered semigroup satisfies both conditions simultaneously we say shortly that it is an SC-ordered semigroup.

Since a semigroup $(S, \cdot)$ with an order $\leqq$ is an ordered semigroup if and only if any left and right translation of it is an isotone selfmap of the ordered set $(S, \leqq)$, it is easy to see that the following lemma holds.

Lemma 2.1. *For an ordered semigroup $(S, \cdot, \leqq)$ the following conditions are equivalent:*

1° *$(S, \cdot, \leqq)$ is an LSC-ordered (RSC-ordered) semigroup.*

2° *Any left translation $\lambda_a\colon (S, \leqq) \to (S, \leqq)$, $a \in S$ (right translation $\varrho_a\colon (S, \leqq) \to (S, \leqq)$, $a \in S$) is a strongly isotone mapping.*

3° *For any pair of elements a, $b \in S$ we have $a \cdot [b)_{\leqq} = [a \cdot b)_{\leqq}$ ($[a)_{\leqq} \cdot b = [a \cdot b)_{\leqq}$).*

4° *For any pair of elements a, $b \in S$ we have $[a \cdot b)_{\leqq} \subseteq a \cdot [b)_{\leqq}$ ($[a \cdot b)_{\leqq} \subseteq [a)_{\leqq} \cdot b$).*

Examples. **1.** If $+, \cdot, \leqq$ means the usual addition, multiplication, ordering of real numbers, respectively, then the ordered semigroups $(\mathbb{N}, +, \leqq)$, $(\mathbb{R}^+, +, \leqq)$ (where $\mathbb{N}$, $\mathbb{R}^+$ is the set of all positive integers, positive real numbers, respectively) and the group $(\mathbb{R}^+, \cdot, \leqq)$ are commutative SC-ordered semigroups, but the ordered semigroup $(\mathbb{N}, \cdot, \leqq)$ satisfies neither the LSC-condition nor the RSC-condition.

2. If $\delta \subseteq \mathbb{N} \times \mathbb{N}$ denotes the divisibility relation (i.e. $(m, n) \in \delta$ if and only if $m|n$) then $(\mathbb{N}, \cdot, \delta)$ is an SC-ordered monoid. Indeed, for arbitrary $x \in [m \cdot n)_\delta$ we have $m \cdot n | x$, i.e. $x = m \cdot n \cdot k$, thus $x = m \cdot p$, where $p = n \cdot k \in [n)_\delta$, hence $x \in m \cdot [n)_\delta$ and the monoid $(\mathbb{N}, \cdot)$ is commutative.

3. For any upper semilattice $(L, \vee)$ considered as an ordered band $(L, \vee, \leqq)$ (where $a \leqq b$ iff $a \vee b = b$) we have that this band is an SC-semigroup. Indeed, for $a, b, c \in L$ the condition $a \vee b \leqq c$ implies $a \vee c = c$, $b \leqq c$ and it remains to put $b' = c$. On the other hand if the ordered band $(L, \wedge, \leqq)$ contains more than one element, then it is not an SC-ordered semigroup. In the case considered, $(L, \wedge)$ contains also a two-element chain, say $\{a, b\} \subseteq L$, $a < b$ and $b \in [a)_{\leqq} = [a \wedge a)_{\leqq}$, $b \notin \{a\} = a \wedge [a)_{\leqq}$.

4. Consider the unit closed interval $I = [0, 1] \subseteq \mathbb{R}$ with the usual ordering $\leqq$. Let us denote $I_1 = (0, 1]$, $a \circ_1 b = \min\{a + b, 1\}$ for any pair a, $b \in I_1$, $I_2 = (0, 1] \cup \{\omega\}$, where $\omega \notin I_2$, $1 < \omega$,

$$a \circ_2 b = \begin{cases} a + b & \text{if} \quad a + b \leqq 1, \\ \omega & \text{if} \quad a + b > 1, \end{cases}$$

$a, b \in I_2$, where $x \in I_2$ implies $x + \omega = \omega + x = \omega$, $I_3 = [\frac{1}{2}, 1] \subseteq \mathbb{R}$, $a \circ_3 b = \max\{\frac{1}{2}, ab\}$, $a, b \in I_3$. Further, for any pair $a, b \in I = [0, 1]$ define

$$a \circ_4 b = a + b - ab, \ a \circ_5 b = (a + b)/(1 + ab).$$

Then $(I_j, \circ_j, \leqq)$, $j = 1, 2, 3$ are commutative linearly ordered semigroups, $(I, \circ_4, \leqq)$, $(I, \circ_5, \leqq)$ are linearly ordered monoids (cf. [6], chapt. X, §2).

We show that $(I_j, \circ_j, \leqq)$ for $j = 1, 2$ and $(I, \circ_k, \leqq)$ for $k = 4, 5$ are SC-ordered, $(I_3, \circ_3, \leqq)$ is not SC-ordered. Indeed, in the first case for $a, b, c \in (0, 1]$, $a + b \leqq c$ we put $b' = c - a$, thus $b \leqq b'$, $a + b' = c$ and similarly in the second case.

Consider $([0, 1], \circ_4, \leqq)$. Let $a, b, x \in [0, 1]$ be numbers such that $a \circ_4 b = a + b - ab \leqq x$. Then for $a < 1$ we have $b \leqq (x - a)/(1 - a) \leqq 1$ and denoting $b' = (x - a)/(1 - a)$ we have $x = a + b' - ab' = a \circ_4 b'$, where $b \leqq b'$. If $a = 1$ then $a \circ_4 b = 1 = x$, hence $[a \circ_4 b, 1] \subseteq a \circ_4 [b, 1]$ for any pair $a, b \in [0, 1]$.

In the case of the ordered monoid $([0,1], \circ_5, \leqq)$ consider $a, b, x \in [0,1]$ such that $a \circ_5 b = (a+b)/(1+ab) \leqq x$. Suppose $x = 1$. Then for $b' = 1$ we have

$$x = 1 = (a+1)/(1+a) = a \circ_5 1 = a \circ_5 b',$$

where $b' \geqslant b$. If $0 \leqq x < 1$ then $ax < 1$ and the equality $(a+b)/(1+ab) \leqq x$ implies $b \leqq (x-a)/(1-ax)$. Let us denote $b' = (x-a)/(1-ax)$. Since $-a \leqq 1$, $0 < 1-x$, we have $-a(1-x) < 1-x$, which implies $x-a < 1-ax$ and $0 \leqq b' = (x-a)/(1-ax) \leqq 1$. Further, $b'-ab'x = x-a$, thus $x = (a+b')/(1+ab') = a\circ_5 b$, hence $[a\circ_5 b, 1] \subseteq a\circ_5[b,1]$.

