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ON THE EMBEDDING OF SEMIGROUPS INTO 2-ENGEL GROUPS

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The following definitions and notation are presented for the convenience of the reader. If x and y are elements of a group G, the element $[x, y] = x^{-1}y^{-1}xy$ is the commutator of x and y. For elements x, y and z belonging to the group G, $[[x, y], z]$ is the commutator of the elements $[x, y]$ and z and is denoted by $[x, y, z]$. $y^{-1}xy$ is the conjugate (or transform) of x by y; it is denoted by x^y. The reader is also referred to P. Hall [1] and E. Schenkman [3, Chapter VI] for basic techniques of the commutator calculus.

A subsemigroup S of a group G is said to satisfy the 2-Engel condition if $[x, y, y] = 1$ in $gp\{S\}$ for all $x, y \in S$, where $gp\{S\}$ denotes the group generated by S in G and 1 is the identity element of G. Accordingly, a group is 2-Engel if it satisfies the 2-Engel condition.

Neumann—Taylor [2, p. 1] have shown that a semigroup S can be embedded in a nilpotent group of class $\leqslant 2$ if, and only if, it is cancellative and satisfies the law $xyzyx = yxzxy$ for all $x, y, z \in S$. It is natural then to consider the law $xyyx = yxxy$ for all x, y belonging to S. This law will be referred to as the E_2 law.

The purpose of this paper is to prove the following.

Theorem. *A semigroup can be embedded in a 2-Engel group if and only if it is cancellative and satisfies the E_2 law.*

Proof. If S is a subsemigroup of a 2-Engel group, then S is cancellative. The commutator $[x, y]$ commutes with y since $1 = [x, y, y] = [x, y]^{-1}y^{-1}[x, y]y$. $[x, y]$ also commutes with x since its inverse $[y, x]$ commutes with x as indicated by the fact that $1 = [y, x, x]$. Consequently, $xy = yx\,[x, y] = [x, y]yx = x^{-1}y^{-1}xyyx$ and $yxxy = xyyx$ for all x, y in S.

Thus, it remains to show that a cancellative semigroup satisfying the E_2 law may be embedded in a 2 Engel group. It will in fact be shown that such a semigroup generates a 2-Engel group of the form $S \, . \, S^{-1}$.

(1) Since S is a cancellative semigroup satisfying a non-tautological law, it follows from Neumann—Taylor [1, p. 1] that $gp\{S\} = S \,.\, S^{-1} = S^{-1} \,.\, S$. Thus the group generated by S exists and its elements are all of the form st^{-1} for $s,\ t \in S$.

(2) The elements of S satisfy the 2-Engel condition.

It suffices to demonstrate that $yxy^{-1}x^{-1}y^{-1}xyy = xy$ since this implies that $[x, y, y] = 1$. Using the E_2 law, $x \,.\, xyxy \,.\, x = xy \,.\, xx \,.\, xy$ and $xyxyx = yxxxy$. Then,

$$
\begin{aligned}
yxxy \,.\, y^{-1}xy &= xyxyx,\\
xyyx \,.\, y^{-1}xy &= xyxyx,\\
yx \,.\, y^{-1}xy &= xyx,\\
yxy^{-1}x^{-1}y^{-1}yx \,.\, xy &= xyx,\\
yxy^{-1}x^{-1}y^{-1} \,.\, xyyx &= xyx.
\end{aligned}
$$

Hence, $yxy^{-1}x^{-1}y^{-1}xyy = xy$ for all x, y in S.

(3) $[ab^{-1}, c, c] = 1$ for all a, b, c in S.

$$
\begin{aligned}
&[ab^{-1}, c, c] = [b^{-1}a[a, b^{-1}], c, c] = [b^{-1}[a, b^{-1}]a, c, c] =\\
&= [b^{-1}[b, a]a, c, c] = [b^{-1}b^{-1}a^{-1}baa, c, c] = [[b^{-1}b^{-1}a^{-1}, c]^{baa}\,[baa, c], c] =\\
&= [[b^{-1}b^{-1}a^{-1}, c]^{baa-c}, c] \,.\, [baa, c, c] = [[c, abb]^{baa}, c] = 1.
\end{aligned}
$$

The last equality follows from the fact that $1 = [x, yz, x]$ implies $[[x, y]^z, x] = 1$ for all x, y, z in S.

(4) It will now be shown that $[a, cd^{-1}, cd^{-1}] = 1$ for all $a, c, d \in S$.

$$
\begin{aligned}
&[a, cd^{-1}, cd^{-1}] = [[a, d^{-1}] \,.\, [a, c]^{d^{-1}}, cd^{-1}] =\\
&= [[a, d^{-1}] \,.\, [a, c]^{d^{-1}}, d^{-1}] \,.\, [[a, d^{-1}] \,.\, [a, c]^{d^{-1}}, c]^{d^{-1}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&[a, cd^{-1}, cd^{-1}]^d = ([[a, c]^{d^{-1}}, d^{-1}] \,.\, \{[[a, d^{-1}], c]^{[a,c]^{d^{-1}}}\})^d =\\
&= d^{-1} \,.\, d[c, a]d^{-1} \,.\, d \,.\, d[a, c]d^{-1} \,.\, d^{-1} \,.\, d\{[[a, d^{-1}], c]^{[a,c]^{d^{-1}}}\}d^{-1} \,.\, d =\\
&= [[a, c], d^{-1}] \,.\, [[a, d^{-1}], c]^{[a,c]^{d^{-1}}}.
\end{aligned}
$$

