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ABSTRACT FORMULATION OF SOME THEOREMS OF MEASURE THEORY II

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In the paper we formulate and prove the following three theorems only with the help of some properties of the systems $\mathscr{N}_n$ of all measurable sets with a measure less than $1/n$: Vitali's covering theorem, the assertion that the system of all measurable sets of finite measure is a complete pseudometric space (with the pseudometric $\varrho(E, F) = \mu(E \triangle F)$) and the theorem on approximation of a measure.

Note that some other theorems of measure theory were generalized by a similar way in the author's paper [1] and in the paper [2] by T. Neubrunn.

We shall assume that there are given a σ-ring $\mathscr{S}$ of subsets of a set X and a sequence $\{\mathscr{N}_n\}_{n=0}^{\infty}$ of subsystems of $\mathscr{S}$. We shall assume, if it is convenient, that $\{\mathscr{N}_n\}_{n=0}^{\infty}$ satisfies some of the following axioms:

(1) $\emptyset \in \mathscr{N}_n$ for all n.

(2) To any positive integer n there is an increasing sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that $\bigcup_{i=1}^{\infty} E_i \in \mathscr{N}_n$ as soon as $E_i \in \mathscr{N}_{k_i}$.

(3) Let $\{E_i\}_{i=1}^{\infty}$ be an arbitrary non decreasing sequence of sets of $\mathscr{N}_0$, and $\bigcap_{i=1}^{\infty} E_i = \emptyset$. Then to any positive integer n there is a positive integer m such that $E_m \in \mathscr{N}_n$.

(4) If $E \subset F$, $E \in \mathscr{S}$, $F \in \mathscr{N}_n$, then $E \in \mathscr{N}_n$ $(n = 0, 1, 2, \ldots)$.

(5) $\mathscr{N}_{n+1} \subset \mathscr{N}_n$ for any positive integer n.

If $(X, \mathscr{S}, \mu)$ is a measure space, $\mathscr{N}_0 = \{E \in \mathscr{S} : \mu(E) < \infty\}$, $\mathscr{N}_n = \{E \in \mathscr{S} : \mu(E) < 1/n\}$, then we easily find out that all the conditions (1)—(5) are satisfied. In section 1 we shall use a more special condition (the condition (V)) connected with Vitali's covering theorem. In section 2 we shall use instead of (2) the following stronger condition:

(2′) There is a sequence $\{k_i\}_{i=0}^{\infty}$ of positive integers such that $\bigcup_{i=N+1}^{\infty} E_i \in \mathscr{N}_{k_N}$ whenever $E_i \in \mathscr{N}_{k_i}$ $(i = N + 1, \ldots)$.

We see that (2) follows from (5) and (2′). It is evident that (2′) is satisfied also for the above choice of $\{\mathcal{N}_n\}$. We shall often use also the following consequence of (1) and (2):

(2'') To any positive integer n there are positive integers p, q such that $E \in \mathcal{N}_p$, $F \in \mathcal{N}_q$ imply $E \cup F \in \mathcal{N}_n$.

1

Vitali's theorem which we are just going to prove, is usually formulated for outer measures. That is why we dinstinguish in the used axioms two σ-rings $\mathcal{S}$ and $\mathcal{B}$. Let $\mathcal{S}$ be the system of all subsets of the k-dimensional Euclidean space X, let $\mathcal{B}$ be the system of all Borel subsets of X. We shall assume that $\mathcal{N}_n$ are subsystems of $\mathcal{S}$, but (3) is satisfied only for such sequences $\{E_i\}_{i=1}^{\infty}$, for which $E_i \in \mathcal{B} \cap \mathcal{N}_0$. Hence, the following property is satisfied:

(3') Let $\{E_i\}_{i=1}^{\infty}$ be any non increasing sequence of sets of $\mathcal{B} \cap \mathcal{N}_0$, and $\bigcap_{i\ 1}^{\infty} E_i = \emptyset$. Then to any n there is an m such that $E_m \in \mathcal{N}_n$.

If E is a sphere in X, then by $5E$ we shall denote the sphere with the same centre but with a 5 times larger diameter.

