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ON SOME PROPERTIES OF THE CANTOR SET

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Introduction

Let x be a number given by $x = \sum_{i=1}^{\infty} \frac{c_i}{3^i}$, where $c_i = 0$ or 2 for all i. Then the set $\{x\}$ is the Cantor set C which is a nondense perfect set; and the set of complementary intervals $\left\{\left(\frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_{n-1}}{3^{n-1}} + \frac{1}{3^n}, \frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_{n-1}}{3^{n-1}} + \frac{2}{3^n}\right)\right\}$, none of which contains a point of C, is everywhere dense in $[0,1]$. Steinhaus [6] proved that given any d in $[0,1]$, it is possible to find points x and y of C such that $y - x = d$. Utz [8] proved Steinhaus' result geometrically in the following way: Given m and d satisfying $\frac{1}{3} \leqslant |m| \leqslant 3$ and $0 \leqslant d \leqslant 1$, there exists a pair of points x and y from the Cantor set such that $y - mx = d$. Randolph [5] proved that any point in the unit interval $[0,1]$ is midway between two Cantor points. Bose Majumdar [1] gave an alternative proof of this theorem. Randolph's results was generalized by Ganguly [3] in the following manner: Given positive real numbers p and q, $0 < \frac{p}{q} < 1$, and d, $0 \leqslant d \leqslant 1$, it is possible to find Cantor points x_1 and x_2 such that $d = \frac{px_2 + qx_1}{p+q}$.

Clearly, we can see that the points 0 and 1 of C are not midway between two distinct Cantor points. In 1936, V. Jarník [4] showed that all Cantor points which represent irrational numbers cannot be expressed as centers of two distinct Cantor points. Here, in Section 1, we extend the result of Jarník. By a non-end point of the Cantor set we mean any Cantor point which is not an end-point of any of the remaining closed intervals in the construction of the Cantor set. We show that no non-end point of the Cantor set is expressible as the center of two distinct Cantor points.

Bose Majumdar [2] proved that any point d in the unit interval can be expressed uniquely as $d = x + y$ where $x \in C$, $y \in C$ if and only if $d = .\delta_1 1 \delta_2 1 \delta_3 1 \ldots \delta_{2k-1} 1 \delta_{2k} 1 \ldots$, where each δ is either a block of 0's and 2's or may be void, but no δ_{2k-1} should contain a "two" and no δ_{2k} should contain a "zero". He also noted that $d = \frac{1}{2} = (.111\ldots)$ is the only point in $0 < d < 1$ which can be uniquely expressed both as $y + x = d$ and $y - x = d$, where $x \in C$, $y \in C$. With $0 \leqslant d \leqslant 1$, we define $\Delta_d = \{(x, y)\colon x \in C, y \in C \text{ and } x + y = d\}$. We now present the following theorems.

Theorem 1.1. *If d is any number satisfying $0 < d < 1$, such that $\overline{\overline{\Delta}}_d = 1$ where $\overline{\overline{\Delta}}_d$ means the cardinality of Δ_d, then $\frac{d}{2}$ is a non-end point of C. Moreover, if x and y are in C and $x + y = d$, then $x = y = \frac{d}{2}$.*

Proof. Since $\overline{\overline{\Delta}}_d = 1$, then according to Bose Majumdar [1], $d = .\delta_1 1 \delta_2 1 \delta_3 1 \ldots \delta_{2k-1} 1 \delta_{2k} \ldots$ where each δ is a block of 0's and 2's or empty; but no δ_{2k-1} contains the digit 2 and no δ_{2k} contains the digit 0.

It is easily seen that $\frac{d}{2} = \frac{1}{2}(.\delta_1 1 \delta_2 1 \delta_3 1 \ldots) = .\alpha_1 \beta_2 \alpha_3 \beta_4 \ldots$ where α_{2k-1} is a block of 0's only and β_{2k} is a block of 2's only. Thus $\frac{d}{2}$ is a non-end point of C.

Since $\overline{\overline{\Delta}}_d = 1$, there exists only one pair $(x, y) \in C \times C$ such that $x + y = d$. However, $\frac{d}{2} \in C$ and $d = \frac{d}{2} + \frac{d}{2}$. Therefore $x = y = \frac{d}{2}$. □

Corollary. *If $0 < d < 1$ is such that the set $\{(x, y)\colon x \in C, y \in C, y = x + d\}$ has cardinal number 1, then $\frac{1-d}{2}$ is a non-end point of C. Furthermore, if x and y are in C such that $|y - x| = d$, then $x = 1 - y = \frac{1-d}{2}$.*

Now we extend the result of Jarník.

