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ON A GENERALIZATION OF PERMUTABLE EQUIVALENCE RELATIONS

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The concept of permutability of two equivalence relations is of great importance in many fields. There are situations where a more general concept of permutability dealing with more than two equivalence relations is needed (e. g. direct representations of algebras [9], [10], subdirect representations of algebras [15], or independence of equational classes [6]). Such concepts are introduced e. g. in [9] („completely permutable“ equivalence relations) and [10] („associable“ equivalence relations).

Lattice — theoretical consequences of pairwise permutability of equivalence relations were studied by several authors (see e. g. [3], [7], [14]). One of the most familiar examples is Dedekind's theorem on the modularity of congruence lattice of a group. The aim of the present note is to study some lattice — theoretical properties of systems of equivalence relations derived from a system of associable equivalence relations [10] and because in some cases a generalization of the concept of equivalence relations is useful, such as symmetric and transitive relation (ST-relation) (see e. g. [3] and [8]), the definitions and theorems of the present paper are given for ST-relations and specialized to equivalence relations. The mentioned results in [3], [7], [14] are obtained as corollaries. Some results of [7] concerning pairwise permutability of equivalence relations are completed (Theorem 2.12, Remark 2.17). Ph. Dwinger [16] proved that the congruence lattice of an algebra with pairwise permutable congruence relations is completely modular. In theorem 2.8 we get a generalization of this assertion. It seems that J. Hashimoto's concept of permutability is less convenient to obtain the lattice — theoretical consequences treated in this paper (even not for equivalence relations, see Remark 2.6).

1. Notations, Definitions and Some Propositions

In the whole paper M will denote a non-empty set. The empty set is denoted by $\emptyset$. Given two binary relations A, B, AB will denote their product (cf. [1, VII, § 3]).

Definition 1.1. *We say that two binary relations A_1, A_2 are permutable if $A_1A_2 = A_2A_1$.*

A partition in a set M is a system of non-empty disjoint subsets of M. Symmetric and transitive relations shall be shortly called ST-relations. There is a one-one correspondence between ST-relations in a set M and partitions in the set M. The symbol $D(A_\gamma)$ will denote a domain of the ST-relation A_γ, that is $\{x\colon x \in M$, there exist $y \in M$ such that $xA_\gamma y\}$. The symbol O will denote the empty ST-relation in M (i. e., xOy does not hold for any $x, y \in M$) . $D(O) = \emptyset$. ST-relations in M with the empty relation form a complete lattice with respect to a partial ordering $\leqq$, defined as follows: $A_1 \leqq A_2$ denotes $xA_1y \Rightarrow xA_2y$. O. Borůvka [2, § 13] has shown that there exists a partion $\bigvee\limits_{\gamma \in \Gamma} A_\gamma$, which is a lattice — theoretical join of partitions A_γ, for an arbitrary system $\{A_\gamma : \gamma \in \Gamma\}$ of partitions in M. The same holds also for ST-relations. We shall use the symbols $\wedge$, $\vee$, $\bigwedge$, $\bigvee$ (and $\cap$, $\cup$) for lattice — theoretical operations (and set-theoretical operations). By a block of an ST-relation A_γ it is meant a set $A^1_\gamma \subset D(A_\gamma)$ such that there exists an element y such that $A^1_\gamma = \{x\colon xA_\gamma y\}$. We shall define some ST-relations by quoting their blocks. E. g., C: $\{1, 2\}$, $\{3\}$ will denote the ST-relation whose blocks are $\{1, 2\}$, $\{3\}$. Blocks of ST-relation A_γ will be denoted by A^i_γ.

Lemma 1.1. [2, § 13]. *Let A_γ be an ST-relation for any $\gamma \in \Gamma$. $x(\bigvee\limits_{\gamma \in \Gamma} A_\gamma)y \Leftrightarrow$ there exists a finite sequence $\iota_1, \iota_2, \ldots, \iota_n \in \Gamma$ such that $xA_{\iota_1} A_{\iota_2} \ldots A_{\iota_n}y$.*

Definition 1.2. *A system $\{A_\gamma : \gamma \in \Gamma\}$ of ST-relations in a set M will be called associable if it has the following property: Let $\{x^\gamma : \gamma \in \Gamma\}$ be a system of elements of M such that $x^\alpha(\bigvee\limits_{\gamma \in \Gamma} A_\gamma)x^\beta$ for any α, $\beta \in \Gamma$. Then one of the next properties is satisfied:*

(1.1) *There exists $x \in M$ such that $x^\gamma A_\gamma x$ for any $\gamma \in \Gamma$.*

(1.2) *There exists $\alpha \in \Gamma$ such that all elements x^γ lie in one block A^1_α of the ST-relation A_α and for any $\gamma \in \Gamma$ either $A^1_\alpha \cap D(A_\gamma) = \emptyset$ or A^1_α is a block of the relation A_γ.*

The following Lemma is obvious.

Lemma 1.2. *A system $\{A_1, A_2\}$ of two ST-relations is associable if and only if A_1* and *A_2 are permutable.*

Remark 1.1. In the case that $\{A_\gamma : \gamma \in \Gamma\}$ is a system of equivalence relations on M the Definition 1.2 is in accord with the Definition of M. Kolibiar [10].

Remark 1.2. The empty relation O is permutable with any ST-relation.

Definition 1.3. *We call a set S of ST-relations on M completely permutable if and only if any subset $\{A_\gamma\} \subset S$ satisfies the following condition*:

(1.3) *If $x^\lambda(C_\lambda \vee C_\eta)\, x^\eta$, where $C_\lambda = \bigwedge\limits_{\nu \neq \lambda} A_\nu$, there exists $x \in M$ such that $x^\nu A_\nu x$.*

Remark 1.3. J. Hashimoto [9] similarly defined the completely permutable system of equivalence relations.

