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A SUFFICIENT DISCONJUGACY CONDITION FOR THE THIRD ORDER DIFFERENTIAL EQUATION

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A linear differential equation of the n-th order is said to be *disconjugate* on an interval J if every nontrivial solution of this equation has at most n-1 zeros (including multiplicity) on J.

In this paper a sufficient condition for the disconjugacy of the differential equation

$$Lx = x''' + p(t)x'' + q(t)x' + r(t)x = 0 \tag{1}$$

on J is established. This condition generalizes results of [1] and [2] dealing with the above — mentioned problem.

Throughout the present paper the interval J denotes any interval at the number axis with the terminal points a and b, where $-\infty < a < b < +\infty$.

The coefficients $p(t)$, $q(t)$, $r(t)$ are assumed to be locally integrable functions on the interval J. Furthermore the functions $p(t)$, $r(t)$ are bounded on the interval J and $q(t)$ is bounded from above on J. (All inequalities are to be understood to hold almost everywhere on J.)

Let

1. $h = b - a$;
2. $|p(t)| \leqq P, \quad q(t) \leqq Q, \quad |r(t)| \leqq R \quad$ for $\quad t \in J$,
where P, Q, R are real numbers;
3. $E_0(t) = e^t - e^{\frac{t}{2}} - \frac{t}{2}, \quad E(t) = te^t - e^t - \frac{t^2}{2} + 1, \quad F(t) = e^t - t - 1;$
4. $C_*^m(J)$ means the set of functions with the absolutely continuous m-th derivative on J.

R. M. Mathsen [1] has proved the following

Theorem A. *Let $J = [a, b]$. Let $p(t)$, $q(t)$, $r(t)$ be continuous functions on the interval J, $q(t) \leqq 0$ for $t \in J$ and $P > 0$,*

$$\frac{R}{P^2}(h + 1)\, E_0(hP) \leqq 1 .$$

Then the differential equation (1) *is disconjugate on the interval J.*

L. K. Jackson [2] has generalized this theorem in the following sense:

Theorem B. *Let* $J = [a, b]$ *and* $p(t)$, $q(t)$, $r(t)$ *be continuous functions in the interval* J *and* $q(t) \leqq 0$ *for* $t \in J$. *If* $P > 0$ *and*

$$\frac{R}{P^2} hE_0(hP) \leqq 1,$$

then the differential equation (1) *is disconjugate on the interval* J.

The following quoted lemma will be used to prove an assertion generalizing Theorem B.

Lemma 1. *Let* $J = [a, b]$ *and let there be functions* $w_1(t)$, $w_2(t)$ *belonging to* $C^2_*(J)$ *with the properties*:

$$w_1(t) > 0 \quad \text{for} \quad t \in (a, b), \quad w_2(t) > 0, \quad \begin{vmatrix} w_1(t), & w_2(t) \\ w_1'(t), & w_2'(t) \end{vmatrix} > 0,$$

$$Lw_1 \geqq 0, \quad Lw_2 \leqq 0 \quad \text{for} \quad t \in (a, b].$$

Then the differential equation $Lx = 0$ *is disconjugate in the interval* J ([3], pp. 77, 80).

Theorem 1. *Let* $P > 0$, $Q \geqq 0$ *and*

$$\frac{R}{P^3} E(hP) + \frac{Q}{P^2} F(hP) \leqq 1. \tag{2}$$

Then the differential equation (1) *is disconjugate in the interval* J.

Proof. Let $j = [\alpha, \beta]$ be an arbitrary, fixed and compact subinterval of the interval J. Define the following functions

$$s_1(t) = \frac{1}{P^2}[e^{P(b-t)} - e^{P(b-a)}] + \frac{1}{P}(t - a)$$

$$s_2(t) = \frac{1}{P^2}[e^{P(b-a)} - e^{P(t-a)}] + \frac{1}{P}(t - b)$$

in the interval J.

It is obvious that $s_1(a) = s_2(b) = 0$, $s_1(t) < 0$, $s_2'(t) < 0$ if $t \in (a, b]$ and $s_2(t) > 0$, $s_1'(t) < 0$ for $t \in [a, b)$.

