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PSEUDOSEMIRINGS INDUCED BY ORTHOLATTICES

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It is well known (see e.g. [1]) that a unitary boolean ring can be assigned to every Boolean algebra and vice versa, and the operations are derived by the rules

$$x + y = (x \wedge y') \vee (x' \wedge y), \quad x \cdot y = x \wedge y$$

and

$$x \vee y = x + y + x \cdot y, \quad x \wedge y = x \cdot y, \quad x' = 1 + x,$$

where $\vee$, $\wedge$, $'$ are operations of the Boolean algebra and $+$, $\cdot$ are ring operations and 1 is the unit element in both of these algebras. This construction can be extended also to relatively complemented distributive lattices with zero and boolean rings which need not be unitary, see [1]. Another generalization is done in a similar manner for the so called q-algebras and boolean semirings, see [2].

The aim of this paper is to follow these considerations also for more general lattices with complementation, namely for ortholattices and orthomodular lattices. Since such lattices need not be modular or distributive, it is clear that the induced operations need not satisfy associativity or distributivity laws. Henceforth, we do not expect to obtain a semiring as a derived algebra but only a weaker form, the so called pseudosemiring.

An algebra $(L; \vee, \wedge, {}^{\perp}, 0, 1)$ of the type (2,2,1,0,0) is an *ortholattice* if $(L; \vee, \wedge)$ is a lattice with the least element 0 and the greatest element 1 satisfying the following identities:

(A) $(x^{\perp})^{\perp} = x$,

(B) $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$ and $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$
(the so called *De Morgan laws*),

(C) $x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$.

If it satisfies also the implication

$(*)$ $x \leqslant y \Rightarrow x \vee (x^{\perp} \wedge y) = y$, then $(L; \vee, \wedge, ^\perp, 0, 1)$ is called an *orthomodular lattice*. It is almost evident that $(*)$ can be replaced by the identity

(D) $(x \wedge y) \vee ((x \wedge y)^\perp \wedge y) = y$.

An example of an ortholattice which is not orthomodular is in Fig. 1.

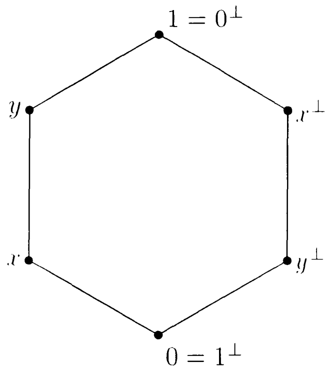

Fig. 1.

An example of an orthomodular lattice which is not modular is depicted in Fig. 2.

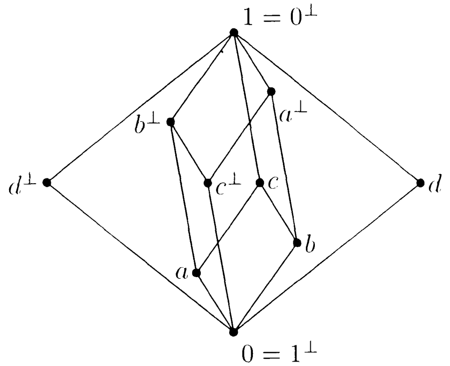

Fig. 2.

It is well known (see e.g. [3]) that if an orthomodular lattice is distributive then it is a boolean lattice. An example of a modular but non distributive orthomodular lattice is in Fig. 3.

Now, let $(A; +, \cdot, 0, 1)$ be an algebra of the type $(2, 2, 0, 0)$. This algebra is called a *pseudosemiring* if the operation $\cdot$ is *associative*, $+$ is *commutative* and the following laws hold:

(i) $x + 0 = x$, $x \cdot 0 = 0 \cdot x$, $x \cdot 1 = x = 1 \cdot x$,

(ii) $x + (1 + y) = (x + 1) + y$ (*weak associativity*).

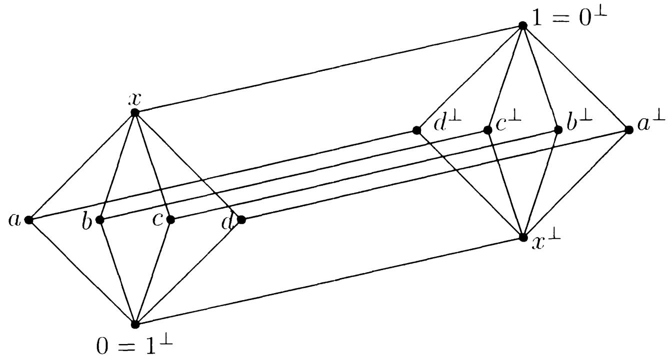

Fig. 3.