Lemma 1.3. [5, Lemma 2.1]. *The mapping h: $A_\gamma \to D(A_\gamma)$ is a lattice homomorphism from the lattice of ST-relations in a set M onto the lattice of all subsets of the set M (onto 2^M).*

Theorem 1.1. [4, Theorem 4.3]. *Let A, B be ST-relations in M. A necessary and sufficient condition for the correspondence $D \to B \vee D$ $(A \geqq D \geqq A \wedge D)$, $C \to A \wedge C$ $(A \vee B \geqq C \geqq B)$ to define an isomorphism of the intervals $[B, A \vee B] \cong [A \wedge B, A]$ is: Any block V of the relation $A \vee B$ either contains no block of the relation A or contains such a block A^1 of the relation A, that any block A^2 (of the relation A), $A^2 \neq A^1$, $A^2 \subset V$, is contained in some block of the relation B.*

2.

Lemma 2.1. *Let A, B be ST-relations in M and let $A \leqq B$. Then $AB = BA$ if and only if the following condition is satisfied*:

(2.1) *If for a block B^1 of the relation B, $B^1 \cap D(A) \neq \emptyset$, then $B^1 \subset D(A)$.*

Proof. Let $AB = BA$, $y \in B^1 \cap D(A)$ and let $x \in B^1 - D(A) \neq \emptyset$. Then $xBAy$, but $xABy$ does not hold which is a contradiction. Conversely, let $A \leqq B$ and the condition (2.1) be fulfilled. Then $xABy \Leftrightarrow xBy$ and $x \in D(A) \Leftrightarrow$ $\Leftrightarrow xBy$ and $y \in D(A) \Leftrightarrow xBAy$.

Corollary 2.1. *Any two comparable equivalence relations are permutable.*

The symbol $A_\gamma | M_1$ denotes the restriction of A_γ to the set M_1.

Lemma 2.2. *A system $\{A_\gamma : \gamma \in \Gamma\}$ of ST-relations in M is associable if and only if a subset $M_1 \subset M$ exists such that the following conditions are satisfied*:

1. *$\{A_\gamma | M_1 : \gamma \in \Gamma\}$ is an associable system of equivalence relations on M_1.*
2. *If for a block A^1_γ of a relation A_γ, $A^1_\gamma \cap (M - M_1) \neq \emptyset$ holds, then $A^1_\gamma \subset$ $\subset M - M_1$,*

3. *If for some blocks* A^1_γ, A^1_δ *of relations* A_γ, A_δ $(\gamma, \delta \in \Gamma)$, $A^1_\gamma \subset M - M_1$, $A^1_\delta \subset M - M_1$, $A^1_\gamma \cap A^1_\delta \neq \emptyset$ *hold then* $A^1_\gamma = A^1_\delta$.

Proof. Let a system $\{A_\gamma : \gamma \in \Gamma\}$ be associable. Let $M_1 = \cap \{D(A_\gamma) : : \gamma \in \Gamma\}$. Then 1. obviously holds. Now we show 2. Let A^1_γ be a block of A_γ and let $a \in A^1_\gamma \cap (M - M_1)$, $b \in A^1_\gamma \cap M_1$. Set $x^\gamma = b$ and $x^\delta = a$ for all $\delta \in \Gamma$, $\delta \neq \gamma$. Then either there exists $x \in M$ such that $aA_\delta x$ for all $\delta \neq \gamma$, or there exists $\alpha \in \Gamma$ such that $a, b \in A^1_\alpha$ and A^1_α is a block of each relation $A_\varkappa$ (because $b \in A^1_\alpha \cap D(A_\varkappa)$). In both cases we get $a \in D(A_\varkappa)$ for all $\varkappa \in \Gamma$, which is a contradiction. Hence 2. holds. Now let A^1_γ, A^1_δ be blocks of A_γ, A_δ contained in $M - M_1$ and let $b \in A^1_\gamma \cap A^1_\delta$, $a \in A^1_\gamma$. Set $x^\varkappa = a$ for all $\varkappa \neq \gamma$, $x^\gamma = b$. Then there exists $\alpha \in \Gamma$ such that $a, b \in A^1_\alpha$ and A^1_α is a block of the relation A_δ. Hence $a \in A^1_\alpha = A^1_\delta$. It implies $A^1_\gamma \subset A^1_\delta$ and symmetrically $A^1_\delta \subset A^1_\gamma$. Hence 3. holds. Conversely, let 1., 2., 3. hold and let $\{x^\gamma : \gamma \in \Gamma\}$ be such a system of elements of M that $x^\alpha(\bigvee_{\gamma \in \Gamma} A_\gamma)x^\beta$ for any $\alpha, \beta \in \Gamma$. From 2. it follows that each block of $\bigvee_{\gamma \in \Gamma} A_\gamma$ is contained either in M_1 or in $M - M_1$. Hence all x^γ are contained either in M_1 or in $M - M_1$. In the first case the condition (1.1) of Definition 1.2 is fulfilled, in the second case (1.2) of Definition 1.2 is fulfilled.