Put

$$w_1(t) = \int_b^t s_1(\tau)\,\mathrm{d}\tau, \quad w_2(t) = \int_a^t s_2(\tau)\,\mathrm{d}\tau \quad (t \in J). \tag{3}$$

Then

$$w_1(t) > 0 \quad \text{in} \quad [a, b), \; w_2(t) > 0 \quad \text{in} \quad (a, b];$$

$$w_1'(t) = s_1(t), \quad w_1''(t) = \frac{1}{P}[1 - e^{P(b-t)}] \leqq 0, \quad w_2'(t) = s_2(t),$$

$$w_2''(t) = -\frac{1}{P}[e^{P(t-a)} - 1] \leqq 0;$$

$$w_1'''(t) = e^{P(b-t)} = 1 - Pw_1''(t), \quad w_2'''(t) = -e^{P(t-a)} = Pw_2''(t) - 1$$

for $t \in J$.

Hence

$$w_1'(t) \geqq s_1(b) = -\frac{1}{P^2}F(hP), \quad w_2'(t) \leqq s_2(a) = \frac{1}{P^2}F(hP)$$

and

$$w_1(t) \leqq w_1(a) = \int_b^a s_1(\tau)\,\mathrm{d}\tau, \quad w_2(t) \leqq w_2(b) = \int_a^b s_2(\tau)\,\mathrm{d}\tau$$

for $t \in J$.

Since

$$\int_b^a s_1(\tau)\,\mathrm{d}\tau = \int_a^b s_2(\tau)\,\mathrm{d}\tau = P^{-3}E(hP),$$

we get

$$w_1(t) \leqq P^{-3}E(hP), \quad w_2(t) \leqq P^{-3}E(hP) \quad (t \in J).$$

From these relations and by the inequality (2), the estimates

$$w_1'''(t) = 1 - Pw_1''(t) \geqq RP^{-3}E(hP) + QP^{-2}F(hP) - p(t)w_1''(t) \geqq$$
$$\geqq Rw_1(t) - Qw_1'(t) - p(t)w_1''(t) \geqq -r(t)w_1(t) - q(t)w_1'(t) - p(t)w_1''(t),$$
$$w_2'''(t) = -1 + Pw_2''(t) \leqq -RP^{-3}E(hP) - QP^{-2}F(hP) - p(t)w_2''(t) \leqq$$
$$\leqq -Rw_2(t) - Qw_2'(t) - p(t)w_2''(t) \leqq -r(t)w_2(t) - q(t)w_2'(t) - p(t)w_2''(t)$$

hold on J, i.e. $Lw_1 \geqq 0$, $Lw_2 \leqq 0$ for $t \in J$. Further, the properties of functions $w_1(t)$, $w_2(t)$ imply

$$w_{12}(t) = \begin{vmatrix} w_1(t), & w_2(t) \\ w_1'(t), & w_2'(t) \end{vmatrix} = w_1(t)w_2'(t) - w_1'(t)w_2(t) = w_1(t)s_2(t) - s_1(t)w_2(t) > 0$$

in $[a, b]$ $(w_{12}(a) = w_1(a)\dot{s}_2(a) > 0,\ w_{12}(b) = -s_1(b)w_2(b) > 0)$.

We see that the functions $w_1(t)$, $w_2(t)$ satisfy the assumptions of Lemma 1 on the interval j. Then the differential equation $Lx = 0$ is disconjugate on j. Since the interval j is an arbitrary compact subinterval of J, the differential equation $Lx = 0$ is disconjugate on J.

Corollary. *Let* $q(t) \leqq 0$ *for* $t \in J$ *and* $P > 0$ *and*

$$\frac{R}{P^3} E(hP) \leqq 1.$$

Then the differential equation (1) *is disconjugate in the interval* J.

In view of $\tau^{-1}E(\tau) < E_0(\tau)$ for $\tau > 0$, this corollary implies Theorem B.

Theorem 1′. *Let* $Q \geqq 0$ *and*

$$R\frac{h^3}{3} + Q\frac{h^2}{2} + Ph \leqq 1. \tag{4}$$

Then the differential equation (1) *is disconjugate on the interval* J.