In the case of the ordered semigroup $(I_3, \circ_3, \leqq)$ we have $\frac{1}{2} \circ_3 \frac{1}{2} = \frac{1}{2}$ and for $c = \frac{3}{4}$ we get $c \neq \frac{1}{2} \circ_3 x$ for any number $x \in [\frac{1}{2}, 1]$.

A verification that the centralizer of an acyclic set transformation, i.e. a mapping of a set into itself (which is acyclic for the sake of simplicity) yields an example of an SC-ordered monoid needs some calculation. So we formulate that fact in a form of a separate proposition. Simultaneously we get an example of a locally finite forest with a (pointwise) SC-ordered monoid of local automorphisms (the terminology is compatible with [23] and [1].

Let $X \neq \emptyset$, let $\varphi\colon X \to X$ be an acyclic mapping, i.e. no iteration φ^n, $n \in \mathbb{N}$ possesses any fixed point. For f, $g \in \mathrm{End}(X,\varphi) = (\{\psi\colon X \to X\,;\, \varphi\circ\psi = \psi\circ\varphi\}, \circ)$ (the monoid of all mappings commuting with φ, with the binary operation which is the composition of mappings) we define $f \leqslant_\varphi g$ if $f(x) \leqslant_\varphi g(x)$ for all $x \in X$, where a, $b \in X$, $a \leqslant_\varphi b$ means that $\varphi^n(a) = b$ for some $n \in \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$.

Proposition 2.1. *Let φ be an acyclic mapping of a nonempty set X into itself. Then $(\mathrm{End}(X,\varphi), \leqslant_\varphi)$ is an SC-ordered monoid.*

Proof. First, suppose that $f, g \in \mathrm{End}(X,\varphi)$ satisfy $f \leqslant_\varphi g$, i.e. $f(x) \leqslant_\varphi g(x)$ for any $x \in X$. Let $h \in \mathrm{End}(X,\varphi)$, $x \in X$ be arbitrary elements. Then $g(x) = \varphi^{n_x}(f(x))$ for some $n_x \in \mathbb{N}_0$ and

$$(h\circ g)(x) = h(\varphi^{n_x}(f(x))) = \varphi^{n_x}(h(f(x))) = \varphi^{n_x}((h\circ f)(x)),$$

thus $(h\circ f)(x) \leqslant_\varphi (h\circ g)(x)$, i.e. $h\circ f \leqslant_\varphi h\circ g$. Further, for the element $y = h(x)$ there exists $n_y \in \mathbb{N}_0$ such that

$$g(h(x)) = \varphi^{n_y}(f(h(x))),$$

which means $(f\circ h)(x) \leqslant_\varphi (g\circ h)(x)$ and thus $f\circ h \leqslant_\varphi g\circ h$. Hence $(\mathrm{End}(X,\varphi), \leqslant_\varphi)$ is an ordered monoid. That means that for an arbitrary pair f, $g \in \mathrm{End}(X,\varphi)$ we have

$$[f)_{\leqslant_\varphi} \circ g \subseteq [f\circ g)_{\leqq_\varphi}, \quad f\circ[g)_{\leqslant_\varphi} \subseteq [f\circ g)_{\leqslant_\varphi}.$$

If we prove the inclusion $[f\circ g)_{\leqslant_\varphi} \subseteq ([f)_{\leqslant_\varphi}\circ g)\cap(f\circ[g)_{\leqslant_\varphi})$, then in view of Lemma 2.1 the proof of Proposition 2.1 will be complete.

Thus, suppose $f, g \in \mathrm{End}(X,\varphi)$. Let

$$(X,\varphi) = \sum_{\alpha\in A}(X_\alpha,\varphi_\alpha)$$

be the orbital decomposition of the mapping $\varphi\colon X \to X$, i.e. $\{X_\alpha\colon \alpha\in A\}$ is the system of all blocks of the KW-equivalence

$$\sim_\varphi = \{(x,y)\in X\times X\colon \exists(m,n)\in\mathbb{N}_0\times\mathbb{N}_0\colon \varphi^m(x)=\varphi^n(y)\}$$

(cf. [14]). Denote $\varphi_\alpha = \varphi\big|X_\alpha$ for any $\alpha\in A$. Suppose $h\in[f\circ g)_{\leqslant_\varphi}$. For any $x\in X$ there exists $n_x\in\mathbb{N}_0$ such that $h(x) = (\varphi^{n_x}\circ f\circ g)(x)$. We show that for any index $\alpha\in A$ and any pair $x,y\in X_\alpha$ we have $n_x = n_y$.

First, suppose $x,y\in X_\alpha$, $x <_\alpha y$, where $\leqslant_\alpha = \leqslant_\varphi \cap(X_\alpha\times X_\alpha)$. Then $y=\varphi^m(x)$ for a suitable integer $m\in\mathbb{N}$ and

$$\begin{aligned} h(y) &= \varphi^{n_y}(f(g(y))) = \varphi^{n_y}(f(g(\varphi^m)(x))) = \varphi^{n_y+m}((f\circ g)(x)),\\ h(y) &= h(\varphi^m(x)) = \varphi^m(h(x)) = \varphi^m(\varphi^{n_x}((f\circ g)(x)))\\ &= \varphi^{m+n_x}((f\circ g)(x)), \end{aligned}$$

i.e.

$$\varphi^{n_y+m}(z) = \varphi^{m+n_x}(z), \tag{1}$$

where $z = (f\circ g)(x)\in X$. Since the mapping φ is acyclic, (1) implies $n_x = n_y$.

Now suppose $x,y\in X_\alpha$ are arbitrary different elements. The ordered set $(X_\alpha,\leqslant_\alpha)$ is an upper semilattice. Putting $t = \sup_\alpha\{x,y\}$ we have $x\leqslant_\varphi t$, $y\leqslant_\alpha t$ and by the above consideration we obtain $n_x = n_t = n_y$.