Theorem 2.1. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M *and* $\Lambda \subset \Gamma$. *Then the system* $\{A_\gamma : \gamma \in \Lambda\}$ *is associable, too. In particular any two ST-relations* A_γ, A_δ $(\gamma, \delta \in \Gamma)$ *are permutable.*

Proof. Let $\{x^\gamma : \gamma \in \Lambda\}$ be a system of elements such that $x^\iota(\bigvee_{\gamma \in \Lambda} A_\gamma)x^\delta$ for any $\iota, \delta \in \Lambda$. Let $\lambda_0 \in \Lambda$ be an arbitrary selected element. We set $x^\iota = x^{\lambda_0}$ for $\iota \in \Gamma - \Lambda$. $x^\eta(\bigvee_{\iota \in \Gamma} A_\iota)x^\nu$ holds for any $\eta, \nu \in \Gamma$ (because $\bigvee_{\gamma \in \Lambda} A_\gamma \leqq \bigvee_{\iota \in \Gamma} A_\iota$). If (1.1) of Definition 1.2 holds then by the assumption there exists $x \in M$ such that $x^\iota A_\iota x$ for any $\iota \in \Gamma$ and thus the condition (1.1) also holds for the system $\{A_\gamma : \gamma \in \Lambda\}$. Let the system $\{x^\gamma : \gamma \in \Gamma\}$ satisfy the condition (1.2). Then $x^{\lambda_0} \in A^1_\alpha$ and, since $x^{\lambda_0}(\bigvee_{\gamma \in \Lambda} A_\gamma)x^{\lambda_0}$, $x^{\lambda_0} \in D(A_{\lambda_1})$ for some $\lambda_1 \in \Lambda$. It follows that A^1_α is a block of A_{λ_1} and consequently, we can suppose $\alpha \in \Lambda$. Now it is obvious that (1.2) is satisfied for the system $\{x^\gamma : \gamma \in \Lambda\}$. Consequently the system $\{A_\gamma : \gamma \in \Lambda\}$ is associable.

The next assertion follows by using Lemma 1.2.

Corollary 2.2. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of equivalence relations in* M *(see Remark 1.1) and* $\Lambda \subset \Gamma$. *Then also the system* $\{A_\gamma : \gamma \in \Lambda\}$ *is associable. In particular any two equivalence relations* A_γ, A_δ $(\gamma, \delta \in \Gamma)$ *are permutable.*

Corollary 2.3. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M. *If for some block* A^1_α *of a relation* A_α *it holds* $A^1_\alpha \cap D(A_\beta) \neq \emptyset$ $(\alpha, \beta \in \Gamma)$, *then* $A^1_\alpha \subset D(A_\beta)$.

Proof. Let $a \in A^1_\alpha \cap A^1_\beta \neq \emptyset$ and $A^1_\alpha \not\subset D(A_\beta)$, i. e. there exists $b \in A^1_\alpha$ such that $b \notin D(A_\beta)$. Then $bA_\alpha A_\beta a$ holds but $bA_\beta A_\alpha a$ does not hold, contrary to Theorem 2.1.

Remark 2.1. Let $\{A_\iota : \iota \in \Gamma\}$ be such a system of ST-relations in M that any two elements of the system are permutable. The system $\{A_\iota : \iota \in \Gamma\}$ need not be associable, not even if it is a system of equivalence relations, as the next example shows: $M = \{1, 2, 3, 4\}$; A: $\{1, 2\}$, $\{3, 4\}$; B: $\{1, 4\}$, $\{2, 3\}$; C: $\{1, 3\}$, $\{2, 4\}$. $AB = BA$, $AC = CA$, $BC = CB$ hold. The system A, B, C is not associable because to the elements $x^A = 1$, $x^B = 2$, $x^C = 3$ there does not exist an element x fulfilling condition (1.1) of Definition 1.2 and condition (1.2) of Definition 1.2 is not satisfied, too.

Theorem 2.2. *Let* A *be an ST-relation permutable with any ST-relation* B_ι, $\iota \in \Gamma$. *Then* A *is also permutable with the ST-relation* $\bigvee_{\iota \in \Gamma} B_\iota$.

Proof. Let us denote $\bigvee_{\iota \in \Gamma} B_\iota = B$. Let $xABy$. Then there exists z such that xAz and zBy hold. By Lemma 1.1, xAz and there exist $\iota_0, \iota_1, \ldots, \iota_n \in \Gamma$ such that $zB_{\iota_0} \ldots B_{\iota_n}y$. Then $xAB_{\iota_0} \ldots B_{\iota_n}y$. It follows $xB_{\iota_0}A \ldots B_{\iota_n}y$. By successive application of permutability we get $xB_{\iota_0} \ldots B_{\iota_n}Ay$. It follows that there exists an element t such that $xB_{\iota_0} \ldots B_{\iota_n}t$ and tAy hold. By Lemma 1.1, xBt and tAy hold. Thus $xBAy$ and we have proved $AB \leqq BA$. By the assertion 3.5 [11] we get $AB = BA$.

Remark 2.2. An analogous statement for two equivalence relations has been proved in the papers [7, § 3, Th. 1, p. 76], [14, Chap. 1, § 8, p. 591].

Remark 2.3. Theorem 2.2 does not hold for $\bigwedge_{\iota \in \Gamma} B_\iota$, not even for a meet of two equivalence relations as an example in [11, § 2] shows.

Theorem 2.3. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M. *Let* $\{B_\iota : \iota \in \Gamma\}$ *be such a system of ST-relations that* $D(B_\iota) = D(A_\iota)$ *and* $A_\iota \leqq B_\iota \leqq \bigvee_{\iota \in \Gamma} A_\iota$ *hold for any* $\iota \in \Gamma$. *Then the system* $\{B_\iota : \iota \in \Gamma\}$ *is associable.*