Theorem 1. *Let $\{\mathcal{N}_n\}_{n=0}^{\infty}$ be a sequence of subsystems of the σ-algebra $\mathcal{S}$ satisfying the conditions* (1), (3') *and* (4). *Let $\mathcal{N}_0$ contain all bounded sets. Let $\mathcal{K}$ be any Vitali covering*(1) *of a bounded set $A \in \mathcal{S}$ by closed spheres. Moreover let $\{\mathcal{N}_n\}$ satisfy the following condition:*

(V) *To any positive integer m there is a positive integer k such that $\bigcup_{i=1}^{\infty} 5E_j \in \mathcal{N}_m$ whenever $\{E_n\}_{n=1}^{\infty}$ is a sequence of sets of $\mathcal{K}$ such that $\bigcup_{j=1}^{\infty} E_j \in \mathcal{N}_k, E_i \cap E_j = \emptyset, i \neq j$.*

Then there is a sequence $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets of $\mathcal{K}$ such that $A - \bigcup_{i=1}^{\infty} E_i \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$.

Proof. Since A is bounded, there is an open sphere F such that $A \subset F$. We may assume that $\mathcal{K}$ is a system of subsets of F. Put $d_1 = \sup \{\operatorname{diam}(E): E \in \mathcal{K}, E \subset F\}$ and choose $E_1 \in \mathcal{K}$ with $\operatorname{diam}(E_1) > d_1/2$. Assume now that we have constructed sets $E_1, \ldots, E_{n-1} \in \mathcal{K}$ such that $E_i \cap E_j = \emptyset$ $(i \neq j)$ and

$$\operatorname{diam} E_i > \tfrac{1}{2} \sup \{\operatorname{diam}(E): E \in \mathcal{K}, E \subset F - \bigcup_{j=1}^{i-1} E_j\} \quad (i = 1, \ldots, n-1).$$

If we put

$$(6) \quad d_n = \sup \{\operatorname{diam}(E): E \in \mathcal{K}, E \subset F - \bigcup_{i=1}^{n-1} E_i\},$$

(1) I. e. to any $r > 0$ and any $x \in A$ there is $E \in \mathcal{K}$ such that $x \in E$ and $\operatorname{diam}(E) < r$.

we can choose E_n such that $E_n \in \mathscr{K}$, $E_n \subset F - \bigcup_{i=1}^{n-1} E_i$ and

(7) $\operatorname{diam}(E_n) > \frac{d_n}{2}$.

By this process we have constructed a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint sets of $\mathscr{K}$ satisfying the conditions (6) and (7).

By a standard way ([3]) we prove that

(8) $A - \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=p}^{\infty} 5E_n$

for any positive integer p. First let L be the Lebesgue measure in X. Since $\sum_{n=1}^{\infty} L(E_n) \leqq L(F) < \infty$, we get $\lim_{n\to\infty} L(E_n) = 0$ and hence also

(9) $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$.

Let $x \in A - \bigcup_{n=1}^{\infty} E_n \subset F - \bigcup_{i=1}^{p} E_i$. Since $F - \bigcup_{i=1}^{p} E_i$ is open and $\mathscr{K}$ is a Vitali covering of A, there exists $E \in \mathscr{K}$ such that $x \in E \subset F - \bigcup_{i=1}^{p} E_i$. According to (9) and (7) there is a q such that $E \cap \bigcup_{i=1}^{q} E_i \neq \emptyset$. Let r be the least positive integer for which $E \cap E_r \neq \emptyset$. Evidently $r > p$ and $\operatorname{diam}(E) < 2 \operatorname{diam}(E_r)$ according to (7). Therefore $E \subset 5E_r$, hence $x \in \bigcup_{n=p}^{\infty} 5E_n$, which proves the inclusion (8).

Let m be any positive integer; choose k according to (V). Put $A_q = \bigcup_{n=q}^{\infty} E_n$. Evidently $A_q \in \mathscr{B}$; $A_q \in \mathscr{N}_0$, since $\mathscr{N}_0$ contains all bounded sets. Besides $\bigcap_{n=p}^{\infty} A_q = \emptyset$ and $A_q \supset A_{q+1}$ ($q = 1, 2, \ldots$). Therefore by (3') there is a p such that $\bigcup_{n=p}^{\infty} E_n = A_p \in \mathscr{N}_k$. The property (V) implies $\bigcup_{n=p}^{\infty} 5E_n \in \mathscr{N}_m$, hence by (8) and (4) we have $A - \bigcup_{n=1}^{\infty} E_n \in \mathscr{N}_m$ for any m, which proves our assertion.