Now for any $\alpha\in A$, $x\in X_\alpha$ let us denote $n_\alpha = n_x$ and define mappings $f_1, g_1\colon X\to X$ in the following way: For any pair $(\alpha,x)\in A\times X_\alpha$ put

$$f_1(x) = \varphi^{n_\alpha}(f(x)),\quad g_1(x) = \varphi^{n_\alpha}(g(x)).$$

Then $f\leqslant_\varphi f_1$, $g\leqslant_\varphi g_1$ and it is easy to see that $f_1, g_1\in\mathrm{End}(X,\varphi)$. Further, we have $h(x) = \varphi^{n_\alpha}((f\circ g)(x)) = \varphi^{n_\alpha}(f(g(x))) = f_1(g(x)) = (f_1\circ g)(x)$, as well as

$$\begin{aligned} h(x) &= \varphi^{n_\alpha}(f(g(x))) = (\varphi^{n_\alpha}\circ f)(g(x)) = (f\circ\varphi^{n_\alpha})(g(x))\\ &= f(\varphi^{n_\alpha}(g(x))) = f(g_1(x)) = (f\circ g_1)(x), \end{aligned}$$

hence $h = f_1\circ g = f\circ g_1$. Therefore $h\in([f)_{\leqslant_\varphi}\circ g)\cap(f\circ[g)_{\leqslant_\varphi})$. □

By [6], chapt. X, §1 an ordered semigroup $(S, \cdot, \leqq)$ is said to be

(i) sharply ordered, if $a, b, c \in S$, $a < b$ implies $a \cdot c < b \cdot c$, $c \cdot a < c \cdot b$,

(ii) strongly ordered (or briefly strong), if for any triad a, b, $c \in S$ such that $a \cdot c \leqq b \cdot c$ (or $c \cdot a \leqq c \cdot b$) we have $a \leqq b$.

It is to be noted that an ordered groupoid is sharply ordered if and only if it satisfies the weak cancellation law: $a \cdot c = b \cdot c$ (or $c \cdot a = c \cdot b$) implies either $a = b$ or a, b are $\leqq$-incomparable. A strongly ordered groupoid is cancellative and thus sharply ordered ([6], chapt. X, §1).

Proposition 2.2. *Let $(S, \cdot, \leqq)$ be an ordered semigroup which is either right cancellative RSC-ordered or left cancellative LSC-ordered. Then $(S, \cdot, \leqq)$ is strongly ordered and thus also sharply ordered.*

Proof. Let $(S, \cdot, \leqq)$ be a RSC-ordered right cancellative semigroup, a, b, $c \in S$ such elements that $a \cdot c \leqq b \cdot c$. Then $b \cdot c \in [a \cdot c)_{\leqq} = [a)_{\leqq} \cdot c$, thus $b \cdot c = a' \cdot c$ for a suitable element $a' \in S$, $a \leqq a'$. This implies $b = a'$, thus $a \leqq b$ which means that the semigroup $(S, \cdot, \leqq)$ is strongly ordered.

Similarly we get the dual assertion using the relationship $c \cdot a \leqq c \cdot b$. □

A useful tool for the investigation of the structure of regular semigroups is the so called natural partial order introduced by S. Nambooripad in [13] and independently by R. Hartwig in [7]. This was generalized by H. Mitsch in [11] for arbitrary semigroups—see also [12], where semigroups with the right compatibility of the natural partial order (called also Nambooripad-Hartwig's order in the case of regular semigroups) with multiplication are characterized—[12], Proposition 3.1. Corollary 3.2 of this Proposition says that for any commutative semigroup the natural partial order is compatible with multiplication. In a non-commutative case it is compatible if and only if the underlying regular semigroup $(S, \cdot)$ is pseudo-inverse (which means e.g. that $e \cdot S \cdot e$ is an inverse semigroup for any idempotent $e \in S$)—[13], Theorem 3.3. The strong compatibility of the Nambooripad-Hartwig's order on a commutative regular semigroup which is then completely regular and inverse, characterizes the complete simplicity of a semigroup in question.

According to [8], Theorem 5.1 the Nambooripad-Hartwig's order on a regular semigroup is determined by a certain condition which in a commutative case appears in the following form:

For x, $y \in S$, $x \leqq y$ whenever there exists $a \in S$ such that $a \cdot x^2 = x$, $a \cdot y^2 = y$ and $a \cdot x \cdot y = x$.

Recall that a primitive semigroup is a regular semigroup in which every non-zero idempotent is minimal among the non-zero elements of S and that a completely simple semigroup is a primitive semigroup without zero.

Theorem 2.1. *Let $(S, \cdot)$ be a commutative regular semigroup without zero, let $\leqq$ be the Nambooripad-Hartwig's order on S. Then the semigroup $(S, \cdot)$ is completely simple if and only if $(S, \cdot, \leqq)$ is an SC-ordered semigroup.*

Proof. Since by [13], Theorem 1.4 a regular semigroup without zero is completely simple if and only if the Nambooripad-Hartwig's order $\leqq$ is the identity relation, then in view of the well-known fact that $\leqq$ is compatible with a binary operation on an inverse semigroup (cf. also [13], Theorem 3.3) it is sufficient to show that if $(S, \cdot, \leqq)$ is SC-ordered then the order $\leqq$ is the identity relation on S. (The opposite implication is evident.)

Suppose the semigroup $(S, \cdot, \leqq)$ is SC-ordered, $x, y \in S$ being such a pair of elements that $x \leqq y$. Then for some $a \in S$ we have $a \cdot x^2 = x$, $a \cdot y^2 = y$, $a \cdot x \cdot y = x$ and

$$y \in [x)_{\leqq} = [a \cdot x \cdot y)_{\leqq} = [x \cdot a \cdot y)_{\leqq} = x \cdot [a \cdot y)_{\leqq},$$

thus $y = x \cdot s$ for some $s \in S$ such that $a \cdot y \leqq s$. Then

$$x = a \cdot x \cdot y = a \cdot x^2 \cdot s = x \cdot s = y.$$

□

By [13] a mapping f of an ordered set $(X, \leqq_X)$ into an ordered set $(Y, \leqq_Y)$ is said to be reflecting orders of for all $y, y' \in f(X)$ with $y' \leqq_Y y$ and $x \in X$ with $f(x) = y$ there exists $x' \in X$, $x' \leqq_X x$ and $f(x') = y'$. An important property of homomorphisms of regular semigroups is that they preserve and reflect Nambooripad-Hartwig orders ([13], Theorem 1.8).