Proof. Let $\{x^\iota : \iota \in \Gamma\}$ be a system of elements of M such that for any λ, $\varkappa \in \Gamma$ $x^\lambda(\bigvee_{\iota \in \Gamma} B_\iota)x^\varkappa$ holds. $\bigvee_{\iota \in \Gamma} B_\iota = \bigvee_{\iota \in \Gamma} A_\iota$ holds and thus $x^\lambda(\bigvee_{\iota \in \Gamma} A_\iota)x^\varkappa$. By assumption, (1.1) or (1.2) of Definition 1.2 holds. If (1.1) holds, then there exists $x \in M$ such that $x^\lambda A_\lambda x$ and thus $x^\lambda B_\lambda x$ holds for any $\lambda \in \Gamma$. It follows that condition (1.1) is fulfilled for the system $\{B_\iota : \iota \in \Gamma\}$, too. Now let condition (1.2) of Definition 1.2 be satisfied, i. e. there exists $\alpha \in \Gamma$ such that all elements x^{γ} lie in one block A^1_{α} of the relation A_{α} and for any $\gamma \in \Gamma$ either $A^1_{\alpha} \cap D(A_{\gamma}) = \emptyset$ holds or A^1_{α} is a block of the relation A_{γ}. We assert: A^1_{α} is a block of the relation B_{α}. Since $A_{\alpha} \leqq B_{\alpha}$, there exists B^1_{α} such that $A^1_{\alpha} \subset B^1_{\alpha}$. If $A^1_{\alpha} \neq B^1_{\alpha}$, then since $D(A_{\alpha}) = D(A_{\beta})$, there must exist $A^2_{\alpha} \neq A^1_{\alpha}$ such that $A^1_{\alpha} \cup A^2_{\alpha} \subset B^1_{\alpha}$. Because $B_{\alpha} \leqq \bigvee_{\iota \in \Gamma} A_{\iota}$, there exists a block A^1_{δ} of a relation A_{δ} ($\delta \in \Gamma$), incident with both blocks A^1_{α}, A^2_{α}, contrary to condition (1.2). Thus A^1_{α} is a block of the relation B_{α}. In the case that A^1_{α} is a block of relation A_{γ} we have to show that it is a block of the relation B_{γ}, too. Let us denote $A^1_{\alpha} = A^1_{\gamma}$. If $A^1_{\gamma} \subsetneq B^1_{\gamma}$, then, since $A_{\gamma} \leqq B_{\gamma}$ and $D(A_{\gamma}) = D(B_{\gamma})$, there must exist $A^2_{\gamma} \neq A^1_{\gamma}$ such that $A^1_{\gamma} \cup A^2_{\gamma} \subset B^1_{\gamma}$. Since $B_{\gamma} \leqq \bigvee_{\iota \in \Gamma} A_{\iota}$, a block A^1_{λ} of a relation A_{λ} exists ($\lambda \in \Gamma$, $\lambda \neq \gamma$) which is incident with both blocks A^1_{γ}, A^2_{γ}. Then $A^1_{\alpha} \cap D(A_{\lambda}) \neq \emptyset$ and A^1_{α} is not a block of relation A_{λ} contrary to condition (1.2) of Definition 1.2. It follows that the block $A^1_{\alpha} = A^1_{\gamma}$ is a block of relation B_{γ}. In this case the system $\{B_{\iota} : \iota \in \Gamma\}$ fulfils condition (1.2) of Definition 1.2, too. It follows that the system $\{B_{\iota} : \iota \in \Gamma\}$ is associable.

Remark 2.4. The condition $D(A_{\iota}) = D(B_{\iota})$ for any $\iota \in \Gamma$ cannot be left out as the next example shows: $A_1 : \{1\}$, $A_2 : \{2, 3\}$, $B_1 : \{1\}, \{2\}$. $A_1 \vee A_2 \geqq B_1 \geqq A_1$. $2B_1A_23$ holds but $3B_1A_22$ does not hold, consequently $B_1A_2 \neq A_2B_1$. It follows that the system $\{B_1, A_2\}$ is not associable, although the system $\{A_1, A_2\}$ is.

Corollary 2.4. *Let A, B, C be ST-relations in M and let $AB = BA$, $A \leqq C \leqq A \vee B$, $D(C) = D(A)$. Then B and C are permutable.*

Corollary 2.5. *Let $\{A_{\iota} : \iota \in \Gamma\}$ be a system of equivalence relations on M. Let $\{B_{\iota} : \iota \in \Gamma\}$ be such a system of equivalence relations that $A_{\iota} \leqq B_{\iota} \leqq \bigvee_{\iota \in \Gamma} A_{\iota}$ hold for any $\iota \in \Gamma$. Then the system $\{B_{\iota} : \iota \in \Gamma\}$ is associable.*

Remark 2.5. An analogous statement to the Corollary 2.4 for equivalence relations (in this case condition $D(C) = D(A)$ is automatically fulfilled) is proved in papers [3, § 5.3], [7, Th. III., p. 77].

Remark 2.6. The assertion of the Theorem 2.3 does not hold if we replace „associable“ by „completely permutable“ (see Definition 1.3) even in the case of equivalence relations as the following example shows: $M = \{1, 2, 3, 4, 5, 6\}$; A_1: $\{1, 2, 3\}, \{4, 5, 6\}$; A_2: $\{1, 2, 4, 5\}, \{3, 6\}$; A_3: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$. $A_1 \vee A_2 \vee A_3$: $\{1, 2, 3, 4, 5, 6\}$. The system $\{A_1, A_2, A_3\}$ is completely permutable, because every two elements of this system are permutable and $C_1 = A_2 = C_2$, $C_3 = A_1 \wedge A_2$: $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}$ and $x^i(C_i \vee C_j)x^j$ implies $x^1 = x^2$, $x^3C_3x^1$, $x^2C_3x^3$. It sufficies to choose $x = x^3$. Let us take the system $\{A_1, A_2, A'_3\}$, where A'_3: $\{1\}, \{2, 5\}, \{3\}, \{4\}, \{6\}$. It is evident that the assumptions of the (modified) Theorem 2.3 are satisfied. $C'_1 = A_2 \wedge A'_3 =$

A'_3, $C'_2 = A_1 \wedge A'_3 = A_3$, $C'_3 = A_1 \wedge A_2 = C_3$. Let us take $x^1 = 2$, $x^2 = 5$, $x^3 = 4$. Then $2(C'_1 \vee C'_2)5$, $5(C'_2 \vee C'_3)4$, $2(C'_1 \vee C'_3)4$ hold but there does not exist an element $x \in M$ such that $2A_1x$, $5A_2x$, $4A'_3x$ would hold. It follows that the system $\{A_1, A_2, A'_3\}$ is not completely permutable.