Theorem 1.2. *If z is a non-end point of C, then z cannot be expressed as the center of two distinct Cantor points.*

Proof. We are to prove that if $z = \frac{x+y}{2}$, $x \in C$, $y \in C$, then $x = y = z$. Let z be a non-end point of C such that $0 < z < \frac{1}{3}$. Then $z = \sum\limits_{i=1}^{\infty} \frac{z_i}{3^i}$ where $z_1 = 0$ and $z_i = \left\{ {0 \atop 2} \right\}$ for $i > 1$ and there is infinite number of 0's and 2's in the expression for z.

Then $z - \frac{1}{2} = \sum\limits_{i=1}^{\infty} \frac{z_i - 1}{3^i} = \sum\limits_{i=1}^{\infty} \frac{\lambda_i}{3^i}$, where $\lambda_i = \left\{ {-1 \atop 1} \right\}$ for all i. As $2z - 1$ is any point in $(-1, 1)$, according to Bose Majumdar [2] $\frac{2z-1}{2}$ can be expressed uniquely as $z - \frac{1}{2} = \sum\limits_{i=1}^{\infty} \frac{\lambda_i}{3^i}$, $\lambda_i = \left\{ {-1 \atop 1} \right\}$ for all i. Now, choose $x_i = 1$, $y_i = 0$ if $\lambda_i = -1$ and

$x_i = 0$ and $y_i = 1$ if $\lambda_i = 1$. Then $2z - 1 = \sum_{i=1}^{\infty} \frac{2\lambda_i}{3^i} = \sum_{i=1}^{\infty} \frac{2y_i}{3^i} - \sum_{i=1}^{\infty} \frac{2x_i}{3^i} = y - x$, where $y = \sum_{i=1}^{\infty} \frac{2y_i}{3^i} \in C$ and $x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \in C$.

Therefore $2z = y + 1 - x = y + x'$ where $x' = 1 - x \in C$ as C is symmetric. But $2z = z + z$, hence $x' = y = z$.

If $\frac{2}{3} < z < 1$, then $0 < 1 - z < \frac{1}{3}$. $1 - z$ is also a non-end point of C as the Cantor set C is symmetric.

If $u + v = 2(1 - z)$, $u, v \in C$, then $u = v = 1 - z$. So if $x + y = 2z$, then $(1 - x) + (1 - y) = 2(1 - z)$, where $x, y \in C$. Hence $1 - x = 1 - y = 1 - z$, i.e. $x = y = z$. □

§2

Now we recall some basic notation and definitions.

Definition 1. If P_1, P, P', P_2 are four collinear points then the expression

$$\frac{\overline{PP_1}}{\overline{PP_2}} \bigg/ \frac{\overline{P'P_1}}{\overline{P'P_2}} = \frac{\overline{PP_1} \cdot \overline{P'P_2}}{\overline{PP_2} \cdot \overline{P'P_1}},$$

which is the ratio of the distance ratios, is called the cross-ratio of the four collinear points. We shall denote this cross-ratio by (P_1P_2, PP').

The family of straight lines in the plane passing through a fixed point is said to form a pencil of lines. The straight lines are called the rays and the common point the centre of the pencil.

Let p_1 and p_2 be two intersecting lines and let p be a straight line passing through the point of intersection of p_1 and p_2. A point P is taken on p. Draw perpendiculars PQ_1, PQ_2 on p_1 and p_2, respectively. The centre of the pencil divides each ray into two halfrays. The angles (p, p_1) and (p, p_2) are measured between the halfray of p on which P lies and those halfrays of p_1 and p_2 on which Q_1 and Q_2 lie, in the directions of $\overline{PQ_1}$ and $\overline{PQ_2}$, respectively.

Definition 2. If p_1, p_2, p, p' are four concurrent straight lines then the expression

$$\frac{\sin(p, p_1)}{\sin(p, p_2)} \bigg/ \frac{\sin(p', p_1)}{\sin(p', p_2)}$$

is called the cross-ratio of the four concurrent straight lines and is denoted by (p_1p_2, pp').

Definition 3. In four concurrent straight lines a, b, c, d are such that $(ab, cd) = -1$, then a, b, c, d are called four harmonic lines.