If f is a surjective mapping of an ordered set $(X, \leqq_X)$ onto $(Y, \leqq_Y)$ then f reflects orders if and only if $(f(x)]_{\leqq_Y} \subseteq f(x]_{\leqq_X}$ for any $x \in X$. Indeed, if a surjective mapping $f\colon (X, \leqq_X) \to (Y, \leqq_Y)$ reflects orders, $x \in X$, $y' \in (f(x)]_{\leqq_Y}$, i.e. $y' \leqq_Y f(x)$, then $y' = f(x')$ for some $x' \in X, x' \leqq_X x$ thus $y' = f(x') \in f(x]_{\leqq_X}$. Hence $(f(x)]_{\leqq_Y} \subseteq f(x]_{\leqq_X}$. Conversely, if the last inclusion holds for any $x \in X$, $y, y' \in f(X) = Y$ with $y' \leqq_Y y$ and $x \in X$ with $f(x) = y$, then $y' \in (f(x)]_{\leqq_Y}$ which implies $y' \in f(x]_{\leqq_X}$ and thus $y' = f(x')$ for some $x' \in (x]_{\leqq_X}$, i.e. $x' \leqq_X x$. Hence f reflects orders. Since a mapping $f\colon (X, \leqq_X) \to (Y, \leqq_Y)$ is isotone if and only if $f(x]_{\leqq_X} \subseteq (f(x)]_{\leqq_Y}$ for any $x \in X$, we get in view of the above formulated facts:

Theorem 2.2. *Let $(S, \cdot)$, $(T, \cdot)$ be pseudo-inverse semigroups, $\leqq_S$, $\leqq_T$ their Nambooripad-Hartwig orders, $f\colon (S, \cdot) \to (T, \cdot)$ be a surjective homomorphism. Then f is a strongly isotone homomorphism of the ordered semigroup $(S, \cdot, \leqq_S^{-1})$ onto the ordered semigroup $(T, \cdot, \leqq_T^{-1})$.*

3. Characterization of regular SC-ordered semigroups

As in papers [19], [20] and elsewhere, by a closure operation on a set S we understand any mapping $\mathbf{U}\colon \exp S \to \exp S$ satisfying the usual axioms:

(i) $\mathbf{U}(\emptyset) = \emptyset$ (ii) $X \subseteq S \Rightarrow X \subseteq \mathbf{U}(X)$, (iii) $X \subseteq Y \subseteq S \Rightarrow \mathbf{U}(X) \subseteq \mathbf{U}(Y)$,
(iv) $\mathbf{U}^2(X) = X$, for any X, $Y \subseteq S$, (i.e. a U-space in the Čech sense).

For $x \in S$ we write briefly $\mathbf{U}(x)$ instead of $\mathbf{U}(\{x\})$. The system of all closure operations on S will be denoted by $\mathscr{C}(S)$ and its subsystem of all totally additive operations by $\mathscr{Q}(S)$. Further,

$$\mathcal{F}(\mathbf{U}) = \{X\,;\, X \subseteq S,\ \mathbf{U}(X) = X\}, \quad \mathcal{F}'(\mathbf{U}) = \mathcal{F}(\mathbf{U}) \setminus \{\emptyset\}$$

and $\mathbf{U} \leqslant \mathbf{V}$ if and only if $\mathbf{U}(X) \subseteq \mathbf{V}(X)$ for any subset $X \subseteq S$.

Let $(S, \cdot)$ be a semigroup. In accordance with [19], [20], for any non-empty subset X of S we put $\mathbf{R}(X) = X \cdot S^1$, $\mathbf{L}(X) = S^1 \cdot X$, $\mathbf{R}(\emptyset) = \mathbf{L}(\emptyset) = \emptyset$. Then $\mathbf{R}, \mathbf{L} \in \mathscr{Q}(S)$ and $\mathcal{F}'(\mathbf{R})$ ($\mathcal{F}'(\mathbf{L})$) is the system of all right (left) ideals of the semigroup $(S, \cdot)$.

Definition. ([19], II-Def. 6.) For a semigroup $(S, \cdot)$, ϱ_S stands for a binary relation on $\mathscr{C}(S)$ defined by

$$\varrho_S = \{(\mathbf{U}, \mathbf{V}) \in \mathscr{C}(S)\,;\, (A, B) \in \mathcal{F}'(\mathbf{U}) \times \mathcal{F}'(\mathbf{V}) \Rightarrow A \cap B = A \cdot B\}.$$

A survey of properties of the relation ϱ_S yields the following theorem summarizing some results of papers [19], [20]:

Theorem 3.1. ([19], L. 2, Th. 9, Th. 10, [20], L. 3). *Let $(S, \cdot)$ be a semigroup, $\mathbf{U}$, $\mathbf{U}_1$, $\mathbf{V}$, $\mathbf{V}_1 \in \mathscr{C}(S)$. Then we have:*

1° *If $\mathbf{U} \leqslant \mathbf{U}_1$, $\mathbf{V} \leqslant \mathbf{V}_1$ and $(\mathbf{U}, \mathbf{V}) \in \varrho_S$, then $(\mathbf{U}_1, \mathbf{V}_1) \in \varrho_S$.*

2° *$(\mathbf{U}, \mathbf{V}) \in \varrho_S$ if and only if $\mathbf{R} \leqslant \mathbf{U}$, $\mathbf{L} \leqslant \mathbf{V}$ and simultaneously $x \in \mathbf{U}(x) \cdot \mathbf{V}(x)$ for any $x \in S$.*

3° *If $\mathbf{U}(x) = \mathbf{U}_1(x)$, $\mathbf{V}(x) = \mathbf{V}_1(x)$ for any $x \in S$ then $(\mathbf{U}, \mathbf{V}) \in \varrho_S$ if and only if $(\mathbf{U}_1, \mathbf{V}_1) \in \varrho_S$.*