(iii) for each $x \in A$ there exists $y \in A$ such that $x + y = 0$
(*inverse element*),

(iv) $(1 + xy)x = x + xyx$ (*weak distributivity*).

A pseudosemiring is called *commutative* or *idempotent* if the operation $\cdot$ is commutative or idempotent, respectively.

A pseudosemiring is called an *orthopseudoring* if it is commutative, idempotent and satisfies the following identities:

(a) $x + x = 0$,

(b) $(1 + x)(1 + xy) = 1 + x$,

(c) $(1 + x(1 + y))(1 + y(1 + x)) = 1 + (x + y)$.

If, moreover, it satisfies also

(d) $(x + xy) + xy = x$,

it will be called an *orthomodular pseudoring.*

Lemma. *Let $(A; +, \cdot, 0, 1)$ be an orthopseudoring. Then the following identities hold:*

$$1 + (1 + x) = x, \tag{1}$$

$$(1 + x)x = 0. \tag{2}$$

Proof. Putting $y = 1$ in (ii), we obtain using also (a) $x = x + 0 = x + (1 + 1) = (x+1)+1 = 1+(1+x)$. If we put $y = 1$ in (iv), we have $(1+x)x = x+x\cdot x = x+x = 0$. □

Theorem 1. *Let* $\mathrm{L} = (L; \vee, \wedge, {}^{\perp}, 0, 1)$ *be an ortholattice. Put* $x + y = (x \wedge y^{\perp}) \vee (x^{\perp} \wedge y)$ *and* $x \cdot y = x \wedge y$. *Then* $\mathrm{P} = (L; +, \cdot, 0, 1)$ *is an orthopseudoring, called an* *orthopseudoring induced by* L. *If, moreover,* L *is an orthomodular lattice then* P *is an orthomodular pseudoring.*

Proof. Commutativity of $+$, associativity of $\cdot$ and the identities of (i) are evident. We can easily show that $1 + x = (1 \wedge x^{\perp}) \vee (0 \wedge x) = 1 \wedge x^{\perp} = x^{\perp}$. Let us prove (ii): $x + (1 + y) = (x \wedge [(1 \wedge y^{\perp}) \vee (0 \wedge y)]^{\perp}) \vee (x^{\perp} \wedge [(1 \wedge y^{\perp}) \vee (0 \wedge y)]) = (x \wedge y) \vee (x^{\perp} \wedge y^{\perp}) = (x^{\perp} \wedge y^{\perp}) \vee (x \wedge y) = x^{\perp} + y = (x + 1) + y$. Since $x + x = (x \wedge x^{\perp}) \vee (x^{\perp} \wedge x) = 0 \vee 0 = 0$ which proves (a), we have proved also (iii). As for (iv), it is enough to prove $(1 + xy)x = x + xy$ because of idempotence and commutativity of $\cdot$. Thus $x + xy = (x \wedge (x \wedge y)^{\perp}) \vee (x^{\perp} \wedge x \wedge y) = x \wedge (x \wedge y)^{\perp}$ and $(1 + xy)x = (x \wedge y)^{\perp} \wedge x$, i.e. (iv) holds.

Hence $(L; +, \cdot, 0, 1)$ is a commutative and idempotent pseudosemiring satisfying (a).

Let us prove (b): $1 + (1 + x)(1 + xy) = 1 + (x^{\perp} \wedge (x \wedge y)^{\perp}) = [x^{\perp} \wedge (x \wedge y)^{\perp}]^{\perp} = x \vee (x \wedge y) = x$; using (ii) we have $(1 + x)(1 + xy) = 0 + (1 + x)(1 + xy) = (1 + 1) + (1 + x)(1 + xy) = 1 + (1 + (1 + x)(1 + xy)) = 1 + x$, which proves (b). For (c), we can count $(1 + x(1 + y))(1 + y(1 + x)) = [(1 \wedge (x \wedge y^{\perp})^{\perp}) \vee (0 \wedge (x \wedge y^{\perp}))] \wedge [(1 \wedge (y \wedge x^{\perp})^{\perp}) \vee (0 \wedge (y^{\perp} \wedge x))] = (x \wedge y^{\perp})^{\perp} \wedge (y \wedge x^{\perp})^{\perp} = [(x \wedge y^{\perp}) \vee (x^{\perp} \wedge y)]^{\perp} = 1 + (x + y)$. We have proved now that $(L; +, \cdot, 0, 1)$ is an orthopseudoring.

