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RELATIONAL STRUCTURES AND DEPENDENCE SPACES

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Dedicated to Professor Josef Novák on the occasion of his 90^{th} birthday

The present paper is an attempt to connect the theory of pseudodimension of relational structures with the theory of dependence spaces. The concept of pseudodimension was introduced in [5] for ordered sets as a generalization of the dimension and especially of the α-dimension as follows. Let G be an ordered set, L a chain of type α, $|L| \geqslant 2$, let $(f_t\,;\, t \in T)$ be a system of mappings of G into L such that for any $x, y \in G$ the following condition is satisfied:

$$x \leqslant y \Longleftrightarrow f_t(x) \leqslant f_t(y) \quad \text{for all} \quad t \in T.$$

Then $(f_t\,;\, t \in T)$ is called an α-realizer of G. Furthermore, put

$$\alpha\text{-pdim}\, G = \min\{|T|\,;\, (f_t\,;\, t \in T) \text{ is an } \alpha\text{-realizer of } G\};$$

this cardinal is called the α-pseudodimension of G.

While α-dim G need not exist, in [5] it is shown that α-pdim G always exists. The theory of **2**-pseudodimension is developed in [6]. In the present paper we extend the concept of α-pseudodimension to arbitrary relational structures; consequently, α is a type of some (fixed) relational structure. Another generalization of dimension of ordered sets can be found in [7].

The second outcome of our paper is the theory of dependence spaces. This concept has appeared in the theory of information systems ([10], [11], [12]) and in mathematical linguistics in connection with constructions of grammars ([9]) in a natural way. We use dependence spaces introduced in [9] that facilitate the investigation of infinite sets.

When investigating these problems the authors remember with gratitude the work of Professor Josef Novák. His outstanding scientific work (cf., e.g., [4]) has inspired them to study relational structures and the present paper may be regarded as a result of his activity. This paper is dedicated to him on the occasion of his 90$^{\text{th}}$ birthday.

1. Basic notions

All sets in this paper are assumed non-empty, if the contrary is not stated. If G is a set, then $|G|$ denotes the cardinality of G and $\mathbf{B}(G)$ is the power set of G, i.e. $\mathbf{B}(G) = \{H;\ H \subseteq G\}$. If G, H are sets, then H^G denotes the set of all mappings $f\colon G \to H$.

Let G be a set and $X \subseteq G \times G$ a binary relation on G. The pair $\mathbf{G} = (G, X)$ will be called a *relational structure*; the set G is said to be the *carrier* of $\mathbf{G}$ and the set X the *relation* of $\mathbf{G}$. Sometimes we use the symbols $G = \mathscr{C}(\mathbf{G})$, $X = \mathscr{R}(\mathbf{G})$ for the carrier and the relation of $\mathbf{G}$.

If $\mathbf{G} = (G, X)$, $\mathbf{H} = (H, Y)$ are relation structures and $h \in H^G$, then h is called a *homomorphism* of $\mathbf{G}$ into $\mathbf{H}$ iff for any $x, y \in G$ the following condition is satisfied:

$$(x, y) \in X \Rightarrow (h(x), h(y)) \in Y.$$

The set of all homomorphisms of $\mathbf{G}$ into $\mathbf{H}$ will be denoted by $\operatorname{Hom}(\mathbf{G}, \mathbf{H})$. A homomorphism $h \in \operatorname{Hom}(\mathbf{G}, \mathbf{H})$ will be called *strong*, iff for any $x, y \in G$ the condition

$$(x, y) \in X \Longleftrightarrow (h(x), h(y)) \in Y$$

is satisfied.

An injective strong homomorphism is called an *embedding* of $\mathbf{G}$ into $\mathbf{H}$. A bijective strong homomorphism of $\mathbf{G}$ onto $\mathbf{H}$ is clearly an *isomorphism* of $\mathbf{G}$ onto $\mathbf{H}$.

A relational structure $\mathbf{G}$ is called *discrete* if $\mathscr{R}(\mathbf{G}) = \emptyset$.

Let $\mathbf{G} = (G, X)$, $\mathbf{H} = (H, Y)$ be relational structures. The *power* $\mathbf{H}^{\mathbf{G}}$ is a relational structure where $\mathscr{C}(\mathbf{H}^{\mathbf{G}}) = \operatorname{Hom}(\mathbf{G}, \mathbf{H})$ and

$$\mathscr{R}(\mathbf{H}^{\mathbf{G}}) = \{(h_1, h_2) \in \mathscr{C}(\mathbf{H}^{\mathbf{G}}) \times \mathscr{C}(\mathbf{H}^{\mathbf{G}});\ (h_1(x), h_2(x)) \in \mathscr{R}(\mathbf{H}) \text{ for any } x \in G\}.$$

If the structure $\mathbf{G}$ is discrete then clearly $\mathscr{C}(\mathbf{H}^{\mathbf{G}}) = H^G$. The arithmetics of relational structures is developed in [1]; for general relational systems see, e.g. [13].

