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COMPACT UNIVERSAL RELATION IN VARIETIES WITH CONSTANTS

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B. Csákány [2] (see also [3]) characterized varieties of algebras having no one element subalgebras. Later on J. Kollár [5] proved that in a variety $\mathcal{V}$ no algebra has a one element subalgebra if and only if the universal relation $A \times A$ on every $A \in \mathcal{V}$ is a compact element of $\operatorname{Con} A$.

In this paper we consider varieties containing at least two constants. Then every universal relation is a compact element of a congruence lattice in such varieties, by Kollár's theorem. Now we formulate a more detailed question under what conditions the finite generating set for the universal relation consists of pairs of constants only. It is shown that this problem can be answered by means of missing skew subalgebras of direct products or of missing congruence classes in algebras. Furthermore, Mal'cev-type conditions characterizing the above properties are presented.

Let $\mathcal{V}$ be a variety. By a constant of $\mathcal{V}$ we mean either a nullary (fundamental) operation or an equationally defined nullary term function.

From now on the type of $\mathcal{V}$ will be considered to be known and by C we denote the set of all constants of $\mathcal{V}$. Throughout the paper, all varieties will be considered with $\operatorname{card} C > 1$. For $A \in \mathcal{V}$, we denote by C_A the set of all constants of A.

Let A, B be sets. A subset U of the Cartesian product $A \times B$ is called *factorable* whenever $U = V \times W$ for some subsets $V \subseteq A$ and $W \subseteq B$; if U is not factorable, it is called *skew*.

A subset S of A is called *proper* whenever $S \neq A$.

The following simple result will be useful in the sequel:

Lemma 1. *Let A, B be non void sets. A subset M of $A \times B$ is factorable if and only if $\langle a, b\rangle \in S$ and $\langle c, d\rangle \in S$ imply $\langle a, d\rangle \in S$ for any elements a, c of A and b, d of B.*

P r o o f. Immediate. □

A binary relation on an algebra A is called *compatible* if it is a subalgebra of the square $A \times A$. As was shown in [1], the set of all compatible relations on A satisfying a given subset of the properties: reflexivity, symmetry, transitivity, forms an algebraic lattice with respect to set inclusion. Hence, for a subset M of $A \times A$, we denote by

$R(M)$ the compatible reflexive relation on A generated by M,

$T(M)$ the tolerance on A generated by M,

$Q(M)$ the compatible quasiorder on A generated by M,

$\Theta(M)$ the congruence on A generated by M.

Lemma 2. *Let S be a subset of an algebra A. Then $T(S \times S) = R(S \times S)$ and $\Theta(S \times S) = Q(S \times S)$.*

The proof follows by the symmetry of the generating set $S \times S$, see e.g. [4] for the details. □

Theorem 1. *Let $\mathcal{V}$ be a variety with a set of constants C, $\operatorname{card} C > 1$. The following conditions are equivalent:*

(1) *for any $A, B \in \mathcal{V}$, the interval $[C_A \times C_B, A \times B]$ contains no skew subalgebra;*

(2) *for any $A \in \mathcal{V}$, $A \times A$ is a finite join of principal tolerances generated by pairs of constants from $C_A \times C_A$;*

(3) *for any $A \in \mathcal{V}$, $A \times A = T(C_A \times C_A) = R(C_A \times C_A)$;*

(4) *there are constants $o_1, \ldots, o_n, i_1, \ldots, i_n \in C$ and an $(n+2)$-ary term p such that*

$$x = p(o_1, \ldots, o_n, x, y) \quad \text{and} \quad y = p(i_1, \ldots, i_n, x, y)$$

are identities in $\mathcal{V}$.

Proof. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are trivial.

Prove (3) $\Rightarrow$ (4): Let $A = F_{\mathcal{V}}(x, y)$, the free algebra of $\mathcal{V}$ with two free generators. Then $\langle x, y \rangle \in A \times A = R(C_A \times C_A)$. By [1], there exist an $(n+2)$-ary term p and constants $o_1, \ldots, o_n, i_1, \ldots, i_n \in C_A$ with

$$\langle x, y \rangle = p\left(\langle o_1, i_1 \rangle, \ldots, \langle o_n, i_n \rangle, \langle x, x \rangle, \langle y, y \rangle\right).$$

Writing it componentwise, we obtain (4).