4° *A semigroup $(S, \cdot)$ is regular if and only if $(\mathbf{R}, \mathbf{L}) \in \varrho_S$.*

Now, let $(S, \cdot, \leqq)$ be a semigroup with an ordering $\leqq$ on its carrier S. For any non-empty subset X of S denote by $\mathbf{C}(X) = [X)_{\leqq}$ (the end or cone generated by the set X within the ordered set $(S, \leqq)$, $\mathbf{C}(\emptyset) = \emptyset$). Evidently by $\mathbf{C} \in \mathscr{Q}(S)$. Further we define mappings $\overline{\mathbf{R}}, \overline{\mathbf{L}}\colon \exp S \to \exp S$ by $\overline{\mathbf{R}}(X) = \mathbf{R}(X) \cup \mathbf{C}(X)$, $\overline{\mathbf{L}}(X) = \mathbf{L}(X) \cup \mathbf{C}(X)$ for any subset X of S.

Lemma 3.1. *Let $(S, \cdot, \leqq)$ be a semigroup with an ordering $\leqq$ on S. Then $\bar{\mathbf{R}} = \mathbf{R}$ if and only if $\mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R} = \mathbf{R}$.*

Proof. Suppose $\bar{\mathbf{R}} = \mathbf{R}$, $X \subseteq S$. Since $\mathbf{R} = \mathbf{R}^2 = \mathbf{R} \circ \bar{\mathbf{R}} = \bar{\mathbf{R}} \circ \mathbf{R}$, we have

$$\mathbf{R}(X) = \mathbf{R}(\bar{\mathbf{R}}(X)) = \mathbf{R}(\mathbf{R}(X) \cup \mathbf{C}(X)) = \mathbf{R}(X) \cup \mathbf{R} \circ \mathbf{C}(X),$$

which implies $\mathbf{R} \circ \mathbf{C} \leqslant \mathbf{R}$ and by virtue of $\mathbf{R} \leqslant \mathbf{R} \circ \mathbf{C}$ we get $\mathbf{R} \circ \mathbf{C} = \mathbf{R}$. Similarly,

$$\mathbf{R}(X) = \bar{\mathbf{R}}(\mathbf{R}(X)) = \mathbf{R}^2(X) \cup \mathbf{C} \circ \mathbf{R}(X) = \mathbf{R}(X) \cup \mathbf{C} \circ \mathbf{R}(X),$$

thus $\mathbf{C} \circ \mathbf{R} \leqslant \mathbf{R}$, consequently $\mathbf{C} \circ \mathbf{R} = \mathbf{R}$.

Now suppose $\mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R} = \mathbf{R}$, $X \subseteq S$. Then

$$\mathbf{C}(X) \subseteq \mathbf{R} \circ \mathbf{C}(X) = \mathbf{R}(X) \quad \text{and} \quad \bar{\mathbf{R}}(X) = \mathbf{R}(X) \cup \mathbf{C}(X) \subseteq \mathbf{R}(X),$$

hence $\bar{\mathbf{R}} = \mathbf{R}$. □

It is to be noted that the just proved assertion holds for an arbitrary reflexive binary relation r on S which need not be an ordering. If $(S, \cdot, \leqq)$ is an ordered monoid satisfying the LSC-condition then $\bar{\mathbf{R}} = \mathbf{R}$. Indeed, for any $X \subseteq S$, $X \neq \emptyset$ and $x \in \mathbf{C}(X)$ there exists $y \in X$ such that $y \leqq x$. Clearly, $y \cdot 1 \leqq x$, thus there exists $z \in S$ such that $x = y \cdot z \in X \cdot S = \mathbf{R}(X)$. Then we have $\mathbf{C}(X) \subseteq \mathbf{R}(X)$, hence $\bar{\mathbf{R}}(X) = \mathbf{R}(X) \cup \mathbf{C}(X) = \mathbf{R}(X)$.

Now we consider ordered semigroups which need not contain identity elements.

Lemma 3.2. *If $(S, \cdot, \leqq)$ is an RSC-ordered semigroup, then $\mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R} \in \mathscr{Q}(S)$.*

Proof. If $x \in S$, then

$$\begin{aligned}\mathbf{R} \circ \mathbf{C}(x) &= [x)_{\leqq} \cdot S^1 = \bigcup_{s \in S} [x)_{\leqq} \cdot s \cup [x)_{\leqq} = \bigcup_{s \in S} [x \cdot s)_{\leqq} \cup [x)_{\leqq} \\ &= \Big[\bigcup_{s \in S} x \cdot \{s\}\Big)_{\leqq} \cup [x)_{\leqq} = [x \cdot S)_{\leqq} \cup [x)_{\leqq} = [x \cdot S^1)_{\leqq} = \mathbf{C} \circ \mathbf{R}(x).\end{aligned}$$

Since $\mathbf{R}$, $\mathbf{C} \in \mathscr{Q}(S)$, we have $\mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R}$ and from the evident fact that the composition of two commuting idempotent mappings is an idempotent mapping we get $\mathbf{C} \circ \mathbf{R} \in \mathscr{Q}(S)$. □

Let $(S, \cdot, \leqq)$ be an ordered semigroup without the identity element. The ordering $\leqq$ can be extended onto $(S^1, \cdot)$ in two quite natural ways such that $(S^1, \cdot)$ becomes an ordered monoid:

$(S^1, \leqq) = \{1\} + (S, \leqq)$ (the cardinal sum),

$(S^1, \leqq) = \{1\} \oplus (S, \leqq)$ (the ordinal sum).

In the first case (when $(S^1, \cdot, \leqq)$ is called the cardinal unit extension of $(S, \cdot, \leqq)$) the element 1 is incomparable with any other element of S, in the second case (when $(S^1, \cdot, \leqq)$ is called the ordinal unit extension of $(S, \cdot, \leqq)$), 1 is the least element of the poset $(S^1, \leqq)$; in this case the monoid $(S^1, \cdot, \leqq)$ is positive ordered, i.e. it is ordered and for any pair a, $b \in S^1$ we have $a \leqq a \cdot b$, $b \leqq a \cdot b$ (cf. [6], chap. X, §1, p. 217). Another unit extension preserving the positivity and the negativity in $(S, \cdot, \leqq)$ under the assumption of equality of one-sided cones is described in [6], chap. X, §3 (Theorem 3).