A binary relation X on a set G is called a *preorder* if it is reflexive and transitive; the relational structure $\mathbf{G} = (G, X)$ is referred to as a *preordered set*. An antisymmetric preorder is an *order*; if X is an order on G, then $\mathbf{G} = (G, X)$ is said to be an *ordered set*. Of course, an order on a set G will be denoted by the standard symbol $\leqslant$. An ordered set $\mathbf{G} = (G, \leqslant)$ is a *chain* (or *linearly ordered set*) if $x \leqslant y$ or $y \leqslant x$ for any $x, y \in G$; it is an *antichain* if for any $x, y \in G$, $x \leqslant y$ implies $x = y$. A symmetric preorder on a set G is an *equivalence relation* on G.

Let B be a set and $\leqslant$ an order on B such that $\mathbf{B} = (B, \leqslant)$ is a complete lattice. Let K be an equivalence relation on B such that every K-block has a greatest element. Then the triple $(B, \leqslant, K)$ is called a *dependence space* ([9], [10], [11], [12]). Such a dependence space will be referred to as *natural* if there exists a set M such that $B \subseteq \mathbf{B}(M)$ and the relation $\leqslant$ on B coincides with inclusion.

Let $(B, \leqslant, K)$ be a dependence space and let $x \in B$ be an element. An element $x' \in B$ is called a *K-reduct* of x if $x' \leqslant x$ and x' is a minimal element in the K-block containing x ([9], [10], [11], [12]). A K-reduct of x need not exist; of course, if $(B, \leqslant)$ satisfies the descending chain condition (especially, if the set B is finite) then the K-reduct exists for any $x \in B$.

Let $(B, \subseteq, K)$ be a natural dependence space and suppose $x \in B$. We put

$$c_K(x) = \min\{|z|;\ (z, x) \in K, z \subseteq x\};$$

this cardinal will be called the *K-character* of x. While a K-reduct of x need not exist, the K-character of x is always defined. Clearly, if $(B, \subseteq, K)$ is a natural dependence space and if $(B, \subseteq)$ satisfies the descending chain condition then any element $x \in B$ has a K-reduct x' such that $c_K(x) = |x'|$.

2. Natural dependence space generated by relational structure

Let G be a set, $\mathbf{L} = (L, H)$ a relational structure and suppose $|G| \geqslant 2$, $|L| \geqslant 2$. If $x, y \in G$, $f \in L^G$ then we put $((x, y);\ f) \in R$ iff $(f(x), f(y)) \in H$.

For any $X \subseteq G \times G$ put

$$\begin{aligned} S(X) &= \{f \in L^G;\ ((x, y); f) \in R \text{ for any } (x, y) \in X\} \\ &= \{f \in L^G;\ (f(x), f(y)) \in H \text{ for any } (x, y) \in X\}. \end{aligned}$$

In other words, we set $S(X) = \mathscr{C}(\mathbf{L}^{\mathbf{G}})$ where $\mathbf{G} = (G, X)$. Furthermore, for any $Y \subseteq L^G$ put

$$\begin{aligned} T(Y) &= \{(x, y) \in G \times G;\ ((x, y); f) \in R \text{ for any } f \in Y\} \\ &= \{(x, y) \in G \times G;\ (f(x), f(y)) \in H \text{ for any } f \in Y\}. \end{aligned}$$

Clearly, the pair of mappings (S, T) forms a Galois connection ([2], p. 124) between $(\mathbf{B}(G \times G), \subseteq)$ and $(\mathbf{B}(L^G), \subseteq)$. Thus $T \circ S$ is a closure operator on $\mathbf{B}(G \times G)$ and $S \circ T$ is a closure operator on $\mathbf{B}(L^G)$.

If $Y_1 \in \mathbf{B}(L^G)$, $Y_2 \in \mathbf{B}(L^G)$ then we set $(Y_1, Y_2) \in K_{\mathbf{L}}$ iff $T(Y_1) = T(Y_2)$. Then $K_{\mathbf{L}}$ is an equivalence relation on $\mathbf{B}(L^G)$.

Theorem 2.1. $(\mathbf{B}(L^G), \subseteq, K_{\mathbf{L}})$ *is a natural dependence space.*

Proof. $(\mathbf{B}(L^G), \subseteq)$ is a complete lattice and $K_{\mathbf{L}}$ is an equivalence relation on $\mathbf{B}(L^G)$. Let C be any $K_{\mathbf{L}}$-block. Let us choose a set $Y \in C$; we show that $S(T(Y))$ is the greatest element in C. As $T(S(T(Y))) = T(Y)$, we obtain $(Y, S(T(Y))) \in K_{\mathbf{L}}$, i.e. $S(T(Y)) \in C$. Let $Y_1 \in C$ be any set; then $(Y_1, Y) \in K_{\mathbf{L}}$, i.e. $S(T(Y_1)) = S(T(Y))$ and hence $Y_1 \subseteq S(T(Y_1)) = S(T(Y))$. □

Note that we have proved the following assertion:

Corollary. *For any* $Y \in \mathbf{B}(L^G)$ *the set* $S(T(Y))$ *is the greatest element in the* $K_{\mathbf{L}}$*-block containing* Y.