(4) $\Rightarrow$ (2): The identities (4) mean that $A \times A = R(\langle o_1, i_1 \rangle, \ldots, \langle o_n, i_n \rangle) = R(o_1, i_1) \vee \ldots \vee R(o_n, i_n)$, $o_1, \ldots, o_n, i_1, \ldots, i_n \in C_A$, for any $A \in \mathcal{V}$, see [1]. Consequently, $A \times A = T(o_1, i_1) \vee \ldots \vee T(o_n, i_n)$.

(4) $\Rightarrow$ (1): Let $A, B \in \mathcal{V}$ and let S be a subalgebra from the interval $[C_A \times C_B, A \times B]$. Take $\langle a, b \rangle \in S$ and $\langle c, d \rangle \in S$. Putting $x = a$, $y = c$ in the first and $x = b$,

$y = d$ in the second identity of (4) we get

$$\langle a, d\rangle = p\left(\langle o_1, i_1\rangle, \ldots, \langle o_n, i_n\rangle, \langle a, b\rangle, \langle c, d\rangle\right),$$

giving $\langle a, d\rangle \in S$. Lemma 1 completes the proof. □

Corollary 1. *Let $\mathcal{V}$ be a variety with exactly two constants* 0 *and* 1*. The following conditions are equivalent:*

(1) *any subalgebra S of $A \times B$, $A, B \in \mathcal{V}$, is factorable whenever $\langle 0_A, 1_A\rangle \in S$ and $\langle 1_A, 0_A\rangle \in S$;*
(2) *for any $A \in \mathcal{V}$, $A \times A = T(0_A, 1_A)$;*
(3) *there is a quaternary term q such that*

$$x = q(0, 1, x, y) \quad \text{and} \quad y = q(1, 0, x, y)$$

are identities in $\mathcal{V}$.

The proof follows directly by Theorem 1. □

Example 1. (a) For a variety of *unitary rings*, we can put $q(a, b, x, y) = a{\cdot}y + b{\cdot}x$.

(b) For a variety of *bounded lattices*, put $q(a, b, x, y) = (a \wedge y) \vee (b \wedge x)$.

(c) By a *ternary ring* we call an algebra $(K, t, 0, 1)$, where t is a ternary operation on K and $0, 1$ are constants satisfying the identities

$$t(x, 0, y) = y = t(0, x, y),$$
$$t(x, 1, 0) = x = t(1, x, 0);$$

moreover, for every $a, b, c \in K$ there exists a unique $d \in K$ with $t(a, b, d) = c$. Consider any variety of ternary rings and put

$$q(a, b, x, y) = t\left(t(b, x, a), t(a, y, b), 0\right).$$

Theorem 2. *Let $\mathcal{V}$ be a variety with a set of constants C, $\operatorname{card} C > 1$. The following conditions are equivalent:*

(1) *for any $A \in \mathcal{V}$, the interval $[C_A, A]$ contains no proper congruence class;*
(2) *for any $A \in \mathcal{V}$, the interval $[C_A \times C_A, A \times A]$ contains no proper congruence on A;*
(3) *for any $A \in \mathcal{V}$, $A \times A$ is a finite join of principal congruences generated by pairs of constants from $C_A \times C_A$;*
(4) *for any $A \in \mathcal{V}$, $A \times A = \Theta(C_A \times C_A) = Q(C_A \times C_A)$;*

(5) *there are constants* $o_1, \dots, o_n, i_1, \dots, i_n \in C$ *and ternary terms* $t_1, \dots, t_m$ *such that*

$$x = t_1(o_1, x, y),$$
$$t_j(i_j, x, y) = t_{j+1}(o_j, x, y) \quad \textit{for} \quad j = 1, \dots, m-1,$$
$$y = t_m(i_m, x, y)$$

are identities in $\mathcal{V}$;

(6) *there are constants* $o, o_1, \dots, o_n, i_1, \dots, i_n \in C$ *and binary terms* $s_1, \dots, s_n$ *such that*

$$x = s_1(o_1, x),$$
$$s_j(i_j, x) = s_{j+1}(o_j, x) \quad \textit{for} \quad j = 1, \dots, n-1,$$
$$o = s_n(i_n, x)$$

are identities in $\mathcal{V}$.