In the next part of this paragraph, we shall suppose that $(S^1, \cdot, \leqq)$ is one on the above mentioned extensions of $(S, \cdot, \leqq)$. Of course, if the cardinal unit extension of an ordered semigroup satisfies either LSC-condition or RSC-condition, then it is an antichain.

Lemma 3.3. *Let $(S, \cdot, \leqq)$ be an SC-ordered semigroup. Then*

$$\bar{\mathbf{R}} = \mathbf{R},\ \bar{\mathbf{L}} = \mathbf{L} \quad \textit{and} \quad \mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R} = \mathbf{R},\ \mathbf{L} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{L} = \mathbf{L}.$$

Proof. For arbitrary $x \in S$ we have by Lemma 3.2

$$\mathbf{C} \circ \mathbf{R}(x) = [x \cdot S^1)_{\leqq} = x \cdot [S^1)_{\leqq} = x \cdot S^1 = \mathbf{R}(x) \subseteq \mathbf{R} \circ \mathbf{C}(x) = \mathbf{C} \circ \mathbf{R}(x),$$

thus $\mathbf{C} \circ \mathbf{R} = \mathbf{R} \circ \mathbf{C} = \mathbf{R}$ and by Lemma 3.1 we have $\bar{\mathbf{R}} = \mathbf{R}$. Similarly we get the dual assertions. □

It follows from the above considerations that characterizations of regularity of SC-ordered semigroups are reduced to the case contained in Theorem 3.1 (i.e. Theorem 10, [19]). The substantial difference between regularity of topological semigroups with continuous closed translations ([20]) and SC-ordered semigroups consists in the fact that in the first case $\mathbf{R} < \bar{\mathbf{R}}$, $\mathbf{L} < \bar{\mathbf{L}}$ in spite of equalities $\mathbf{R}(x) = \bar{\mathbf{R}}(x)$, $\mathbf{L}(x) = \bar{\mathbf{L}}(x)$ holding in both cases. If $(S, \cdot)$ is a monoid or $(S, \cdot, \leqq)$ is an SC-ordered monoid, we have that $(S, \cdot)$ is regular if and only if $\mathbf{H}(x) = \mathbf{R}(x) \cdot \mathbf{L}(x)$ $(= \bar{\mathbf{R}}(x) \cdot \bar{\mathbf{L}}(x))$ for any $x \in S$, where $\mathbf{H}(X) = \mathbf{R}(X) \cap \mathbf{L}(X)$ for each subset X of S.

Indeed, according to Iséki theorem 1.1 we have $A \cap B = A \cdot B$ for any pair $(A, B) \in \mathcal{F}'(\mathbf{R}) \times \mathcal{F}'(\mathbf{L})$. In particular, by virtue of Lemma 3.3 we have for any element $x \in S$:

$$\mathbf{H}(x) = \mathbf{R}(x) \cap \mathbf{L}(x) = \mathbf{R}(x) \cdot \mathbf{L}(x) = \bar{\mathbf{R}}(x) \cdot \bar{\mathbf{L}}(x).$$

On the other hand, let $a \in S$ be an arbitrary element. Since

$$a \in a \cdot S^1 \cap S^1 \cdot a = \mathbf{R}(a) \cap \mathbf{L}(a) = \mathbf{H}(a) = \\ = \mathbf{R}(a) \cdot \mathbf{L}(a) = a \cdot S^1 \cdot S^1 \cdot a = a \cdot S^1 \cdot a = a \cdot S \cdot a,$$

the element a is a regular element of the monoid $(S, \cdot)$.

Example 5. Consider the set $\mathbb{Z}[i]$ of all Gauss integers and define a mapping $\varphi\colon \mathbb{Z}[i] \to \mathbb{Z}[i]$ by

$$\varphi(a + b\,i) = \begin{cases} a+1 & \text{if} \quad b = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since $(\mathbb{Z}[i], \varphi)$ is a connected unar which is a line with short tails (in the terminology of [22], [1]), its endomorphism monoid $\mathrm{End}(\mathbb{Z}[i], \varphi)$—which is not a group—is regular by [22], Theorem 1 (cf. also [2], Theorem 1) and by Proposition 2.1 of this paper $(\mathrm{End}(\mathbb{Z}[i], \varphi), \leqslant_\varphi)$ is SC-ordered. Evidently the monoid $\mathrm{End}(\mathbb{Z}[i], \varphi)$ (i.e. a centralizer of φ within $(\mathbb{Z}[i]^{\mathbb{Z}[i]}, \circ)$) is noncommutative. Let $k \in \mathbb{Z}$, $k > 1$ be chosen arbitrary but fixed and put $f = \varphi^k$.

First, we define a mapping $g \in \mathrm{End}(\mathbb{Z}[i], \varphi)$ which will be a regular conjugate of f, i.e. such that (f, g) forms a regular pair ([21], p. 264), which means $f \circ g \circ f = f$, $g \circ f \circ g = g$. Denote $M = \{a + b\,i \in \mathbb{Z}[i];\, b \neq 0\}$, $M_0 = M \cup \{0\}$. Let $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ be the successor function, i.e. $\sigma(p) = p + 1$. If S_0 is a set formed by mappings $\xi\colon \mathbb{Z}[i] \to \mathbb{Z}[i]$ such that $\xi\big|\mathbb{Z} = \mathrm{id}_{\mathbb{Z}}$ and $\xi\big|M_0$ is a selfmapping of M_0 with the fixed point 0 and S_1 is formed by mappings $\psi\colon \mathbb{Z}[i] \to \mathbb{Z}[i]$ such that $\psi|\mathbb{Z} = \sigma^n$ for a suitable $n \in \mathbb{Z} \setminus \{0\}$ and $\psi\big|M_0$ is a constant mapping with the value n, we have

$$\mathrm{End}(\mathbb{Z}[i], \varphi) = S_0 \cup S_1, \quad S_0 \cap S_1 = \emptyset.$$