In the following lemmas and theorems we assume that G is a set, $\mathbf{L} = (L, H)$ is a relational structure and $|G| \geqslant 2$, $|L| \geqslant 2$.

Lemma 2.1. *If the relation* H *is reflexive (symmetric, transitive), then for any* $Y \subseteq L^G$ *the relation* $T(Y)$ *on* G *is reflexive (symmetric, transitive).*

Proof. Let H be reflexive and $Y \subseteq L^G$. If $x \in G$ then $(f(x), f(x)) \in H$ for any $f \in Y$ so that $(x, x) \in T(Y)$ and $T(Y)$ is reflexive.

Let H be symmetric and $Y \subseteq L^G$. If $x, y \in G$, $(x, y) \in T(Y)$, then $(f(x), f(y)) \in H$ for any $f \in Y$, thus $(f(y), f(x)) \in H$ for any $f \in Y$ and $(y, x) \in T(Y)$. Hence $T(Y)$ is symmetric.

Let H be transitive, let $Y \subseteq L^G$. If $x, y, z \in G$, $(x, y) \in T(Y)$, $(y, z) \in T(Y)$, then $(f(x), f(y)) \in H$, $(f(y), f(z)) \in H$ for any $f \in Y$, thus $(f(x), f(z)) \in H$ for any $f \in Y$ and $(x, z) \in T(Y)$. Therefore $T(Y)$ is transitive. □

Corollary. *If* H *is a preorder on* L *then* $T(Y)$ *is a preorder on* G *for any* $Y \subseteq L^G$. *If* H *is an equivalence relation on* L *then* $T(Y)$ *is an equivalence relation on* G *for any* $Y \subseteq L^G$.

Lemma 2.2. *If the relation* H *is antisymmetric and if* $Y \subseteq L^G$ *contains at least one injective mapping then the relation* $T(Y)$ *on* G *is antisymmetric.*

Proof. Let H be antisymmetric, let $Y \subseteq L^G$ and let $f \in Y$ be injective. If $x, y \in G$, $(x, y) \in T(Y)$, $(y, x) \in T(Y)$, then $(f(x), f(y)) \in H$, $(f(y), f(x)) \in H$, thus $f(x) = f(y)$ and $x = y$ as f is injective. Thus $T(Y)$ is antisymmetric. □

Corollary. *If H is an order on L then $T(Y)$ is an order on G for any $Y \subseteq L^G$ containing at least one injective mapping.*

Lemma 2.3. *Let $f \in L^G$ be surjective. If the relation $T(\{f\})$ on G is reflexive (symmetric, transitive, antisymmetric), then the relation H on L is reflexive (symmetric, transitive, antisymmetric).*

Proof. Let $T(\{f\})$ be reflexive and suppose that $l \in L$ is any element. Choose $x \in G$ such that $f(x) = l$. By hypothesis $(x, x) \in T(\{f\})$ so that $(l, l) = (f(x), f(x)) \in H$ and H is reflexive. In the other cases the proof is similar. □

Corollary. *1. Let $|G| \geqslant |L|$. The relation $T(Y)$ on G is reflexive (symmetric, transitive) for any $Y \subseteq L^G$ iff the relation H on L is reflexive (symmetric, transitive).*
2. Let $|G| = |L|$. The relation $T(Y)$ on G is antisymmetric for any $Y \subseteq L^G$ containing at least one injective mapping iff the relation H on L is antisymmetric.

In particular, we have:

Theorem 2.2. *Let $|G| \geqslant |L|$. The relation $T(Y)$ is a preorder (an equivalence relation) on G for any $Y \subseteq L^G$ iff the relation H is a preorder (an equivalence relation) on L.*

Theorem 2.3. *Let $|G| = |L|$. The relation $T(Y)$ is an order on G for any $Y \subseteq L^G$ containing at least one injective mapping iff the relation H is an order on L.*

3. Realizer and pseudodimension of a relational structure

Let G be a set, $\mathbf{L} = (L, H)$ a relational structure, $|G| \geqslant 2$, $|L| \geqslant 2$ and suppose that $X \subseteq G \times G$ is a relation on G. A set $Y \subseteq L^G$ is said to be an $\mathbf{L}$-*realizer* of the structure (G, X) if $T(Y) = X$.

Theorem 3.1. *Let $X \subseteq G \times G$ be a relation on G. The structure (G, X) has an $\mathbf{L}$-realizer iff $T(S(X)) = X$.*

Proof. If $T(S(X)) = X$ then $S(X)$ is an $\mathbf{L}$-realizer of (G, X). On the other hand, if $Y \subseteq L^G$ is an $\mathbf{L}$-realizer of (G, X) then $T(Y) = X$ and, therefore, $T(S(X)) = T(S(T(Y))) = T(Y) = X$. □

Corollary. *An* **L**-*realizer of* (G, X) *exists for any* $X \subseteq G \times G$ *iff* $T \circ S = \mathrm{id}_{\mathbf{B}(G\times G)}$.