Theorem 2.4. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M. *Let* $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$ *and let* B *be such an ST-relation that* $B \leqq A_\iota$ *holds for any* $\iota \in \Gamma_1$ *and* $D(B) \subset D(A_\varkappa)$ *for any* $\varkappa \in \Gamma_2$. *Then the system* $\{A_\iota : \iota \in \Gamma_1\} \cup \{B \vee A_\varkappa : \varkappa \in \Gamma_2\}$ *is associable.*

Proof. If $\varkappa \in \Gamma_2$ then $A_\varkappa \leqq B \vee A_\varkappa \leqq \bigvee_{\iota \in \Gamma} A_\iota$. $D(A_\varkappa) \subset D(A_\varkappa \vee B) = D(B) \cup D(A_\varkappa) = D(A_\varkappa)$ (Lemma 1.3). Thus $D(A_\varkappa) = D(B \vee A_\varkappa)$ and consequently the assumptions of Theorem 2.3 for the considered system are fulfilled.

Corollary 2.6. *Let* A, B, C *be ST-relations in* M. *Let* $AB = BA$, $C \leqq A$ *and* $D(C) \subset D(B)$ *hold. Then* A *and* $C \vee B$ *are permutable.*

Remark 2.7. An analogous statement to Corollary 2.6 for equivalence relations (here the condition $D(C) \subset D(B)$ is automatically satisfied) was proved by O. Borůvka [3, § 5.3].

Corollary 2.7. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of equivalence relations on* M. *Let* $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$ *and let* B *be such an equivalence relation on* M *that* $B \leqq A_\iota$ *holds for any* $\iota \in \Gamma_1$. *Then the system* $\{A_\iota : \iota \in \Gamma_1\} \cup \{B \vee A_\varkappa : \varkappa \in \Gamma_2\}$ *of equivalence relations is associable.*

Theorem 2.5. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M. *Let* $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ *and let* B_1, B_2 *be such ST-relations that* $B_1 \leqq A_\iota$ *for any* $\iota \in \Gamma_1$, $B_2 \leqq A_\varkappa$ *for any* $\varkappa \in \Gamma_2$ *and* $D(B_1) \subset D(A_\varkappa)$, $D(B_2) \subset D(A_\iota)$ *for any* $\varkappa \in \Gamma_2$ *and any* $\iota \in \Gamma_1$. *Then the system* $\{B_2 \vee A_\iota : \iota \in \Gamma_1\} \cup \{B_1 \vee A_\varkappa : \varkappa \in \Gamma_2\}$ *is associable.*

Proof. It suffices to use the Theorem 2.4 twice.

Corollary 2.8. *Let* A, B, A', B' *be ST-relations in* M, $AB = BA$, $A' \leqq A$, $B' \leqq B$, $D(A') \subset D(B)$, $D(B') \subset D(A)$. *Then* $A \vee B'$ *and* $A' \vee B$ *are permutable.*

Remark 2.8. The assumption about the domains of the considered ST-relations in Theorem 2.5 and Corollary 2.8 can be omitted if all these ST-relations are equivalence relations. In this case Corollary 2.8 is symmetric to the Ore's assertion (see Remark 2.10).

Theorem 2.6. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M. *Let* $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$ *and let* B *be such an ST-relation in* M *that* $A_\iota \leqq B$ *holds for any* $\iota \in \Gamma_1$. *Then the system* $\{A_\iota : \iota \in \Gamma_1\} \cup \{B \wedge A_\varkappa :$

$: \varkappa \in \Gamma_2\}$ *is associable. In particular it holds if all* A_ι $(\iota \in \Gamma)$ *are equivalence relations.*

Proof. Let $\{x^\gamma : \gamma \in \Gamma\}$ be such a system of elements of M that for all $\gamma, \delta \in \Gamma$

(2.2) $x^\gamma(\bigvee_{\iota \in \Gamma_1} A_\iota \vee \bigvee_{\iota \in \Gamma_2} (B \wedge A_\varkappa))x^\delta$ holds.

It follows

(2.3) $x^\gamma(\bigvee_{\lambda \in \Gamma} A_\lambda)x^\delta$ holds for all $\gamma, \delta \in \Gamma$,

(2.4) $x^\gamma B x^\delta$ holds for all $\gamma, \delta \in \Gamma$.

With respect to (2.3) and to the fact that the system $\{A_\iota : \iota \in \Gamma\}$ is associable, one of the conditions (1.1), (1.2) of Definition 1.2 is fulfilled. If condition (1.1) is satisfied then it suffices to show $x^\varkappa B x$ for any $\varkappa \in \Gamma_2$. But this follows directly: Since $\Gamma_1 \neq \emptyset$, there exists $\delta \in \Gamma_1$. Then $x^\delta A_\delta x$, thus $x^\delta B x$ which follows by using (2.4), $x^\varkappa B x$ for any $\varkappa \in \Gamma_2$. Now let condition (1.2) be satisfied. Let B^1 be a block of the relation B containing x^α (by (2.4) such a block exists). By (2.4) $x^\lambda \in B^1$ holds for all $\lambda \in \Gamma$, thus all elements x^λ belong to the block $B^1 \cap A^1_\alpha$ of the relation $B \wedge A_\alpha$. (If $\alpha \in \Gamma_1$ then obviously $B^1 \cap A^1_\alpha = A^1_\alpha$.) Now we shall verify condition (1.2) for the system $\{A_\iota : \iota \in \Gamma_1\} \cup \{B \wedge A_\varkappa : \varkappa \in \Gamma_2\}$. If $\gamma \in \Gamma_1$ this is trivial. Let $\gamma \in \Gamma_2$ and $A^1_\alpha \cap D(A_\gamma \wedge B) \neq \emptyset$. It follows $A^1_\alpha \cap D(A_\gamma) \neq \emptyset$. Then A^1_α is a block of the relation A_γ, thus $B^1 \cap A^1_\alpha$ is a block of the relation $B \wedge A_\gamma$, too. Consequently, the considered system is associable.