If, moreover, L is an orthomodular lattice, then $(x + xy) + xy = [(x \wedge (x \wedge y)^{\perp}) \vee (x^{\perp} \wedge x \wedge y)] + (x \wedge y) = [(x \wedge (x \wedge y)^{\perp} \wedge (x \wedge y)^{\perp}] \vee [(x \wedge (x \wedge y)^{\perp})^{\perp} \wedge (x \wedge y)] = (x \wedge (x \wedge y)^{\perp}) \vee [(x^{\perp} \vee (x \wedge y)) \wedge (x \wedge y)] = (x \wedge (x \wedge y)^{\perp}) \vee (x \wedge y)$. By (d), it is equal to x, which completes the proof. □

Theorem 2. *Let* $\mathrm{P} = (L; +, \cdot, 0, 1)$ *be an orthopseudoring. Put* $x \vee y = 1 + (1 + x)(1 + y)$, $x \wedge y = x \cdot y$ *and* $x^{\perp} = 1 + x$. *Then* $\mathrm{L} = (L; \vee, \wedge, \perp, 0, 1)$ *is an ortholattice. If, moreover,* P *is also an orthomodular pseudoring, the* L *is an orthomodular lattice.* L *is called an orthomodular lattice induced by* P.

Proof. Commutativity, associativity and idempotence of $\wedge$ as well as commutativity of $\vee$ are trivial. Let us prove associativity of $\vee$:

$$\begin{aligned} x \vee (y \vee z) &= 1 + (1 + x)(1 + (1 + (1 + y)(1 + z))) && \text{by (1)} \\ &= 1 + (1 + x)(1 + y)(1 + z) && \text{by (1) once more} \\ &= 1 + (1 + z)(1 + (1 + (1 + x)(1 + y))) \\ &= z \vee (x \vee y) = (x \vee y) \vee z. \end{aligned}$$

Further, by (1),

$$(x^{\perp})^{\perp} = 1 + (1 + x) = x,$$

by (2) we obtain $x \wedge x^\perp = x(1+x) = 0$ and, by (1) and (2), $x \vee x^\perp = 1 + (1+x)(1+(1+x)) = 1 + (1+x)x = 1 + 0 = 1$.
Let us prove De Morgan laws: by (1) we have

$$(x \vee y)^\perp = 1 + (1 + (1+x)(1+y)) = (1+x)(1+y) = x^\perp \wedge y^\perp.$$

Using this law with $(x^\perp)^\perp = x$, we obtain the second law:

$$x^\perp \vee y^\perp = ((x^\perp \vee y^\perp)^\perp)^\perp = (x \wedge y)^\perp.$$

It remains to prove the absorption laws: by (b) and (1) we immediately obtain

$$x \vee (x \wedge y) = 1 + (1+x)(1+xy) = 1 + (1+x) = x.$$

Using this law and the De Morgan law, we infer

$$x \wedge (x \vee y) = [x^\perp \vee (x^\perp \wedge y^\perp)]^\perp = (x^\perp)^\perp = x,$$

thus L is an ortholattice.

If, moreover, P is orthomodular, we infer by (d) $(x \wedge (x \wedge y)^\perp) \vee (x \wedge y) = (x \wedge (x \wedge y)^\perp) \vee (1 \wedge (x \wedge y)) = ((x \wedge (x \wedge y)^\perp) \wedge (x \wedge y)^\perp) \vee ((x \wedge (x \wedge y)^\perp)^\perp \wedge (x \wedge y)) = [(x \wedge (x \wedge y)^\perp) \vee (x^\perp \wedge x \wedge y)] + xy = (x + xy) + xy = x$. According to (d), L is an orthomodular lattices. □

Remark 1. Orthomodular pseudorings do not satisfy $(x+y)+y = x+(y+y)$ or $x(1+y) = x + xy$ in the general case. If e.g. L is the orthomodular (and modular) lattice visualized in Fig. 3 and P the orthomodular pseudoring induced by L, then $(a + c^\perp) + c^\perp = [(a \wedge c) \vee (a^\perp \wedge c^\perp)] + c^\perp = x^\perp + c^\perp = (x^\perp \wedge c) \vee (x \wedge c^\perp) = b$, but $a + (c^\perp + c^\perp) = a + 0 = a \neq b$. Also $a(1 + c^\perp) = a \wedge c = 0 \neq a = a + ac^\perp$. Of course, if L is distributive, then it is boolean and both of the foregoing identities are satisfied in the induced pseudoring (which is in this case a boolean ring).