Theorem 3.2. *Let* $X \subseteq G \times G$ *be a relation such that* (G, X) *has an* **L**-*realizer. A set* $Y \subseteq L^G$ *is an* **L**-*realizer of* (G, X) *iff* $(Y, S(X)) \in K_{\mathbf{L}}$.

Proof. By Theorem 3.1, $T(S(X)) = X$ holds. A set $Y \subseteq L^G$ is an **L**-realizer of (G, X) iff $T(Y) = X = T(S(X))$, i.e. iff $(Y, S(X)) \in K_{\mathbf{L}}$. □

Let $Y \subseteq L^G$ be any set. By the *evaluation map* for Y ([3], p. 116) we mean the mapping $e\colon G \to L^Y$ given by

$$e(x)(f) = f(x).$$

Theorem 3.3. *Let* $X \subseteq G \times G$, $Y \subseteq L^G$. *Then the following statements are equivalent:*

(i) Y *is an* **L**-*realizer of* (G, X).

(ii) *The evaluation map for* Y *is a strong homomorphism of* (G, X) *into* $\mathbf{L}^{\mathbf{Y}}$ *where* $\mathbf{Y} = (Y, \emptyset)$ *is a discrete structure.*

Proof. Let (i) hold and suppose $x, y \in G$, $(x, y) \in X$. As $T(Y) = X$, we obtain $(f(x), f(y)) \in H$ for any $f \in Y$, i.e. $(e(x)(f), e(y)(f)) \in H$ for any $f \in Y$, which implies $(e(x), e(y)) \in \mathscr{R}(\mathbf{L}^{\mathbf{Y}})$. On the contrary, if $x, y \in G$ and $(e(x), e(y)) \in \mathscr{R}(\mathbf{L}^{\mathbf{Y}})$ then $(e(x)(f), e(y)(f)) = (f(x), f(y)) \in H$ for any $f \in Y$ and as Y is an **L**-realizer of (G, X), this implies $(x, y) \in X$. Thus e is a strong homomorphism of (G, X) into $\mathbf{L}^{\mathbf{Y}}$ and (ii) holds.

Let (ii) hold and suppose $x, y \in G$. Then we have $(x, y) \in X \iff (e(x), e(y)) \in \mathscr{R}(\mathbf{L}^{\mathbf{Y}}) \iff (e(x)(f), e(y)(f)) \in H$ for any $f \in Y \iff (f(x), f(y)) \in H$ for any $f \in Y \iff (x, y) \in T(Y)$. Thus $X = T(Y)$, Y is an **L**-realizer of (G, X) and (i) holds. □

Let $\mathbf{G} = (G, X)$ be a relational structure, $\mathbf{L} = (L, H)$ a relational structure of type α and suppose $|G| \geqslant 2$, $|L| \geqslant 2$ and $T(S(X)) = X$. We put

$$\alpha\text{-pdim}\,\mathbf{G} = \min\{|Y|;\ Y \subseteq L^G \text{ is an } \mathbf{L}\text{-realizer of } \mathbf{G}\};$$

this cardinal will be called the *α-pseudodimension* of the structure **G**.

Theorem 3.4. *Let* $\mathbf{G} = (G, X)$ *be a relational structure,* $\mathbf{L} = (L, H)$ *a relational structure of type* α *and let* $T(S(X)) = X$. *Then* $\alpha\text{-pdim}\,\mathbf{G} = c_{K_{\mathbf{L}}}(\mathscr{C}(\mathbf{L}^{\mathbf{G}}))$.

Proof. By definition we have $c_{K_{\mathbf{L}}}(\mathscr{C}(\mathbf{L}^{\mathbf{G}})) = \min\{|Y|;\ Y \subseteq L^G, (Y, \mathscr{C}(\mathbf{L}^{\mathbf{G}})) \in K_{\mathbf{L}}\}$. As $(\mathscr{C}(\mathbf{L}^{\mathbf{G}})) = S(X)$, it follows that $(Y, \mathscr{C}(\mathbf{L}^{\mathbf{G}})) \in K_{\mathbf{L}} \iff (Y, S(X)) \in K_{\mathbf{L}} \iff T(Y) = T(S(X)) = X \iff Y$ is an **L**-realizer of **G**. Thus $c_{K_{\mathbf{L}}}(\mathscr{C}(\mathbf{L}^{\mathbf{G}})) = \min\{|Y|;\ Y \subseteq L^G \text{ is an } \mathbf{L}\text{-realizer of } \mathbf{G}\} = \alpha\text{-pdim}\,\mathbf{G}$. □

4. Preorders and orders

In this section we suppose that $\mathbf{L} = (L, H)$ is a preordered set such that there exist $l_1, l_2 \in L$ with $(l_1, l_2) \in H$, $(l_2, l_1) \notin H$. Furthermore, let G be any set; by Corollary of Lemma 2.1, $T(Y)$ is a preorder on G for any $Y \subseteq L^G$.