Corollary 2.9. *Let* A, B, C *be ST-relations in* M. *Let* $AB = BA$ *and let* $A \leqq C$ *hold. Then* A *and* $B \wedge C$ *are permutable.*

Remark 2.9. An analogous statement to the Corollary 2.9 for equivalence relations is proved in papers [3, § 5.3], [7, Th. II., p. 76], and [14, Chap. I. § 8].

Theorem 2.7. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of ST-relations in* M. *Let* $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$. *Let* B_1, B_2 *be such ST-relations that* $A_\iota \leqq B_1$ *holds for any* $\iota \in \Gamma_1$ *and* $A_\varkappa \leqq B_2$ *holds for any* $\varkappa \in \Gamma_2$. *Then the system* $\{B_1 \wedge A_\varkappa : \varkappa \in \Gamma_2\} \cup \{B_2 \wedge A_\iota : \iota \in \Gamma_1\}$ *is associable. In particular this holds if all* A_ι $(\iota \in \Gamma)$ *are equivalence relations on* M.

Proof. It suffices to use Theorem 2.6 twice.

Corollary 2.10. *Let* A, B, A_1, B_1 *be ST-relations in* M. *Let* $AB = BA$, $A \leqq A_1$, $B \leqq B_1$ *hold. Then* $A_1 \wedge B$ *and* $B_1 \wedge A$ *are permutable.*

Remark 2.10. An analogous statement to this Corollary for equivalence relations is in [14, Chap. I., § 8].

Theorem 2.8. *Let* A_ι, B_ι *(for* $\iota \in \Gamma$*) be ST-relations in* M. *Let any two ele-*

ments of the system $\{A_\iota : \iota \in \Gamma\}$ *be permutable and let*

(2.5) $A_\iota \leqq B_\varkappa$ *hold for any* $\iota \neq \varkappa$.

Then $(\bigvee_{\iota\in\Gamma} A_\iota) \wedge \bigwedge_{\iota\in\Gamma} B_\iota = \bigvee_{\iota\in\Gamma} (A_\iota \wedge B_\iota)$. *In particular this holds if* A_ι, B_ι $(\iota \in \Gamma)$ *are equivalence relations on* M.

Proof. $(\bigvee_{\iota\in\Gamma} A_\iota) \wedge \bigwedge_{\iota\in\Gamma} B_\iota \geqq \bigvee_{\iota\in\Gamma} (A_\iota \wedge B_\iota)$ holds for the elements fulfilling (2.5) in an arbitrary complete lattice. We shall show the converse inequality. Let $x[(\bigvee_{\iota\in\Gamma} A_\iota) \wedge \bigwedge_{\iota\in\Gamma} B_\iota]y$ hold. Then $x(\bigvee_{\iota\in\Gamma} A_\iota)y$ and $xB_\iota y$ for any $\iota \in \Gamma$. This means that there exists a finite sequence $z_0, z_1, \ldots, z_n$, $z_0 = x$, $z_n = y$ and to any $i \in \{0, 1, \ldots, n\}$ there exists $\iota(i) \in \Gamma$ such that $z_i A_{\iota(i)} z_{i+1}$. Because of the permutability we can suppose $\iota(i) \neq \iota(j)$ for $i \neq j$. Let $i \in \{0, 1, \ldots, n\}$. Then $z_i A_{\iota(i)} z_{i+1}$. If $i \neq j$ then $A_{\iota(j)} \leqq B_{\iota(i)}$, consequently $z_j B_{\iota(i)} z_{j+1}$ holds for all $j \neq i$. Then $z_i B_{\iota(i)} x$ and $z_{i+1} B_{\iota(i)} y$. But $x B_{\iota(i)} y$, thus $z_i B_{\iota(i)} z_{i+1}$. From this and from $z_i A_{\iota(i)} z_{i+1}$ it follows $z_i(A_{\iota(i)} \wedge B_{\iota(i)}) z_{i+1}$. Hence $x[\bigvee_{\iota\in\Gamma} (A_\iota \wedge B_\iota)]y$.

Corollary 2.11 [16]. *Let* $\mathfrak{A}$ *be an algebra such that each two congruence relations of* $\mathfrak{A}$ *are permutable. Then the lattice of all congruence relations of* $\mathfrak{A}$ *is completely modular (i. e. satisfies the assertion of Theorem 2.8). In particular the lattice of all normal subgroups of a group is completely modular.*[1]

Corollary 2.12. *Let* A_ι, B_ι $(\iota \in \Gamma)$ *be ST-relations in* M. *Let the system* $\{A_\iota : \iota \in \Gamma\}$ *be associable and let* $A_\iota \leqq B_\varkappa$ *hold for any* $\iota \neq \varkappa$. *Then* $(\bigvee_{\iota\in\Gamma} A_\iota) \wedge \bigwedge_{\iota\in\Gamma} B_\iota = \bigvee_{\iota\in\Gamma} (A_\iota \wedge B_\iota)$. *This holds in particular if* A_ι, B_ι $(\iota \in \Gamma)$ *are equivalence relations on* M.