Now we show that $S_1 \subseteq \mathrm{End}(\mathbb{Z}[i], f) = \mathrm{End}(\mathbb{Z}[i], \varphi^k)$. Suppose $\psi \in S_1$ and $p \in \mathbb{Z} \setminus \{0\}$ such that $\psi\big|\mathbb{Z} = \sigma^p$, $\psi(M) = \{p\}$. Let $z = a + bi \in \mathbb{Z}[i]$. If $b = 0$ then

$$(f \circ \psi)(z) = \varphi^k(\sigma^p(a)) = \sigma^{p+k}(a) = a + p + k = \sigma^p(a + k) = \psi(\varphi^k(z)) \\ = (\psi \circ f)(z),$$

if $b \neq 0$ then

$$(f \circ \psi)(z) = \varphi^k(p) = p + k = \sigma^p(k) = \psi(k) = \psi(\varphi^k(a + bi)) \\ = (\psi \circ f)(z).$$

Since $f \circ S_0 = \varphi^k \circ S_0 = \{\varphi^k\} = S_0 \circ \varphi^k = S_0 \circ f$, we have

$$\mathbf{R}(f) = f \circ (S_0 \cup S_1) = (f \circ S_0) \cup (f \circ S_1) = (S_0 \circ f) \cup (S_1 \circ f) = (S_0 \cup S_1) \circ f$$
$$= \mathbf{L}(f),$$

and thus

$$\mathbf{H}(f) = \mathbf{R}(f) \cap \mathbf{L}(f) = \mathbf{R}(f) = \mathbf{L}(f). \tag{2}$$

By the definition of an ideal we have $\mathbf{R}(f) \circ \mathbf{R}(f) \subseteq \mathbf{R}(f)$.

On the other hand, since $\varphi^{2k} \in \varphi^{2k} \circ S_1 \circ S_1$, $\varphi^{2k} \circ S_1 \subseteq \varphi^{2k} \circ S_1 \circ S_1$, we get

$$\mathbf{R}(f) \circ \mathbf{R}(f) = (\{\varphi^k\} \cup (\varphi^k \circ S_1)) \circ (\{\varphi^k\} \cup (\varphi^k \circ S_1))$$
$$= \{\varphi^{2k}\} \cup (\varphi^{2k} \circ S_1) \cup (\varphi^{2k} \circ S_1 \circ S_1) = \varphi^{2k} \circ S_1 \circ S_1.$$

Suppose $g \in \mathbf{R}(f)$. If $g = \varphi^k$, then $g \in \varphi^{2k} \circ S_1$. Indeed, if $\psi \in S_1$ is such a mapping that $\psi|\mathbb{Z} = \sigma^{-k}$, then $g = \varphi^k = \varphi^{2k} \circ \psi \in \varphi^{2k} \circ S_1 \subseteq \mathbf{R}(f) \circ \mathbf{R}(f)$. If $g = \varphi^{2k} \circ \psi_1$, where $\psi_1 \in S_1$, $\psi_1|\mathbb{Z} = \sigma^n$, then $\varphi_1^{2k} \circ \psi_1 = \varphi^{2k} \circ \xi \in \varphi^{2k} \circ S_1$, where $\xi \colon \mathbb{Z}[i] \to \mathbb{Z}[i]$ satisfies $\xi|\mathbb{Z} = \sigma^{n-k}$, thus

$$\mathbf{R}(f) \subseteq \varphi^{2k} \circ S_1 \circ S_1 = \mathbf{R}(f) \circ \mathbf{R}(f),$$

therefore—with respect to (2)—we have

$$\mathbf{R}(f) \circ \mathbf{L}(f) = \mathbf{R}(f) \circ \mathbf{R}(f) = \mathbf{R}(f) = \mathbf{H}(f).$$

In what follows, we are going to show that within the class of positive ordered semigroups the regularity of an SC-ordered semigroup can be established with the use of the end operation $\mathbf{C}$ only. In order to prove that, we formulate some characterizations of semigroups belonging to the class of positive SC-ordered semigroups.

Recall that an element a of an ordered semigroup $(S, \cdot, \leqq)$ is said to be positive if $x \leqq a \cdot x$, $x \leqq x \cdot a$ for all $x \in S$, and $(S, \cdot, \leqq)$ is called positive ordered if all its elements are positive ([6], chap. X, §1). An ordered semigroup $(S, \cdot, \leqq)$ is said to be naturally ordered if it is positive ordered and for any pair $a, b \in S$ such that $a < b$ there exist $x, y \in S$ with the property

$$a \cdot x = y \cdot a = b.$$

Lemma 3.4. *If $(S, \cdot, \leqq)$ is a positive ordered semigroup, then $\overline{\mathbf{R}} = \mathbf{R}$ if and only if $\mathbf{R} = \mathbf{C}$.*

Proof. Suppose $\overline{\mathbf{R}} = \mathbf{R}$. Then $\mathbf{C} \leqslant \mathbf{R} \circ \mathbf{C}$ and by Lemma 3.3, $\mathbf{R} \circ \mathbf{C} = \mathbf{R}$, thus $\mathbf{C} \leqslant \mathbf{R}$. For any $y \in \mathbf{R}(x) = x \cdot S^1$, where $x \in S$, we have $y = x \cdot s$ with a suitable $s \in S^1$, hence $x \leqq x \cdot s = y$, consequently $\mathbf{R}(x) \subseteq \mathbf{C}(x)$ and we have $\mathbf{R} = \mathbf{C}$.

If $\mathbf{R} = \mathbf{C}$, then $\mathbf{R} = \mathbf{R}^2 = \mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R}$ and again by Lemma 3.3 we get $\overline{\mathbf{R}} = \mathbf{R}$. □

Theorem 3.2. *Let $(S, \cdot, \leqq)$ be a positive ordered semigroup. Then the following conditions are equivalent:*

1° *$(S, \cdot, \leqq)$ is naturally ordered.*

2° *$(S, \cdot, \leqq)$ is an SC-ordered semigroup.*

3° $\overline{\mathbf{R}} = \mathbf{R}$, $\overline{\mathbf{L}} = \mathbf{L}$.

4° $\mathbf{R} = \mathbf{C} = \mathbf{L}$.

5° $\mathbf{R} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{R} = \mathbf{R}$, $\mathbf{L} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{L} = \mathbf{L}$.