Lemma 4.1. *If X is a preorder on G and $x, y \in G$, $(x, y) \notin X$ then there exists an $f \in \mathscr{C}(\mathbf{L}^\mathbf{G})$ such that $(f(x), f(y)) \notin H$.*

Proof. Suppose $l_1, l_2 \in L$, $(l_1, l_2) \in H$, $(l_2, l_1) \notin H$. Let us define a mapping $f\colon G \to L$ as follows: for $t \in G$ put

$$f(t) = \begin{cases} l_1 \text{ if } (t, y) \in X, \\ l_2 \text{ if } (t, y) \notin X. \end{cases}$$

We will show that $f \in \operatorname{Hom}(\mathbf{G}, \mathbf{L})$. Let us have $t_1, t_2 \in G$, $(t_1, t_2) \in X$ and suppose $(f(t_1), f(t_2)) \notin H$. Then $f(t_1) = l_2$, $f(t_2) = l_1$ so that $(t_2, y) \in X$. Transitivity of X implies $(t_1, y) \in X$ and then $f(t_1) = l_1$, a contradiction. Hence $(t_1, t_2) \in X \implies (f(t_1), f(t_2)) \in H$, i.e. $f \in \operatorname{Hom}(\mathbf{G}, \mathbf{L}) = \mathscr{C}(\mathbf{L}^\mathbf{G})$. From the definition of f we have $f(x) = l_2$, $f(y) = l_1$ so that $(f(x), f(y)) = (l_2, l_1) \notin H$. □

Theorem 4.1. *If $X \subseteq G \times G$ is a preorder on G then $(T(S(X)) = X$.*

Proof. As $T \circ S$ is a closure operator, we obtain $X \subseteq T(S(X))$. Conversely, let $(x, y) \in T(S(X))$ and suppose $(x, y) \notin X$. By Lemma 4.1 there exists $f \in \mathscr{C}(\mathbf{L}^\mathbf{G}) = S(X)$ such that $(f(x), f(y)) \notin H$. But then $(x, y) \notin T(S(X))$, a contradiction. Thus $(x, y) \in X$ and $T(S(X)) \subseteq X$. We have proved $T(S(X)) = X$. □

Corollary. *For any preordered set $\mathbf{G}$ there exists α-pdim $\mathbf{G}$ where α is the type of $\mathbf{L}$.*

Theorem 4.2. *Let $X \subseteq G \times G$ be any relation on G. Then $T(S(X))$ is the least preorder on G containing X.*

Proof. We have $X \subseteq T(S(X))$ and $T(S(X))$ is a preorder on G by Corollary of Lemma 2.1. Let $X_1 \subseteq G \times G$ be a preorder on G such that $X \subseteq X_1$. Then $T(S(X_1)) = X_1$ by Theorem 4.1 and, therefore, $T(S(X)) \subseteq T(S(X_1)) = X_1$. □

From now on, we suppose that $\mathbf{L} = (L, \leqslant)$ is an ordered set which is not an antichain, and α is its type. From Theorem 4.1 we immediately obtain

Theorem 4.3. *Let $X \subseteq G \times G$ be an order on a set G. Then $T(S(X)) = X$.*

Corollary. *Let α be a type of an ordered set which is not an antichain. Then for any ordered set* $\mathbf{G}$ *there exists* α-pdim $\mathbf{G}$.

Furthermore, Theorem 3.3 implies

Theorem 4.4. *Let $\mathbf{G} = (G, X)$ be an ordered set and suppose $Y \subseteq L^G$. Then the following statements are equivalent:*
(i) *Y is an* $\mathbf{L}$*-realizer of* $\mathbf{G}$.
(ii) *The evaluation map for Y is an embedding of* $\mathbf{G}$ *into* $\mathbf{L}^{\mathbf{Y}}$ *where* $\mathbf{Y} = (Y, \emptyset)$.

Proof. Let (i) hold. By Theorem 3.3 it suffices to show that the evaluation map $e\colon G \to L^Y$ is injective. Let $x, y \in G$, $x \neq y$. As X is antisymmetric, we obtain either $(x, y) \notin X$ or $(y, x) \notin X$; let us suppose $(x, y) \notin X$. As Y is an $\mathbf{L}$-realizer of $\mathbf{G}$, there exists an $f \in Y$ such that $f(x) \not\leqslant f(y)$, i.e. $e(x)(f) \not\leqslant e(y)(f)$. Then $e(x) \not\leqslant e(y)$, in particular $e(x) \neq e(y)$ and e is injective.

Let (ii) hold. Then e is a strong homomorphism of $\mathbf{G}$ into $\mathbf{L}^{\mathbf{Y}}$ and Y is an $\mathbf{L}$-realizer of $\mathbf{G}$ by Theorem 3.3. □

Let us note that if α is a type of a chain containing at least two elements and $\mathbf{G}$ is an ordered set then α-pdim $\mathbf{G}$ coincides with the notion introduced in [5].