Corollary 2.13. *Let* A, B, C *be ST-relations in* M. *Let* $AB = BA$ *and* $A \leqq C$ *hold. Then* B *is modular with respect to* C *and* A *i. e.* $C \wedge (A \vee B) = A \vee (C \wedge B)$.

Remark 2.11. An analogous statement to the Corollary 2.13 for equivalence relations is proved in the papers [3, § 5.4], [7, Th. VII., p. 81], and [14, Chap. I, § 8]. The converse statement to Corollary 2.13 [i. e. that the implication $A \leqq C \Rightarrow C \wedge (A \vee B) = A \vee (C \wedge B)$ follows $AB = BA$] does not hold, not even for equivalence relations as the example in [3, § 5.4] shows.

Corollary 2.14. *Let* A, B, C, D *be ST-relations in* M. *Let* $AB = BA$, $A \leqq C$, $D(C) = D(A)$, $B \leqq D$ *and* $D(D) = D(B)$ *hold. Then* $A_1 = A \vee (C \wedge B) = C \wedge (A \vee B)$ *and* $B_1 = B \vee (A \wedge D) = D \wedge (A \vee B)$ *are permutable.*

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[1] The concept of „complete modularity" is due to A. G. Kuroš [13]. The last assertion on the lattice of normal subgroups is given in [12, Chap. XI., § 44].

Proof. It suffices to use Corollary 2.8 by setting $A' = A \wedge D$, $B' = C \wedge B$. $D(D \wedge A) = D(D) \cap D(A) = D(B) \cap D(A) \subset D(B)$ (Lemma 1.3) and similarly $D(C \wedge B) \subset D(A)$.

Remark 2.12. An analogous statement to this Corollary for equivalence relations (the conditions $D(C) = D(A)$, $D(D) = D(B)$ are automatically fulfilled) is in [3, § 5.4].

Theorem 2.9. *Let A, B, C be ST-relations in M, $AB = BA$, $C \leqq A \vee B$, $D(C) \subset D(A) \cap D(B)$ and $C = (A \vee C) \wedge (B \vee C)$. Then $CA = AC$ and $CB = BC$ hold.*

Proof. Since $AB = BA$, $A \leqq A \vee C \leqq A \vee B$ and $D(C) \subset D(A)$ hold, by Lemma 1.3 $D(A \vee C) = D(A) \cup D(C) = D(A)$; then by Corollary 2.4 $A \vee C$ and B are permutable. Combining this with $B \leqq B \vee C$ we get, using Corollary 2.9, that $C = (A \vee C) \wedge (B \vee C)$ and B are permutable. $CA = AC$ can be proved symmetrically.

Remark 2.13. The following example shows that even for equivalence relations the following statement, being the converse of Theorem 2.9, does not hold: Let A, B, C be equivalence relations on M, $AB = BA$, $C \leqq A \vee B$, $CA = AC$, $CB = BC$. Then $C = (A \vee C) \wedge (B \vee C)$. This statement does not hold even if we suppose $A \wedge B \leqq C$. Example: $M = \{1, 2, 3, 4\}$; A: $\{1, 2\}$, $\{3, 4\}$; B: $\{1, 4\}$, $\{2, 3\}$; C: $\{1, 3\}$, $\{2, 4\}$; $A \wedge B$: $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$; $A \vee B = B \vee C$: $\{1, 2, 3, 4\}$; $CA = AC$, $CB = BC$, but $C \neq (A \vee C) \wedge (B \vee C)$ because $1(A \vee C) \wedge (B \vee C)2$ holds but $1C2$ does not hold.

Corollary 2.15. *Let A, B, C be ST-relations in M, $AB = BA$, C be between A and B [i. e. $(A \wedge C) \vee (B \wedge C) = C = (A \vee C) \wedge (B \vee C)$], $D(C) \subset D(A) \cap D(B)$. Then $CA = AC$ and $CB = BC$ hold.*

Lemma 2.3. *Let A, B, C be such ST-relations in M that $CB = BC$ and $A \wedge B \leqq C \leqq A$ hold. Then $C = A \wedge (C \vee B)$.*

Proof. By Corollary 2.13, $A \wedge (C \vee B) = C \vee (A \wedge B) = C$.

Lemma 2.4. *Let A, B, C be ST-relations in M such that $AB = BA$ and $A \wedge B \leqq C \leqq A$ hold. Then: $BC = CB \Leftrightarrow C = A \wedge C'$ for some C' such that $B \leqq C' \leqq A \vee B$. The above-mentioned assumptions imply that $C' = B \vee C$ holds.*

Proof. The assumptions $BC = CB$, $C \leqq A$ imply by Corollary 2.13 $A \wedge (B \vee C) = C \vee (A \wedge B) = C$. Conversely, let $C = A \wedge C'$, $B \leqq C' \leqq A \vee B$. By Corollary 2.13, it follows $C' = C' \wedge (A \vee B) = B \vee (C' \wedge A) = B \vee C$. By Corollary 2.9, $C = A \wedge C'$ and B are permutable.

Remark 2.14. The implication „$\Leftarrow$" for the equivalence relations is proved in [7, Th. VII., p. 78].