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are evident.

(4) $\Rightarrow$ (5): Take $A = F_{\mathcal{V}}(x, y)$. Then $\langle x, y\rangle \in Q(C_A \times C_A)$ from which

$$\langle x, y\rangle \in R(o_1, i_1) \circ \cdots \circ R(o_m, i_m)$$

(where $\circ$ denotes the relational product) for some constants $o_1, \dots, o_m, i_1, \dots, i_m \in C$ (see e.g. [1] or [4] for some details). By [4] there exist ternary terms $t_1, \dots, t_m$ satisfying the identities (5).

(5) $\Rightarrow$ (6): Choose an arbitrary $o \in C$ and put $y = o$ in the identities (5).

(6) $\Rightarrow$ (5): We observe that (6) gives

$$x = s_1(o_1, x),$$
$$s_j(i_j, x) = s_{j+1}(o_j, x) \quad \text{for} \quad j = 1, \dots, n-1,$$
$$s_n(i_n, x) = o = s_n(i_n, y),$$
$$s_{n-j+1}(o_{n-j+1}, y) = s_{n-j}(i_{n-j}, y) \quad \text{for} \quad j = 1, \dots, n-1,$$
$$y = s_1(o_1, y).$$

Setting $m = 2n + 1$ and $t_{n+1} = o$,

$$t_j(a, x, y) = s_j(a, x) \quad \text{for} \quad j = 1, \dots, n,$$
$$t_{n+j+1}(a, x, y) = s_{n-j+1}(a, y) \quad \text{for} \quad j = 1, \dots, n,$$

we obtain the identities of (5).

It remains to prove (5) $\Rightarrow$ (2): Let $a, b \in A \in \mathcal{V}$ and let Θ be a congruence from the interval $[C_A \times C_A, A \times A]$. Putting $x = a, y = b$ in the identities (5), we conclude $\langle a, b\rangle \in \Theta$. Hence $\Theta = A \times A$ is apparent. $\square$

Corollary 2. *Let $\mathcal{V}$ be a variety with exactly two constants* 0 *and* 1. *The following conditions are equivalent:*

(1) *for any $A \in \mathcal{V}$, $A \times A = \Theta(0_A, 1_A)$;*

(2) *there are quat?rnary terms $q_1, \ldots, q_m$ such that*

$$x = q_1(0, 1, x, y),$$
$$q_i(1, 0, x, y) = q_{i+1}(0, 1, x, y) \quad \text{for} \quad j = 1, \ldots, m - 1,$$
$$y = q_m(1, 0, x, y)$$

are identities in $\mathcal{V}$;

(3) *there are ternary terms $r_1, \ldots, r_n$ such that*

$$x = r_1(0, 1, x),$$
$$r_i(1, 0, x) = r_{i+1}(0, 1, x) \quad \text{for} \quad j = 1, \ldots, n - 1,$$
$$0 = r_n(1, 0, x)$$

are identities in $\mathcal{V}$.

Remark. There exists a variety $\mathcal{V}$ with two constants 0 and 1 for which $\Theta(0_A, 1_A) = A \times A$ for each $A \in \mathcal{V}$ but $T(0_A, 1_A) \neq A \times A$ for some A in $\mathcal{V}$.

Example 2. Let $\mathcal{V}$ be a variety of $\wedge$-semilattices with 0 (the least element) and 1 (the greatest element). Consider the three-element chain $A = \{0_A, a, 1_A\}$, i. e. $0_A < a < 1_A$. Evidently,

$$T(0_A, 1_A) = (A \times A) \setminus \{\langle a, 1_A \rangle, \langle 1_A, a \rangle\}.$$

On the other hand, we can put $n = 1$ and $r_1(a, b, x) = b \wedge x$. Then

$$x = 1 \wedge x = r_1(0, 1, x),$$
$$0 = 0 \wedge x = r_1(1, 0, x),$$

satisfying (3) of Corollary 2. Hence, $A \times A = \Theta(0_A, 1_A)$ for each A of $\mathcal{V}$.

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