5. Equivalence relations

In this section we assume that $\mathbf{L} = (L, H)$ is a structure such that $|L| \geqslant 2$ and $H = \mathrm{id}_L$; the type of this structure will be denoted by m where $m = |L|$. Furthermore, let G be a set such that $|G| \geqslant m$. As id_L is an equivalence relation on L, Corollary of Lemma 2.1 implies that $T(Y)$ is an equivalence relation on G for any $Y \subseteq L^G$. As an analogue to Lemma 4.1 we have

Lemma 5.1. *Let $X \subseteq G \times G$ be an equivalence relation on G and suppose $x, y \in G$, $(x, y) \notin X$. Then there exists $f \in \mathscr{C}(\mathbf{L}^{\mathbf{G}})$ such that $(f(x), f(y)) \notin H$ (i.e. $f(x) \neq f(y)$).*

Proof. Choose $l_1, l_2 \in L$, $l_1 \neq l_2$ and define a mapping $f\colon G \to L$ as follows: for any $t \in G$ put

$$f(t) = \begin{cases} l_1 \text{ if } (t, x) \in X, \\ l_2 \text{ if } (t, x) \notin X. \end{cases}$$

We show that $f \in \mathrm{Hom}(\mathbf{G}, \mathbf{L})$. Suppose $t_1, t_2 \in G$, $(t_1, t_2) \in X$. If $(t_1, x) \in X$ then $(t_2, x) \in X$ so that $f(t_1) = l_1 = f(t_2)$ and $(f(t_1), f(t_2)) \in H$. If $(t_1, x) \notin X$ then $(t_2, x) \notin X$ and $f(t_1) = l_2 = f(t_2)$, i.e. $(f(t_1), f(t_2)) \in H$. Thus $f \in \mathscr{C}(\mathbf{L}^{\mathbf{G}})$ and by definition $f(x) = l_1 \neq l_2 = f(y)$. □

From Lemma 5.1 we get

Theorem 5.1. *Let $X \subseteq G \times G$ be an equivalence relation on G. Then $T(S(X)) = X$.*

Proof. We have $X \subseteq T(S(X))$. Suppose $(x,y) \in T(S(X))$; if $(x,y) \notin X$ then by Lemma 5.1 there exists $f \in \mathscr{C}(\mathbf{L}^{\mathbf{G}}) = S(X)$ such that $(f(x), f(y)) \notin H$. Then $(x,y) \notin T(S(X))$, a contradiction. Thus $(x,y) \in X$; we have proved $T(S(X)) \subseteq X$ and thus $T(S(X)) = X$. □

Corollary. *Let $X \subseteq G \times G$ be an equivalence relation on G and $\mathbf{G} = (G, X)$. Then there exists m-pdim $\mathbf{G}$.*

Let us denote by $E(X)$ the least equivalence relation on G containing X for a given $X \subseteq G \times G$. The proof of the following theorem is analogous to the proof of Theorem 4.2 and is therefore omitted.

Theorem 5.2. *Let $X \subseteq G \times G$ be any relation. Then $E(X) = T(S(X))$.*

Let us have $f \in L^G$. Put $\ker f = \{(x,y) \in G \times G;\ f(x) = f(y)\} = \{(x,y) \in G \times G;\ (f(x), f(y)) \in H\} = T(\{f\})$. If $Y \subseteq L^G$, $Y \neq \emptyset$ then $T(Y) = \{(x,y) \in G \times G;\ (f(x), f(y)) \in H$ for all $f \in Y\} = \bigcap(T(\{f\});\ f \in Y) = \bigcap(\ker f;\ f \in Y)$. Thus we obtain

Theorem 5.3. *Let $X \subseteq G \times G$ be an equivalence relation on G and suppose $Y \subseteq L^G$. Then Y is an $\mathbf{L}$-realizer of $\mathbf{G} = (G, X)$ iff $X = \bigcap(\ker f;\ f \in Y)$.*

Regarding Theorem 5.2 we get further

Theorem 5.4. *Let $X \subseteq G \times G$ be a relation on G. Then $E(X) = \bigcap(\ker f;\ f \in S(X))$.*

Let E be an equivalence relation on G. If $|G/E| \leqslant m$ then E will be called an *m-equivalence.* Clearly, for any $f \in L^G$, $\ker f$ is an m-equivalence.

Let $\mathscr{S} = (E_i;\ i \in I)$ be a system of equivalence relations on G and let E be an equivalence relation on G. If $E = \bigcap(E_i;\ i \in I)$ then we say that *$\mathscr{S}$ generates E.* If X is an equivalence relation on G and $Y \subseteq L^G$ is an $\mathbf{L}$-realizer of (G, X) then $(\ker f;\ f \in Y)$ generates X by Theorem 5.3. Conversely, let $(E_i;\ i \in I)$ be a system of m-equivalences on G that generates X. Denote by φ_i the natural projection of G onto G/E_i $(i \in I)$ and by ψ_i any (arbitrarily chosen) injective mapping of G/E_i into L. Put $f_i = \psi_i \circ \varphi_i$ and $Y = (f_i;\ i \in I)$. Then $Y \subseteq L^G$ and it is easy to see that it is an $\mathbf{L}$-realizer of $\mathbf{G} = (G, X)$ where $X = \bigcap(E_i;\ i \in I)$. Thus we have

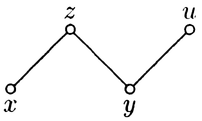
Theorem 5.5. *Let $X \subseteq G \times G$ be an equivalence relation on G and $\mathbf{G} = (G, X)$. Then m-pdim $\mathbf{G}$ is the minimum of cardinalities of systems of m-equivalences on G generating X.*

In view of Theorem 3.4 this assertion can be formulated as follows.