Theorem 2.10. *Let A, B be ST-relations in M such that $AB = BA$. Then the mapping $\varphi : C' \to A \wedge C'$ is an isomorphism from the interval $[B, A \vee B]$ onto some sublattice P of the interval $[A \wedge B, A]$. The sublattice P consists of exactly those ST-relations of $[A \wedge B, A]$ which are permutable with B.*

Proof. Let us take $C'_1, C'_2 \in [B, A \vee B]$. $A \wedge (C'_1 \wedge C'_2) = (A \wedge C'_1) \wedge (A \wedge C'_2)$. Let us denote $C_i = A \wedge C'_i$ for $i = 1, 2$. From the facts $C'_i \geqq B$ and $AB = BA$ we get by Lemma 2.4, $C'_i = B \vee C_i$ for $i = 1, 2$. Then $A \wedge (C'_1 \vee C'_2) = A \wedge (B \vee C_1 \vee C_2) = A \wedge [B \vee (C_1 \vee C_2)]$. By Corollary 2.9 $BC_i = C_iB$ for $i = 1, 2$. By Theorem 2.2, $B(C_1 \vee C_2) = (C_1 \vee C_2)B$. Using Corollary 2.13, we get $A \wedge [B \vee (C_1 \vee C_2)] = (A \wedge B) \vee [A \wedge (C_1 \vee C_2)] = A \wedge (C_1 \vee C_2) = C_1 \vee C_2 = (A \wedge C'_1) \vee (A \wedge C'_2)$. Now we show that φ is injective. Let $C'_1, C'_2 \in [B, A \vee B]$, $C'_1 \neq C'_2$. Let us take $C_i = A \wedge C'_i$ for $i = 1, 2$. If $C_1 = C_2$, then $C'_1 = B \vee C_1 = B \vee C_2 = C'_2$, contrary to the assumption. The remaining assertion about the sublattice P follows from Lemma 2.4.

Remark 2.15. An analogous statement for equivalence relations is in the paper [7, § 5, p. 82].

Remark 2.16. In paper [7] the following Theorem is proved (Theorem V, p. 78): A necessary and sufficient condition that any equivalence relation $C \in [A \wedge B, A]$ be permutable with the equivalence relation B is that A and B be „semi-consécutive“. (The equivalence relations A, B are called semi-consécutive if any block of the relation $A \wedge B$ is either block of the relation A or B.) If we introduce an analogous concept of semi-consécutivity for ST-relations in M then the mentioned Theorem need not hold, as the following example shows: $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$, B: $\{1, 2\}$, $\{5, 6, 7, 8\}$, A: $\{1, 2, 3, 4\}$, $\{5, 6\}$; $A \wedge B$: $\{1, 2\}$, $\{5, 6\}$. Let us consider the ST-relation C: $\{1, 2, 3\}$, $\{5, 6\}$. The assumptions of the said Theorem are fulfilled, but $CB \neq BC$, because $3CB1$ holds and $3BC1$ does not hold.

Theorem 2.11. *Let A, B be ST-relations in M. The necessary and sufficient condition that all ST-relations $C \in [A \wedge B, A]$ be permutable with B is: $AB = BA$ and any block V of the relation $A \vee B$ either contains no block of the relation A or contains such a block A^1 of the relation A that any block A^2 (of the relation A), $A^2 \neq A^1$, $A^2 \subset V$, is contained in some block of the relation B.*

Proof. The assertion follows from Theorem 2.10 and Theorem 1.1.

Theorem 2.12. *Let A, B be permutable ST-relations in M and let the system $\{C_\gamma : \gamma \in \Gamma\}$ of ST-relations in M have the property: $A \wedge B \leqq C_\gamma \leqq A$ holds for any $\gamma \in \Gamma$ and any C_γ is permutable with B. Then $\bigvee_{\gamma\in\Gamma} C_\gamma$ and $\bigwedge_{\gamma\in\Gamma} C$ are permutable with B, thus the set of all ST-relations of the interval $[A \wedge B, A]$*

which are permutable with B forms a complete lattice which is a closed sublattice (cf. [1]) *of the interval* $[A \wedge B, A]$.

Proof. By Theorem 2.2, $(\bigvee_{\gamma \in \Gamma} C_\gamma)B = B(\bigvee_{\gamma \in \Gamma} C_\gamma)$. Now let $aB(\bigwedge_{\gamma \in \Gamma} C_\gamma)b$. Then there exists an element u such that aBu and $u(\bigwedge_{\gamma \in \Gamma} C_\gamma)b$, thus $uC_\gamma b$ for any $\gamma \in \Gamma$. Then $aBC_\gamma b$ for any $\gamma \in \Gamma$ and with respect to $BC_\gamma = C_\gamma B$ for any $\gamma \in \Gamma$, there exist elements s_γ such that:

(0) $aC_\gamma s_\gamma$ holds for any $\gamma \in \Gamma$

(00) $s_\gamma Bb$ holds for any $\gamma \in \Gamma$.

Thus $s_\gamma B s_\varkappa$ for any $\gamma, \varkappa \in \Gamma$. Obviously aAs_γ for any $\gamma \in \Gamma$, thus $s_\gamma A s_\varkappa$ for any $\gamma, \varkappa \in \Gamma$. Hence $s_\gamma(B \wedge A)s_\varkappa$ for any $\gamma, \varkappa \in \Gamma$, which follows $s_\gamma C_\gamma s_\varkappa$ for any $\gamma, \varkappa \in \Gamma$. Combining this with (0) we get $aC_\gamma s_\varkappa$ for any $\gamma, \varkappa \in \Gamma$, thus $a(\bigwedge_{\gamma \in \Gamma} C_\gamma)s_\varkappa$. Combining this with (00) we get $a(\bigwedge_{\gamma \in \Gamma} C_\gamma)Bb$. We have proved $B(\bigwedge_{\gamma \in \Gamma} C_\gamma) \leqq (\bigwedge_{\gamma \in \Gamma} C_\gamma)B$ and by the statement 3.5 [11], $B(\bigwedge_{\gamma \in \Gamma} C_\gamma) = (\bigwedge_{\gamma \in \Gamma} C_\gamma)B$ follows.

Remark 2.17. In paper [7, Th. VI., p. 79] it is shown that the set of equivalence relations from $[A \wedge B, A]$ which are permutable with the equivalence relation B forms a sublattice of the interval $[A \wedge B, A]$.

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