Theorem 2.1. *Let two positive numbers p and q be chosen arbitrarily with $0 < \frac{p}{q} < 1$. For any interior point R of the unit square $S = [(0,0), (1,0); (1,1), (0,1)]$ we can always find a rectangle $A_1B_1C_1D_1$ lying in S, with its vertices on the Cantor product set C^2, such that R lies on the diagonal A_1C_1 dividing it in the ratio $p : q$ and the Cross-ratios of the pencil of four concurrent lines RD_1, RP, RQ and RB_1 is the same for all positions of R in S, where P and Q lie on the other diagonal D_1B_1 dividing it in the ratios $p : q$ and $q : p$, respectively.*

Proof. Let us consider the product set $C^2 = C \times C$ in the unit square S, C being the Cantor set. Hence, if $(x, y) \in C^2$ then $x \in C$, $y \in C$. □

Here p and q are two given positive real numbers such that $0 < \frac{p}{q} < 1$. Consider any interior point $R(x, y)$ of S. Then by a theorem due to Ganguly [3] we can find a rectangle $A_1B_1C_1D_1$ with vertices on C^2 lying in S, where the coordinates of A_1, B_1, C_1 and D_1 are respectively (c_1, c_3), (c_2, c_3), (c_2, c_4), (c_1, c_4) where $c_i \in C$ ($i = 1, 2, 3, 4$) with the property that the point $R(x, y)$ lies on the diagonal A_1C_1 and $x = \frac{pc_2+qc_1}{p+q}$, $y = \frac{pc_4+qc_3}{p+q}$. Now, draw the diagonal B_1D_1 and through the point $R(x, y)$ draw lines parallel to Y and X-axes, respectively, intersecting B_1D_1 at P and Q where $P = (x', y')$ and $Q = (x'', y'')$, say. It is obvious that $\frac{D_1P}{PB_1} = \frac{p}{q}$ and $\frac{D_1Q}{QB_1} = \frac{q}{p}$. Therefore, $x' = x$, $y' = \frac{pc_3+qc_4}{p+q}$, $y'' = y$, $x'' = \frac{pc_1+qc_2}{p+q}$. Here one of the 24 cross-ratios of four collinear points B_1, Q, P, D_1 is

$$(1) \qquad (D_1Q, PB_1) = \frac{\overline{PD_1}}{\overline{PQ}} \Big/ \frac{\overline{B_1D_1}}{\overline{B_1Q}} = \frac{\overline{PD_1}}{\overline{PQ}} \cdot \frac{\overline{B_1Q}}{\overline{B_1D_1}}.$$

Obviously, $\overline{D_1P} = \frac{p}{p+q}\overline{D_1B_1}$ and $\overline{B_1Q} = \frac{p}{p+q}\overline{B_1D_1}$. Also

$$\begin{aligned} PQ^2 &= (x'' - x')^2 + (y'' - y')^2 \\ &= \Big(\frac{pc_1 + qc_2}{p+q} - \frac{pc_2 + qc_1}{p+q}\Big)^2 + \Big(\frac{pc_4 + qc_3}{p+q} - \frac{pc_3 + qc_4}{p+q}\Big)^2 \\ &= \Big(\frac{p-q}{p+q}\Big)^2 \{(c_2 - c_1)^2 + (c_4 - c_3)^2\} \\ &= \Big(\frac{p-q}{p+q}\Big)^2 \{(A_1B_1)^2 + (C_1B_1)^2\} = \Big(\frac{p-q}{p+q}\Big)^2 \cdot (B_1D_1)^2. \end{aligned}$$

Hence, $\overline{PQ} = \frac{q-p}{p+q} \cdot (\overline{D_1B_1})$. Then (1) implies

$$(D_1Q, PB_1) = \frac{\frac{p}{p+1} \cdot \frac{p}{p+q}\overline{B_1D_1} \cdot \overline{B_1D_1}}{-\frac{(q-p)}{p+q}\overline{B_1D_1} \cdot \overline{B_1D_1}} = -\frac{p^2}{q^2 - p^2},$$

which is independent of the position of $R(x, y)$.

In the same manner it follows that each of the 24 cross-ratios is independent of the position of R.

Since the cross-ratio is unaltered by projection and section [7] it follows that the cross-ratios of the four concurrent lines RD_1, RP, RQ and RB_1 are also independent of the position of R.

Note. If $\frac{p}{q} = \frac{1}{\sqrt{2}}$, then the cross-ratio $(D_1Q,\ PB_1) = -1$ and we have the following theorem.

Theorem 2.2. *For any interior point R of the unit square S we can always find a rectangle $A_1B_1C_1D_1$ lying in S, with its vertices on C^2, such that R lies on the diagonal A_1C_1 dividing it in the ratio $1 : \sqrt{2}$ and the lines RD_1, RP, RQ and RB_1 always form a harmonic pencil, P, Q being on the other diagonal D_1B_1 dividing it in the ratio $1 : \sqrt{2}$ and $\sqrt{2} : 1$, respectively.*

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