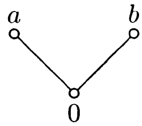
Theorem 5.6. *Let $X \subseteq G \times G$ be an equivalence relation on G and $\mathbf{G} = (G, X)$. Then $c_{K_{\mathbf{L}}}(\mathscr{C}(\mathbf{L}^{\mathbf{G}}))$ is the minimum of cardinalities of systems of m-equivalences on G generating X.*

6. Examples

Example 1. Let $\mathbf{G} = (G, X)$ be an ordered set with the following Hasse diagram:



We find $\mathbf{3}$-pdim $\mathbf{G}$, $\mathbf{2}$-pdim $\mathbf{G}$, α-pdim $\mathbf{G}$ where $\mathbf{3}$ ($\mathbf{2}$) is the type of the 3-element chain (2-element chain) and α is the type of the ordered set



(1) Let $\mathbf{L} = (\{0, 1, 2\}; 0 < 1 < 2)$. Define mappings $f_1, f_2 \colon G \to \{0, 1, 2\}$ by

	x	y	z	u
f_1	0	1	1	2
f_2	1	0	1	0

If $Y = \{f_1, f_2\}$ then it is easy to see that $T(Y) = X$, i.e. Y is an $\mathbf{L}$-realizer of $\mathbf{G}$. Thus $\mathbf{3}$-pdim $\mathbf{G} \leqslant 2$. As trivially $\mathbf{3}$-pdim $\mathbf{G} > 1$ we have $\mathbf{3}$-pdim $\mathbf{G} = 2$.

(2) Suppose $\mathbf{L} = (\{0,1\}; 0 < 1)$. We find all isotonic mappings of $\mathbf{G}$ into $\mathbf{L}$. They are given by the following table:

	x	y	z	u
f_1	0	0	0	0
f_2	0	0	1	0
f_3	1	0	1	0
f_4	0	0	0	1
f_5	0	0	1	1
f_6	1	0	1	1
f_7	0	1	1	1
f_8	1	1	1	1

We are looking for an $\mathbf{L}$-realizer of $\mathbf{G}$. As z, u are incomparable in $\mathbf{G}$, any $\mathbf{L}$-realizer of $\mathbf{G}$ must contain f_4 and either f_2 of f_3. As x, y are incomparable, any $\mathbf{L}$-realizer of $\mathbf{G}$ must contain f_7 and either f_3 or f_6. Putting $Y = \{f_3, f_4, f_7\}$ we obtain that $T(Y) = X$ and Y is an $\mathbf{L}$-realizer of $\mathbf{G}$. Since no two-element subset of Y is an $\mathbf{L}$-realizer of $\mathbf{G}$ we have $\mathbf{2}\text{-pdim}\,\mathbf{G} = 3$.

(3) Suppose $\mathbf{L} = (\{0,a,b\}, 0 < a, 0 < b)$. Let us define mappings $f_1, f_2 \colon G \to \{0,a,b\}$ by

	x	y	z	u
f_1	0	a	a	a
f_2	a	0	a	b

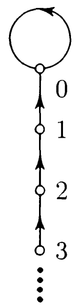
If $Y = \{f_1, f_2\}$ then $T(Y) = X$. As $\alpha\text{-pdim}\,\mathbf{G} > 1$, we have $\alpha\text{-pdim}\,\mathbf{G} = 2$.

Example 6.2. Let $\mathbf{G} = (G, X)$ be an ordered set and let $\mathbf{L} = (L, \leqslant)$ be a chain of type $\mathbf{2}$. Let $Y \subseteq L^G$ be a $\mathbf{2}$-realizer of $\mathbf{G}$. Any $f \in Y$ may be interpreted as a characteristic function of a filter in $\mathbf{G}$, i.e. Y may be interpreted to be a set of filters in $\mathbf{G}$. In particular, $\mathscr{C}(\mathbf{L}^{\mathbf{G}})$, the greatest $\mathbf{L}$-realizer of $\mathbf{G}$, is the set of all filters in $\mathbf{G}$ which is a complete ring of sets ([6]).