Corollary. *Let μ be an outer measure in X, that is a measure on $\mathscr{B}$, finite on bounded sets. Let A be a bounded set, $\mathscr{K}$ be a system of closed spheres covering A in the Vitali sense. Let $\mathscr{K}$ satisfy the following condition:*

(V') *There is $\alpha > 0$ such that $\mu(5E) \leqq \alpha\mu(E)$ for all $E \in \mathscr{K}$.*

Then there is a sequence $\{E_n\}$ of pairwise disjoint sets of $\mathscr{K}$ such that $\mu(A - \bigcup_{n=1}^{\infty} E_n) = 0$.

Note. Vitali's covering theorem (a variant of which has been just presented) has an interesting character. While its assumptions are topological, its assertion is metric. A. Denjoy in some papers (e. g. [4], [5]) formulated and proved Vitali's theorem only in metric terms. Conversely our Theorem 1 is formulated only in topological terms.

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In this section we shall assume that X is an arbitrary space and $\mathcal{S}$ is a σ-ring of subsets of X.

Theorem 2. *Let $\{\mathcal{N}_n\}_{n=0}^{\infty}$ satisfy the conditions* (1), (2″), (4) *and* (5). *Then the system $\mathcal{U}$ of all sets of the form $\{(E, F): E \triangle F \in \mathcal{N}_n\}$ $(n = 1, 2, \ldots)$ is a base of a uniformity of $\mathcal{N}_0$. Since the uniform space has a countable base, $\mathcal{N}_0$ is pseudometrizable.*(²)

Moreover if $\{\mathcal{N}_n\}_{n=0}^{\infty}$ satisfies (2′) *and* (3) *and $\mathcal{N}_0$ is closed under the sums, then $\mathcal{N}_0$ is complete.*

Proof. First we prove that any element of $\mathcal{U}$ contains the diagonal:

(10) $\{(E, E): E \in \mathcal{N}_0\} \subset U$ for each $U \in \mathcal{U}$.

This follows from the condition (1), since $E \triangle E = \emptyset \in \mathcal{N}_n$ for all n. From the definition of $\mathcal{U}$ we get

(11) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$,

where $U^{-1} = \{(E, F): (F, E) \in U\}$. In our case $U^{-1} = U$. Let $U \in \mathcal{U}$, $U = \{(E, F): E \triangle F \in \mathcal{N}_n\}$. Choose p, q according to (2″) and put $m = \max(p, q)$, $V = \{(E, F): E \Delta F \in \mathcal{N}_m\}$. Then $M, N \in \mathcal{N}_m \Rightarrow M \cup N \in \mathcal{N}_n$ according to (2″) and (5). If as usually we denote by $V \circ V$ the set $\{(E, F)$: there is $G \in \mathcal{N}_0$, $(E, G) \in V$, $(G, F) \in V\}$, we get

$$V \circ V = \{(E, F): \text{there is } G \in \mathcal{N}_0,\ E \triangle G \in \mathcal{N}_m,\ G \triangle F \in \mathcal{N}_m\} \subset$$
$$\subset \{(E, F): \text{there is } G \in \mathcal{N}_0,\ (E \triangle G) \cup (G \triangle F) \in \mathcal{N}_n\}.$$

Since $E \triangle F \subset (E \triangle G) \cup (G \triangle F) \in \mathcal{N}_n$, we have by (4) $E \triangle F \in \mathcal{N}_n$, hence $V \circ V \subset U$. We have proved the following:

(12) To any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \circ V \subset U$.

From (5) we get also the following property:

(13) $U \cap V \in \mathcal{U}$ for any $U, V \in \mathcal{U}$.

From the properties (10)—(13) it follows that $\mathcal{U}$ is a base of a uniformity ([6], chap. VI, th. 2, p. 177).

(²) [6], chap. VI, th. 13, p. 186.

In order to prove the completness of $\mathcal{N}_0$ it suffices to prove that any Cauchy sequence is convergent ([6], chap. VI, th. 24, p. 193). Let $\{E_n\}_{n=1}^{\infty}$ be a Cauchy sequence, i. e. to any k there is an N such that $E_n \triangle E_m \in \mathcal{N}_k$ for $m, n > N$.