Proof. By Lemma 3.4 and by the assertion which is dual to it we have that 3° is equivalent to 4°. By Lemma 3.1 (and by assertion dual to it) we have that 3° is equivalent to 5°, thus 4° is equivalent to 5°. (The implication 4° ⇒ 5° is trivial.)

1° ⇒ 2°: Suppose $a, b, c \in S$ are elements such that $a \cdot b \leqq c$. If $a \cdot b = c$, we put $c' = a$, $c'' = b$ and thus $c' \cdot b = a \cdot c'' = c$. Suppose $a \cdot b < c$. Then there exist $x, y \in S$ such that $x \cdot a \cdot b = a \cdot b \cdot y = c$. Denote $c' = x \cdot a$, $c'' = b \cdot y$. Then $c' \cdot b = a \cdot c'' = c$ and since the semigroup $(S, \cdot, \leqq)$ is positive ordered, we have

$$a \leqq x \cdot a = c', \quad b \leqq b \cdot y = c'',$$

consequently, this semigroup is SC-ordered.

The implication 2° ⇒ 3° is contained in Lemma 3.3.

3° ⇒ 1°: Suppose $a, b \in S$, $a < b$. Since 3° is equivalent to 4°, we have

$$b \in [a)_{\leqq} = a \cdot S^1 = S^1 \cdot a,$$

thus $a \cdot x = y \cdot a = b$ for some pair $x, y \in S$, hence 1° holds. □

Corollary. *If a positive ordered semigroup satisfies either $\overline{\mathbf{R}} = \mathbf{R}$ or $\overline{\mathbf{L}} = \mathbf{L}$, then it is a double semigroup (i.e. its any one-sides ideal is two-sided), in particular, a positive SC-ordered semigroup is double.*

By a natural partial ordering on a band $(S, \cdot)$ we understand—in accordance with [3], §1.8—an ordering $\leqslant$ defined by $a \leqslant b$ if and only if $a \cdot b = b \cdot a = a$. Thus a commutative band is a lower semilattice with respect to $\leqslant$—[3], Theorem 1.12.

Theorem 3.3. *Let $(S, \cdot, \leqq)$ be a naturally ordered semigroup or a positive SC-ordered semigroup. Then the following conditions are equivalent:*

1° $(S,\cdot)$ *is regular.*

2° $\mathbf{C}\varrho_S\mathbf{C}$.

3° $(S,\cdot)$ *is a commutative band, i.e. a lower semilattice* $(S,\leqslant)$, *where* $\leqslant\,=\,\leqq^{-1}$.

P r o o f. The implication 1° ⇒ 2° follows from Theorem 3.1, 4° and Theorem 3.2.

2° ⇒ 3°: The relation $\mathbf{C}\varrho_S\mathbf{C}$ means $X\cap Y=X\cdot Y$ for any pair of dual ideals (or ends) X, Y of the poset $(S,\leqq)$. Since $(S,\cdot,\leqq)$ is SC-ordered, we have

$$\mathbf{C}(a)\cdot\mathbf{C}(b)=\mathbf{C}(a\cdot b)\quad\text{for any pair } a,b\in S. \tag{3}$$

Indeed,

$$\mathbf{C}(a)\cdot\mathbf{C}(b)=[a)_{\leqq}\cdot[b)_{\leqq}=\Big(\bigcup_{a\leqq s}\{s\}\Big)\cdot[b)_{\leqq}=\bigcup_{a\leqq s}s\cdot[b)_{\leqq}$$
$$=\bigcup_{a\leqq s}[s\cdot b)_{\leqq}=\Big[\Big(\bigcup_{a\leqq s}\{s\}\Big)\cdot b\Big)_{\leqq}=[\,[a)_{\leqq}\cdot b)_{\leqq}=\mathbf{C}^2(a\cdot b)=\mathbf{C}(a\cdot b).$$

Then for any $a\in S$ we have

$$\mathbf{C}(a)=\mathbf{C}(a)\cap\mathbf{C}(a)=\mathbf{C}(a)\cdot\mathbf{C}(a)=\mathbf{C}(a^2),$$

thus $a^2=a$, i,e, $(S,\cdot)$ is a band. Since $(S,\cdot,\leqq)$ is positive ordered, for any pair $a,b\in S$ we have

$$a\cdot b\leqq(b\cdot a)\cdot(b\cdot a)=b\cdot a\leqq(a\cdot b)\cdot(a\cdot b)=a\cdot b,$$

hence $a\cdot b=b\cdot a$.

Finally, for $a,b\in S$ we have $a\leqq b$ if and only if either $a=b$ or $a<b$ and by (3)

$$\mathbf{C}(b)=\mathbf{C}(a)\cap\mathbf{C}(b)=\mathbf{C}(a)\cdot\mathbf{C}(b)=\mathbf{C}(a\cdot b)=\mathbf{C}(b\cdot a),$$

thus $a\cdot b=b\cdot a=b$, i.e. $b\leqslant a$ and we get 3°.

Since the implication 3° ⇒ 1° is evident, the proof is complete. □

According to [19], Lemma 6 we get

Corollary 1. *If* $(S,\cdot,\leqq)$ *is a positive SC-ordered semigroup satisfying the condition* $\mathbf{C}\varrho_S\mathbf{C}$, *then* $(S,\cdot)$ *is a semilattice of groups.*

From Theorems 3.1, 3.2, 3.3 and in view of the evident fact that an upper semilattice is positive SC-ordered with respect to binary join operator (Example 3) we also obtain:

Corollary 2. *For an ordered semigroup $(S,\cdot,\leqq)$ the following conditions are equivalent:*

1° $(S,\cdot,\leqq)$ *is a regular naturally ordered semigroup.*

2° $(S,\cdot,\leqq)$ *is a positive SC-ordered semigroup satisfying the condition* $\mathbf{C}\varrho_S\mathbf{C}$.

3° $(S,\cdot,\leqq)$ *is a positive SC-ordered semigroup satisfying the condition* $\mathbf{R}\varrho_S\mathbf{L}$.

4° $(S,\cdot)$ *is a commutative band, i.e. a lower semilattice* $(S,\leqslant)$*, where* $\leqslant\; = \;\leqq^{-1}$.

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