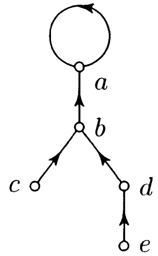
Let Y be a set of filters in $\mathbf{G}$. By definition, Y is a $\mathbf{2}$-realizer of $\mathbf{G}$ iff the condition $x, y \in G$, $(x,y) \notin X$ is equivalent to the existence of a set $M \in Y$ such that $x \in M$, $y \notin M$. In particular, if $x, y \in G$, $x \neq y$ then there exists $M \in Y$ such that either $x \in M$, $y \notin M$ or $y \in M$, $x \notin M$. It follows that two $\mathbf{L}$-realizers Y_1, Y_2 of $\mathbf{G}$ have the same separation property: For any $x, y \in G$, $x \neq y$ there exists $M_1 \in Y_1$ with $x \in M_1$, $y \notin M_1$ iff there exists $M_2 \in Y_2$ with $x \in M_2$, $y \notin M_2$ (see [6], Theorem 2.5.).

Example 6.3. Let G be a finite set, $|G| \geqslant 2$ and let X be an equivalence relation on G. We find $2\text{-pdim}\,\mathbf{G}$ where $\mathbf{G} = (G, X)$. By Theorem 5.5, $2\text{-pdim}\,\mathbf{G}$ is the minimum of cardinalities of systems of 2-equivalences on G generating X. If $(E_1, \ldots, E_m)$ is a system of 2-equivalences on G and $X = \bigcap(E_i\,; i = 1, \ldots, m)$ then clearly $|G/X| \leqslant 2^m$ (as each E_i has at most 2 blocks). If $|G/X| = n$ then there exists an integer $m \geqslant 1$ such that $2^{m-1} < n \leqslant 2^m$; then 2-pdim $\mathbf{G} = m$.

Example 6.4. A *monounary algebra* is a set $G \neq \emptyset$ and a mapping $f: G \to G$; it will be denoted by (G, f). (See, e.g. [8].) As usual, f may be regarded as a binary relation on G by putting $(x, y) \in f$ iff $y = f(x)$. If (G, f), (H, g) are monounary algebras, then the homomorphisms of the algebra (G, f) into (H, g) coincide with the homomorphisms of the relation structure $\mathbf{G} = (G, f)$ into the relational structure $\mathbf{H} = (H, g)$. A monounary algebra (G, f) is called a *connected monounary algebra with a one-element cycle* if there exists exactly one element $c \in G$ such that $f(c) = c$ and that for any $x \in G$ there exists an integer $m \geqslant 0$ such that $f^m(x) = c$ where f^m denotes the m-th iteration of f. Let $\mathbf{L}$ be the set of all nonnegative integers with the operation H given by $H(0) = 0$, $H(n + 1) = n$ for any $n \geqslant 0$:



Let $G = \{a, b, c, d, e\}$ and let X be a binary relation on G such that (G, X) is a connected monounary algebra with a one-element cycle with the following diagram:



Define mappings f_1, f_2 of $\mathbf{G} = (G, X)$ into $\mathbf{L}$ by

	a	b	c	d	e
f_1	0	0	0	1	2
f_2	0	1	2	2	3

It is easy to see that f_1, f_2 are homomorphisms of $\mathbf{G}$ into $\mathbf{L}$. Furthermore, $T(\{f_1, f_2\}) = X$; hence $\{f_1, f_2\}$ is an $\mathbf{L}$-realizer of $\mathbf{G}$.

References

[1] *G. Birkhoff*: Generalized Arithmetics. Duke Math. Journ. 9 (1942), 283–302.
[2] *G. Birkhoff*: Lattice Theory, third edition. Providence, Rhode Island, 1967.
[3] *J.L. Kelley*: General Topology. Van Nostrand, New York, 1955.
[4] *J. Novák*: On some Ordered Continua of Power $2^{\aleph_0}$ containing a Dense Subset of Power $\aleph_1$. Czechoslovak Math. J. *76* (1951), 63–79.
[5] *V. Novák*: On the Pseudodimension of Ordered Sets. Czechoslovak Math. J. *13 (88)* (1963), 587–598.

[6] *V. Novák*: Some Cardinal Characteristics of Ordered Sets. To appear.

[7] *V. Novák, M. Novotný*: Abstrakte Dimension von Strukturen. Ztschr. f. math. Logik u. Grundl. d. Math. *20* (1974), 207–220.

[8] *M. Novotný*: Monounary Algebras in the Work of Czechoslovak Mathematicians. Arch. Math. (Brno) *26* (1990), 155–164.

[9] *M. Novotný*: Reducts versus Reducing Operators. Mathematical aspects of natural and formal languages (G. Păun, ed.). World Scientific Series in Computer Science, Vol. 43, Singapore-New York-London-Hong Kong, 1994, pp. 359–374.

[10] *M. Novotný*: Dependence Spaces of Information Systems. To appear in: Logical and algebraic investigations in rough set theory, ed. E. Orlowska.

[11] *M. Novotný, Z. Pawlak*: Algebraic Theory of Independence in Information Systems. Fund. Inform. *14* (1991), 454–476.

[12] *M. Novotný, Z. Pawlak*: On a Problem concerning Dependence Spaces. Fund. Inform. *16* (1992), 275–287.

[13] *J. Šlapal*: Cardinal Arithmetic of General Relational Systems. Czechoslovak Math. J. *43 (118)* (1993), 125–139.

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