Let $\{k_i\}$ be a sequence according to (2′). Since $\{E_n\}$ is a Cauchy sequence there is an increasing sequence of positive integers $\{n_i\}$ such that $E_{n_i} \triangle E_{n_{i+1}} \in \in \mathcal{N}_{k_i}$ $(i = 1, 2, \ldots)$. Put $F_i = E_{n_i}$ (hence $F_i \triangle F_{i+1} \in \mathcal{N}_{k_i}$) and put

$$E = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} F_i .$$

The set E is a member of $\mathcal{N}_0$, since $\bigcup_{i=1}^{\infty} (F_i \triangle F_{i+1}) \in \mathcal{N}_{k_0}$ (according to (2′)) and $E \subset F_1 \cup \bigcup_{i=1}^{\infty} (F_i \triangle F_{i+1})$. Evidently

$$(14) \quad E \triangle F_n \subset (E \triangle \bigcap_{i=n}^{\infty} F_i) \cup (\bigcap_{i=n}^{\infty} F_i \triangle F_n).$$

Let m be any positive integer. Choose p, q according to (2″). Put $V_n = E - \bigcap_{i=n}^{\infty} F_i$. Evidently $\bigcap_{n=1}^{\infty} V_n = \emptyset$, $V_n \supset V_{n+1}$, $V_1 \in \mathcal{N}_0$. Therefore by (3) there is N_1 such that

$$(15) \quad E - \bigcap_{i=n}^{\infty} F_i \in \mathcal{N}_p$$

for all $n > N_1$. Now notice that $F_n - \bigcap_{i=n}^{\infty} F_i \subset \bigcup_{i=n}^{\infty} (F_i - F_{i+1})$. Hence by (2′) we have

$$F_n - \bigcap_{i=n}^{\infty} F \in \mathcal{N}_{k_{n-1}} .$$

Choose N_2 such that $k_{N_2-1} > q$. Then $\mathcal{N}_{k_{n-1}} \subset \mathcal{N}_q$ for $n > N_2$, hence

$$(16) \quad F_n - \bigcap_{i=n}^{\infty} F_i \in \mathcal{N}_q .$$

From the relations (14)—(16) we get $E \triangle F_n \in \mathcal{N}_m$ for all $n > N_3 = \max(N_1, N_2)$. Hence we proved that to any m there is an N_3 such that

$$E \triangle E_{n_i} \in \mathcal{N}_m$$

for all $n > N_3$. Let u be an arbitrary positive integer, r, s be such that $G \cup H \in \in \mathcal{N}_u$ whenever $G \in \mathcal{N}_r$, $H \in \mathcal{N}_s$. By the foregoing there is an N_3 such that

$$E \triangle E_{n_i} \in \mathcal{N}_r$$

for all $i > N_3$. Since $\{E_n\}$ is a Cauchy sequence, there is an N_4 such that

$E_i \triangle E_k \in \mathcal{N}_s$ for all $i, k > N_4$. Put $N = \max(N_3, N_4)$. Then $n_i > N_4$ for $i > N$, hence $E_i \triangle E_{n_i} \in \mathcal{N}_s$. It follows that

$$E \triangle E_i \subset (E \triangle E_{n_i}) \cup (E_{n_i} \triangle E_i) \in \mathcal{N}_u .$$

It means that $E_i \to E$ in the uniform topology of the space $\mathcal{N}_0$, hence $\mathcal{N}_0$ is complete.

Corollary. *The pseudometric space of all sets of finite measure with the pseudometric $\varrho(E, F) = \mu(E \triangle F)$ is complete.*

3

Now we generalize the theorem on the approximation of a measure ([7]).

Theorem 3. *Let $\{\mathcal{N}_n\}_{n=0}^{\infty}$ satisfy the conditions* (1), (2), (3) *and* (4). *Let $E \cup F \in \mathcal{N}_0$ whenever $E, F \in \mathcal{N}_0$. Let $\mathcal{R}$ be a ring, $\mathcal{S}$ be the σ-ring generated by $\mathcal{R}$. Let to any $E \in \mathcal{R}$ exist $E_i \in \mathcal{R} \cap \mathcal{N}_0$ $(i = 1, 2, \ldots)$ such that $E \subset \bigcup_{i=1}^{\infty} E_i$.*

Then to any n and any $E \in \mathcal{N}_0$ there is $F \in \mathcal{R}$ such that $E \triangle F \in \mathcal{N}_n$.