Theorem 3. *Let* L *be an ortholattice,* P(L) *the induced orthopseudoring and* L(P(L)) *the ortholattice induced by* P(L). *Then*

$$\mathrm{L} = \mathrm{L}(\mathrm{P}(\mathrm{L})).$$

Let P *be an orthopseudoring,* L(P) *the induced ortholattice and* P(L(P)) *the orthopseudoring induced by* L(P). *Then*

$$\mathrm{P} = \mathrm{P}(\mathrm{L}(\mathrm{P})).$$

Proof. Denote by $\curlyvee$, $\curlywedge$ and $*$ the operations in L(P(L)) and by $\vee$, $\wedge$, $\perp$ those L. Evidently, $\curlywedge = \wedge$ and $x^* = (1+x) = x^\perp$, which proves $* = \perp$. Further, $x \curlyvee y = 1 + (1+x)(1+y) = (1 \wedge [(1+x)(1+y)]^\perp) \vee (0 \wedge [(1+x)(1+y)]^\perp) = [(1+x)(1+y)]^\perp = (x^\perp \wedge y^\perp)^\perp = x \vee y$, thus also $\curlyvee = \vee$, which proves the first assertion.

Further, denote by $\oplus$ and $\odot$ the operations in P(L(P)) and by $+$, $\cdot$ those in P. Trivially, $\odot = \cdot$ and, by (c) and (1), we obtain $x \oplus y = (x \wedge y^\perp) \vee (x^\perp \wedge y) = x(1+y) \vee y(1+x) = 1 + (1 + x(1+y))(1+y(1+x)) = 1 + (1 + (x+y)) = x + y$, thus proving the second assertion. □

Remark 2. We cannot introduce the operation $\vee$ in the induced ortholattice by the rule

$$x \vee y = (x+y) + xy$$

similarly as in boolean rings. In such a case, we have

$$x \curlyvee y = (x+y) + xy = (x \wedge y^\perp) \vee (x^\perp \wedge y) \vee (x \wedge y)$$

but the last term can be different from $x \vee y$. For instance for the ortholattice in Fig. 2 we have $a \vee d = 1$ but

$$(a \wedge d^\perp) \vee (a^\perp \wedge d) \vee (a \wedge d) = 0 \vee 0 \vee 0 = 0.$$

Because of lack of associativity of $+$, we can try to introduce $\vee$ in another way similar to that of boolean rings, namely $x \vee y = x + (y + xy)$. In also leads to a contradiction, as we can see in the ortholattice given in Fig. 1:

$$y = x \curlyvee y = x + (y + xy) = x + (y + x) = x + [(y \wedge x^\perp) \vee (y^\perp \wedge x)] = x + 0 = x.$$

Remark 3. Let $(A; +, \cdot, 0, 1)$ be an orthopseudoring. If $\operatorname{card} A = 2$, i.e. $A = \{0, 1\}$, then, by (i) and (ii), $(A; +, \cdot, 0, 1)$ is the two element boolean ring and hence the induced ortholattice is the two element boolean lattice.

If $\operatorname{card} A > 2$ then the operation $+$ need not be associative and it need not satisfy the distributive law. However, the groupoid $(A; +)$ is a union of four element Klein groups. Namely, for any $a \in A$ we have

$+$	0	a	$a^\perp$	1
0	0	a	$a^\perp$	1
a	a	0	1	$a^\perp$
$a^\perp$	$a^\perp$	1	0	a
1	1	$a^\perp$	a	0

where $a^\perp$ denotes $1 + a$.

Example. Let A $=\{0,x,y,x^{\perp},y^{\perp},1\}$ and L $=(A;\vee,\wedge,\perp,0,1)$ be the ortholattice whose diagram is in Fig. 1. Denote by $+,\cdot$ the operations of the induced orthopseudoring P(A). Then the operation $+$ is not associate since $x+(x+y)=x+0=x\neq y=0+y=(x+x)+y$ and P(A) is not distributive:

$$x^{\perp}(1+y)=x^{\perp}\wedge y^{\perp}=y^{\perp}\neq x^{\perp}=x^{\perp}+0=x^{\perp}\cdot 1+x^{\perp}\cdot y.$$

The operation table of $+$ is the following:

$+$	0	x	y	$x^{\perp}$	$y^{\perp}$	1
0	0	x	y	$x^{\perp}$	$y^{\perp}$	1
x	x	0	0	1	1	$x^{\perp}$
y	y	0	0	1	1	$y^{\perp}$
$x^{\perp}$	$x^{\perp}$	1	1	0	0	x
$y^{\perp}$	$y^{\perp}$	1	1	0	0	y
1	1	$x^{\perp}$	$y^{\perp}$	x	y	0

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