Proof of this theorem is analogous to the proof of Theorem 1, however, instead of the functions $w_1(t)$, $w_2(t)$ defined in (3) we have to take the functions

$$\frac{1}{2}(b-t)\left[(b-a)^2 - \frac{(b-t)^2}{3}\right], \quad \frac{1}{2}(t-a)\left[(b-a)^2 - \frac{(t-a)^2}{3}\right] \quad (t \in J),$$

respectively.

Corollary. *Let* $q(t) \leqq 0$ *for* $t \in J$ *and*

$$R\frac{h^3}{3} \leqq 1.$$

Then the differential equation

$$x''' + q(t)x' + r(t)x = 0$$

is disconjugate on the interval J.

Remark 1. Let $Q \geqq 0$, $P > 0$ and let the inequality (2) hold, then the number h satisfies the inequality

$$\frac{R}{P^3}\sum_{i=3}^{n}\frac{(Ph)^i}{i(i-2)!} + \frac{Q}{P^2}\sum_{i=2}^{m}\frac{(Ph)^i}{i!} < 1,$$

where n and m are arbitrary integers such that $n \geqq 3$, $m \geqq 2$.

Especially, if $n = 3$ and $m = 2$

$$R\frac{h^3}{3} + Q\frac{h^2}{2} < 1.$$

Remark 2. Let $Q \geqq 0$, $P > 0$ and let the inequality (4) hold, then (2) is true. Hence Theorem 1′ for $P > 0$ is a special case of Theorem 1.

Further, we shall show that the condition (2) (with $R = 0$) secures the disconjugacy of the differential equation of the second order

$$lx \equiv x'' + p(t)x' + q(t)x = 0$$

on the interval J.

Lemma 2 [4]. *Let there be a function $w(t) \in C^1_*(J)$ such that $w(t) > 0$, $lw \leqq 0$ for $t \in J - \{a\}$ or $t \in J - \{b\}$. Then the differential equation of the second order $lx = 0$ is disconjugate on the interval J* (see [3], too).

Theorem 2. *Let*

$$q(t) \leqq Q, \quad p(t) \leqq P \quad (p(t) \geqq -P) \quad \text{for} \quad t \in J^{(1)},$$

where P, Q are real numbers and let $P > 0$,

$$\frac{Q}{P^2}F(hP) \leqq 1.$$

Then the differential equation of the second order $lx = 0$ is disconjugate on the interval J.

Proof. The following two cases are possible: $Q > 0$, or $Q \leqq 0$, respectively.

Consider the first case $Q > 0$.

Put

$$w(t) = \frac{1}{P^2}[e^{P(b-a)} - e^{P(b-t)}] - \frac{1}{P}(t - a)$$

$$\left(w(t) = \frac{1}{P^2}[e^{P(b-a)} - e^{P(t-a)}] + \frac{1}{P}(t - b)\right)$$

for $t \in J$, $p(t) \leqq P$ $(p(t) \geqq -P)$ in J.

Hence we easily see that $w(t) > 0$ on $J - \{a\}$ ($w(t) > 0$ on $J - \{b\}$) and $lw \leqq 0$ for $t \in J$.

If $Q \leqq 0$, put

$$w(t) = 1, \quad t \in J.$$

Then $l1 = q(t) \leqq 0$ in J.

(1) In this Theorem the assumption of the boundedness from above (or from below) of $p(t)$ on J is sufficient.

In both cases there is a function $w(t) > 0$ in $J - \{a\}$, or $J - \{b\}$, respectively such that $lw \leqq 0$ for $t \in J$. Consequently, by means of Lemma 2 the differential equation $lx = 0$ is disconjugate in the interval J.

Corrollary. *If $p(t) \leqq 0$ $(\geqq 0)$, $q(t) \leqq Q$ for $t \in J$, where Q is a real number and*

$$Q \frac{h^2}{2} \leqq 1\,,$$

then the differential equation of the second order $lx = 0$ is disconjugate in the interval J.

Remark 3. If the hypotheses of Theorem 1 hold, then the hypotheses of Theorem 2 are satisfied too. Hence Theorem 1 may give a positive result only if the differential equation of the second order $lx = 0$ is disconjugate on the interval J and

$$\frac{Q}{P^2} F(hP) \leqq 1\,.$$

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