Proof. First consider $\mathcal{P} = \mathcal{R} \cap \mathcal{N}_0$. $\mathcal{P}$ is a ring and the σ-ring $\mathcal{S}(\mathcal{P})$ generated by $\mathcal{P}$ is $\mathcal{S}$. Take a fixed $G \in \mathcal{P}$ and consider the system M of all $H \in \mathcal{S}$ with the following property: To any n there is $F \in \mathcal{P}$ such that $(H \cap G) \triangle F \in \mathcal{N}_n$. Clearly $M \supset \mathcal{R}$. Namely if $H \in \mathcal{R}$, then put $F = H \cap G$ and use (1). Prove that M is a σ-ring hence that $M \supset \mathcal{S}$.

Let $H_i \in M$ $(i = 1, 2, \ldots)$, n be an arbitrary positive integer. First construct p, q such that $A \cup B \in \mathcal{N}_n$ whenever $A \in \mathcal{N}_p$, $B \in \mathcal{N}_q$. To the number p construct a sequence $\{k_i\}$ according to (2). Since $H_i \in M$, there are $F_i \in \mathcal{R}$ such that

$$(H_i \cap G) \triangle F_i \in \mathcal{N}_{k_i} .$$

Hence we get

$$(17)\quad \bigcup_{i=1}^{\infty} (H_i \cap G) \triangle F_i \in \mathcal{N}_p .$$

Further consider the sequence $B_k = \bigcup_{i=1}^{\infty} (G \cap F_i) - \bigcup_{i=1}^{k} (G \cap F_i)$. Evidently $B_k \in \mathcal{N}_0$, $B_k \supset B_{k+1}$, $\bigcap_{k=1}^{\infty} B_k = \emptyset$. Therefore by (3) there is an N such that

$$(18)\quad B_N = \bigcup_{i=1}^{\infty} (G \cap F_i) - \bigcup_{i=1}^{N} (G \cap F_i) \in \mathcal{N}_q .$$

We easily check the inclusion

$$(19) \quad [(\bigcup_{i=1}^{\infty} H_i) \cap G] \vartriangle (\bigcup_{i=1}^{N} F_i) \subset [\bigcup_{i=1}^{\infty} (H_i \cap G) \vartriangle F_i] \cup [\bigcup_{i=1}^{\infty} (G \cap F_i) - \bigcup_{i=1}^{N} (G \cap F_i)].$$

The relations (17)—(19) imply $[(\bigcup_{i=1}^{\infty} H_i) \cap G] \vartriangle (\bigcup_{i=1}^{N} F_i) \in \mathcal{N}_n$, hence $\bigcup_{i=1}^{\infty} H_i \in M$ according to $\bigcup_{i=1}^{N} F_i \in \mathcal{R}$. The fact that M is closed with respect to differences can be proved similarly by the help of the inclusion

$$[(E_1 - E_2) \cap G] \vartriangle [(F_1 - F_2)] \cap G] \subset [(E_1 \cap G) \vartriangle F_1] \cup [(E_2 \cap G) \vartriangle F_2].$$

The inclusion $M \supset \mathcal{S}$ implies the following: To any $G \in \mathcal{P}$ and $E \in \mathcal{S}$ there is $F \in \mathcal{R}$ such that $(E \cap G) \vartriangle F \in \mathcal{N}_n$.

Let $E \in \mathcal{N}_0$. Let $\mathcal{K}$ be now the system of all sets $G \in \mathcal{S}$ with the following property: To any n there is $F \in \mathcal{R}$ such that $(E \cap G) \Delta F \in \mathcal{N}_n$. By the foregoing we have $\mathcal{K} \supset \mathcal{P}$. We show that $\mathcal{K}$ is a σ-ring similarly as we showed it to M. Hence $\mathcal{K} \supset \mathcal{S}$, and to any n there is $F \in \mathcal{R}$ such that $E \vartriangle F = (E \cap E) \vartriangle F \in \mathcal{N}_n$. The Theorem is proved.

Corollary. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathcal{R} \subset \mathcal{S}$ be a ring, $\mathcal{S}$ be generated by $\mathcal{R}$, μ be σ-finite on $\mathcal{R}$. Then to any set $E \in \mathcal{S}$ of finite measure and any $\varepsilon > 0$ there is $F \in \mathcal{R}$ such that $\mu(E \vartriangle F) < \varepsilon$.*

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