Both sides are now conjugated by $[c, a]^{d^{-1}}$ so that

$$
\begin{aligned}
&[a, cd^{-1}, cd^{-1}]^{d \,.\, [c,a]^{d^{-1}}} =\\
&= [a, c]^{d^{-1}}[[a, c], d^{-1}][c, a]^{d^{-1}} \,.\, [[a, d^{-1}], c] =\\
&= d[a, c]d^{-1}[c, a]d[a, c]d^{-1} \,.\, d[c, a]d^{-1}[a, d^{-1}, c] =\\
&= [d^{-1}, [c, a]] \,.\, [a, d^{-1}, c] = [[c, a], d] \,.\, [d, a, c].
\end{aligned}
$$

Part (3) is used to show that $[d^{-1}, [c, a]] = [[c, a], d]$ and a slight modification of the proof to Part (6) which follows implies that $[d, a, c] = [a, c, d]$.

Hence,

$$
\begin{aligned}
&[a, cd^{-1}, cd^{-1}]^{d \,.\, [c,a]^{d^{-1}}} = [[c, a], d] \,.\, [[a, c], d] =\\
&= [a, c]d^{-1}[c, a]d \,.\, [c, a]d^{-1}[a, c]d = [a, c]d^{-1}[c, a]d \,.\, d^{-1}[a, c]d[c, a] =\\
&= 1 \text{ which gives the result.}
\end{aligned}
$$

The second to the last equality is true because $[a, c]^d$ commutes with a and c, a fact indicated by the last statement in the proof of part (3).

(5) $[[g, b]^c, b] = 1$ and $[[b, g]^c, g] = 1$ for $g \in gp\{S\}$ and $b,\ c \in S$.

$$1 = [gc, b, b] = [[g, b]^c[c, b], b] = [\, g, b]^c, b]$$

using (3) and the fact that S satisfies the E_2-law. Using (4) and the E_2-law,

$$1 = [bc, g, g] = [[b, g]^c[c, g], g] = [[b, g]^c, g].$$

(6) $[b, g, c] = [g, c, b]$ for $g \in gp\{S\}$ and $b, c \in S$.

From Part (5) it is known that $[[g, b]^c, b] = 1$. According to (3)

$$1 = [g, bc, bc] = [[g, c][g, b]^c, bc] =$$

$$= [[g, c][g, b]^c, c][[g, c][g, b]^c, b]^c.$$

Using (3) once again and the initial statement above, the following is obtained:

$$1 = [[g, b]^c, c]\{[g, c, b]^{[g,b]^c}\}^c.$$

Transforming this last expression by c^{-1},

$1 = [g, b, c][g, c, b]^{[g,b]^c}$. A second conjugation by $[b, g]^c$ yields $1 = [c,[b, g]] \,.$ $.\, [g, c, b]$ or $[b, g, c] = [g, c, b]$.

(7) If S is a cancellative semigroup which satisfies the E_2 law, then S generates a 2-Engel group.

According to Part (1) $gp\{S\} = S \,.\, S^{-1}$ and it will be shown that $[ab^{-1}, cd^{-1}, cd^{-1}] = 1$ for all a, b, c, d belonging to S.

$$[ab^{-1}, cd^{-1}, cd^{-1}] = [[ab^{-1}, d^{-1}][ab^{-1}, c]^{d^{-1}}, cd^{-1}] =$$

$$= [[ab^{-1}, d^{-1}][ab^{-1}, c]^{d^{-1}}, d^{-1}][[ab^{-1}, d^{-1}][ab^{-1}, c]^{d^{-1}}, c]^{d^{-1}} =$$

$$= [[ab^{-1}, c]^{d^{-1}}, d^{-1}]\{[[ab^{-1}, d^{-1}], c]^{[ab^{-1},c]^{d-1}}\}^{d^{-1}}.$$

In the above calculations, $[ab^{-1}, d^{-1}, d^{-1}] = 1$ follows from the fact that $gxxg = $ $= xggx$ for $g \in gp\{S\}$ and $x \in S$ using Parts (3) and (4); and $[[ab^{-1}, c]^{d^{-1}}, c] = 1$ is true since $1 = [c, ab^{-1} \,.\, d^{-1}, c]$ by (3).

Now,

$$[ab^{-1}, cd^{-1}, cd^{-1}]^{d[c,ab^{-1}]^{d-1}} = [d^{-1}, [c, ab^{-1}]][ab^{-1}, d^{-1}, c] =$$

$$= [[c, ab^{-1}], d][d, ab^{-1}, c] = [[c, ab^{-1}], d][[ab^{-1}, c], d] = 1,$$

where the last two major steps are applications of parts (5) and (6). Consequently, $[ab^{-1}, cd^{-1}, cd^{-1}] = 1$ and $gp\{S\}$ is 2-Engel. The theorem is now proved.

A well known theorem of Levi (see Schenkman [3]) states that a 2-Engel group is nilpotent of class $\leqslant 3$. This fact in conjunction with the Theorem produces the following.

Corollary. *If S is a cancellative semigroup satisfying the E_2 law, then S generates a group which is nilpotent of class $\leqslant 3$.*

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